Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 472–489.
https://doi.org/10.2298/AADM200412036M

INTEGRAL NORM ESTIMATES FOR THE POLAR DERIVATIVE OF LACUNARY-TYPE COMPLEX POLYNOMIALS

Dedicated to Professor Gradimir Milovanović on the occasion of his 70th birthday

Abdullah Mir

In this paper, we prove some integral-norm inequalities for the polar derivative of lacunary-type complex polynomials having zeros in closed exterior or closed interior of a circle. The results obtained besides derive polar derivative analogues of some classical Bernstein and Turán-type inequalities for the uniform-norm also include several interesting generalizations and refinements of some integral-norm inequalities for polynomials as well.

1. INTRODUCTION

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n in the complex plane and $P'(z)$ its derivative. Various versions of Bernstein and Turán-type inequalities for polynomials and related classes of functions are a classical topic in analysis. These inequalities are important tools in obtaining inverse theorems in Approximation theory. One basic result of Bernstein [**5**] that relates the uniform-norm of the derivative of the polynomial on the unit circle to that of the polynomial itself on the same circle states that: if $P(z)$ is a polynomial of degree n, then it is true that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

2010 Mathematics Subject Classification. 30A10, 30C10, 30D15.
Keywords and Phrases. Polar derivative of a polynomial, Bernstein's inequality, Uniform-norm.

It is important to mention that the equality in (1) holds if and only if $P(z)$ has all its zeros at the origin, one would expect a relationship between the bound n and the distance of the zeros of the polynomial $P(z)$ from the origin. This fact was observed by Erdös and later verified by Lax [**14**] by proving that, if $P(z)$ does not vanish in $|z|<1$, then (1) can be replaced by

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\max_{|z|=1}|P(z)|, \tag{2}$$

whereas, if $P(z)$ has no zeros in $|z|>1$, then Turán [**24**] proved that

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{3}$$

In 1969, Malik [**15**] obtained extensions of (2) and (3), under the condition that $P(z)\neq 0$ in $|z|<k,\ \ k\geq 1$, we have

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k}\max_{|z|=1}|P(z)|, \tag{4}$$

whereas, if $P(z)\neq 0$ in $|z|>k,\ \ k\leq 1$, we have

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{5}$$

Chan and Malik [**7**] generalized (4) in a different direction and proved that, if $P(z)=a_0+\sum\limits_{j=\mu}^{n}a_jz^j,\ \mu\geq 1$, is a polynomial of degree n which does not vanish in $|z|<k,\ k\geq 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^{\mu}}\max_{|z|=1}|P(z)|. \tag{6}$$

On the other hand, Aziz and Shah [**4**] generalized (5) and proved that, if $P(z)=a_nz^n+\sum\limits_{j=\mu}^{n}a_{n-j}z^{n-j},\ \mu\geq 1$, is a polynomial of degree n having all zeros in $|z|\leq k,\ k\leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^{\mu}}\max_{|z|=1}|P(z)|. \tag{7}$$

Inequality (6) was independently proved by Qazi ([**21**], Lemma 1), who also under the same hypothesis proved that

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+\xi_0(\mu)}\max_{|z|=1}|P(z)|, \tag{8}$$

where

$$\xi_0(\mu)=k^{\mu+1}\left(\frac{\left(\frac{\mu}{n}\right)\left|\frac{a_\mu}{a_0}\right|k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}+1}\right). \tag{9}$$

If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j,\ \mu \geq 1$, and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, the (see [**21**]) $\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu \leq 1$, which can also be taken as equivalent to $\xi_0(\mu) \geq k^\mu$. Hence, the inequality (8) is an improvement of (6). As an refinement of (6), Jain [**12**] proved that, if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ \mu \geq 1$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^\mu}\left\{\max_{|z|=1} |P(z)| - \frac{1}{k^{n-\mu}} \min_{|z|=k} |P(z)|\right\}. \tag{10}$$

Further, a similar type of modification to inequality (8) was given by Dewan, Singh and Yadav [**8**] (see also Shah and Liman [**23**]), who under the same hypothesis proved that

$$\max_{|z|=1} |P'(z)| \leq \left(\frac{n}{1+\xi_0(\mu)}\right) \max_{|z|=1} |P(z)| - \frac{n}{k^n}\left(1 - \frac{1}{1+\xi_0(\mu)}\right) \min_{|z|=k} |P(z)|. \tag{11}$$

For a polynomial $P(z)$ of degree n, now we define so-called the polar derivative of $P(z)$ with respect to the point α as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

This polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ in the following sense:

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} := P'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.
Many of the generalizations of Bernstein and Turán-type inequalities involve the comparison of uniform-norm and L^γ-norm estimates of $D_\alpha P(z)$ with various choices of $P(z)$, α and other parameters. More information on the polar derivative of polynomials can be found in the books of Milovanović et al. [**17**], Rahman and Schmeisser [**22**] and Marden [**16**]. One can also see in the literature (for example, refer [**10**], [**13**], [**18**]- [**20**]), the latest research and development in this direction. In 2014, Mir and Dar [**20**] besides proving some other results also established the polar derivative generalizations of (10) and (11). In fact, they proved that, if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n$, is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq n\left(\frac{|\alpha| + \xi_0(\mu)}{1+\xi_0(\mu)}\right) \max_{|z|=1} |P(z)| - \frac{n(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))} \min_{|z|=k} |P(z)|, \tag{12}$$

where $\xi_0(\mu)$ is as defined by the formula (9).
By using the fact that $\xi_0(\mu) \geq k^\mu$ for $\mu \geq 1$, they also established from (12) under the same hypothesis that

$$\max_{|z|=1} |D_\alpha P(z)| \leq n\left(\frac{|\alpha| + k^\mu}{1+k^\mu}\right) \max_{|z|=1} |P(z)| - \frac{n(|\alpha|-1)}{k^{n-\mu}(1+k^\mu)} \min_{|z|=k} |P(z)|, \tag{13}$$

giving the corresponding polar derivative generalization of (10). As mentioned earlier, Bernstein and Turán-type inequalities have been extended and generalized in different domains, different norms and for different classes of functions. Zygmund [**25**] extended the Bernstein-inequality (1) to L^γ-norms of $P(z)$ as

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leq n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \quad \gamma \geq 1,$$

whereas, Arestov [**1**] validated it for $0 < \gamma < 1$, as well. As an extension of (2) to L^γ-norms, de-Bruijn [**6**] proved an analogue of Zygmund's result for the class of polynomials not vanishing in $|z| < 1$. Govil and Rahman [**11**] generalized and sharpened the inequality due to de-Bruijn for polynomials of degree n not vanishing in $|z| < k$, $k \geq 1$ and for any $\gamma \geq 1$. Gardner and Weems [**9**] not only generalized the above result of Govil and Rahman to lacunary-tpye of polynomials but also showed that it holds for $0 < \gamma < 1$, as well. Recently, Kumar [**13**] gave a polar derivative generalization of (8) by first proving an L^γ-norm estimate of the polar derivative $D_\alpha P(z)$ with $|\alpha| \geq 1$ and the result so obtained produced the desired generalization of (8). More precisely, Kumar proved the following result.

Theorem 1.1. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $\mu \geq 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for any $\gamma > 0$, and for every complex number α with $|\alpha| \geq 1$,

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leq n(|\alpha| + \xi_0(\mu))C_\gamma\left(\xi_0(\mu)\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{14}$$

where

$$C_\gamma\left(\xi_0(\mu)\right) = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\xi_0(\mu) + e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{-1}{\gamma}} \tag{15}$$

and $\xi_0(\mu)$ is as defined by the formula (9).

If we let $\gamma \to \infty$ in (14), noting that $C_\gamma\left(\xi_0(\mu)\right) \to \frac{1}{1+\xi_0(\mu)}$, we get

$$\max_{|z|=1}|D_\alpha P(z)| \leq n\left(\frac{|\alpha| + \xi_0(\mu)}{1 + \xi_0(\mu)}\right)\max_{|z|=1}|P(z)|,$$

which clearly represents a generalization of (8).

The similar type of extensions to inequalities (6) and (7) were recently established by Mir [**18**]. The main aim of this paper is to establish some general L^γ-norm inequalities for the polar derivative of a polynomial. Our method of proof may be quite different from the previous methods and the results obtained here derive polar derivative analogues of some classical Bernstein and Turán-type inequalities for the uniform-norm on the unit circle as well.

2. MAIN RESULTS

Here, we first prove the following L^γ-analogue of (12) for every $\gamma > 0$ which generalizes (14) as well. As special cases, some known inequalities that relate the uniform-norm of the derivative of a polynomial on the unit circle to that of the polynomial itself, follows as consequences from this result.

Theorem 2.1. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for every complex number α with $|\alpha| \geq 1$ and $\gamma > 0$,

$$\left\{ \int_0^{2\pi} \left[\left|D_\alpha P(e^{i\theta})\right| + \frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))} \right]^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n\Big(|\alpha| + \xi_0(\mu)\Big) C_\gamma\left(\xi_0(\mu)\right) \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{16}$$

where here and throughout $m = \min_{|z|=k} |P(z)|,\ \xi_0(\mu)$ is as defined by the formula (9) and $C_\gamma\left(\xi_0(\mu)\right)$ is as defined by the formula (15).

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality in (16) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is multiple of μ and α real number with $\alpha \geq 1$.

Remark 2.2. If we let $\gamma \to \infty$ in (16), noting that $C_\gamma\left(\xi_0(\mu)\right) \to \frac{1}{1+\xi_0(\mu)}$, we get (12). Recall that $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, therefore, as mentioned before, we have $\xi_0(\mu) \geq k^\mu,\ \mu \geq 1$. Also, for every $\alpha \in \mathbb{C}$, the function

$$x \mapsto \frac{mn(|\alpha|-1)x}{k^n(1+x)}, \quad (x \geq 0)$$

is increasing for $|\alpha| \geq 1$, it follows that

$$\frac{mn(|\alpha|-1)}{k^{n-\mu}(1+k^\mu)} \leq \frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))}. \tag{17}$$

Further, on taking $a = |\alpha| \geq 1,\ b = \xi_0(\mu)$ and $c = k^\mu$ in Lemma 3.3, we get for each $\gamma > 0$

$$\frac{|\alpha| + \xi_0(\mu)}{\left\{ \int_0^{2\pi} \left|\xi_0(\mu) + e^{i\theta}\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}} \leq \frac{|\alpha| + k^\mu}{\left\{ \int_0^{2\pi} \left|k^\mu + e^{i\theta}\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}}. \tag{18}$$

Using (17) and (18) in Theorem 2.1, we get the following L^γ-norm extension of (13).

Corollary 2.3. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for every complex number α with $|\alpha| \geq 1$

and $\gamma > 0$,

$$\left\{ \int_0^{2\pi} \left[\left| D_\alpha P(e^{i\theta}) \right| + \frac{mn(|\alpha|-1)}{k^{n-\mu}(1+k^\mu)} \right]^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n\big(|\alpha| + k^\mu\big) C_\gamma(k^\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{19}$$

where

$$C_\gamma(k^\mu) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| k^\mu + e^{i\theta} \right|^\gamma d\theta \right\}^{\frac{-1}{\gamma}}.$$

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality in (19) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is multiple of μ and α real number with $\alpha \geq 1$.

Remark 2.4. Dividing both sides of (16) and (19) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the corresponding L^γ-norm extensions of (11) and (10) respectively.

Several other interesting results easily follow from Theorem 2.1 and Corollary 2.3. Here, we mention a few of these. If we use the fact that for every complex number δ with $|\delta| \leq 1$,

$$\left| e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{mn\delta(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))} \right| \leq \left| D_\alpha P(e^{i\theta}) \right| + \frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))}$$

and

$$\left| e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{mn\delta(|\alpha|-1)}{k^{n-\mu}(1+k^\mu)} \right| \leq \left| D_\alpha P(e^{i\theta}) \right| + \frac{mn(|\alpha|-1)}{k^{n-\mu}(1+k^\mu)},$$

we get respectively the following results from Theorem 2.1 and Corollary 2.3.

Corollary 2.5. If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j, 1 \leq \mu < n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for every complex numbers $\alpha,\ \delta$ with $|\alpha| \geq 1,\ |\delta| \leq 1$ and $\gamma > 0$,

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{\delta mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n\Big(|\alpha| + \xi_0(\mu)\Big) C_\gamma\left(\xi_0(\mu)\right) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{20}$$

where $\xi_0(\mu)$ is defined by the formula (9) and $C_\gamma\left(\xi_0(\mu)\right)$ is defined by the formula (15).

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality in (20) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is multiple of μ and α real number with $\alpha \geq 1$.

Corollary 2.6. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for every complex numbers $\alpha,\ \delta$ with $|\alpha| \geq 1,\ |\delta| \leq 1$ and $\gamma > 0$,

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{\delta mn(|\alpha|-1)}{k^{n-\mu}(1+k^\mu)} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\alpha| + k^\mu) C_\gamma(k^\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{21}$$

where $C_\gamma(k^\mu)$ is as defined in Corollary 2.3.

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality in (21) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is multiple of μ and α real number with $\alpha \geq 1$.

Remark 2.7. If we take $\delta = 0$ in (20), we get (14). In the same way, if we put $\delta = 0$ in (21), we get the polar derivative analogue of an integral inequality due to Gardner and Weems [**9**].

As an application of Theorem 2.1, we shall also prove the following result which provides a generalization and refinement of (7) and many related results.

Theorem 2.8. If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, 1 \leq \mu < n,\ a_0 \neq 0$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \leq 1$, then for every complex number β with $|\beta| \leq 1$ and $\gamma > 0$,

$$\left\{ \int_0^{2\pi} \left[\left| D_\beta P(e^{i\theta}) \right| + \frac{(1-|\beta|)mn}{1+\zeta(\mu)} \right]^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n\Big(|\beta| + \zeta(\mu) \Big) C_\gamma(\zeta(\mu)) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{22}$$

where

$$\zeta(\mu) = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}$$

and

$$C_\gamma(\zeta(\mu)) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \zeta(\mu) + e^{i\theta} \right|^\gamma d\theta \right\}^{\frac{-1}{\gamma}}.$$

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality in (22) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is multiple of μ and β a real number with

$\beta \leq 1$.

Remark 2.9. Since $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, 1 \leq \mu < n$, has all its zeros in $|z| \leq k,\ k \leq 1$, and $a_0 \neq 0$, therefore, the reciprocal polynomial $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \geq \frac{1}{k},\ \frac{1}{k} \geq 1$. Thus (for example see [**21**]), we have

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \leq k^{\mu},$$

which can also be taken as equivalent to $\zeta(\mu) \leq k^{\mu}$. Taking $a = \frac{1}{|\beta|} \geq 1,\ b = \frac{1}{\zeta(\mu)},\ c = \frac{1}{k^{\mu}}$ in Lemma 3.3, we get for each $\gamma > 0$,

$$\frac{\frac{1}{|\beta|} + \frac{1}{\zeta(\mu)}}{\left\{\int\limits_0^{2\pi} \left|e^{i\theta} + \frac{1}{\zeta(\mu)}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}} \leq \frac{\frac{1}{|\beta|} + \frac{1}{k^{\mu}}}{\left\{\int\limits_0^{2\pi} \left|e^{i\theta} + \frac{1}{k^{\mu}}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}},$$

or

$$\frac{\zeta(\mu) + |\beta|}{\left\{\int\limits_0^{2\pi} |e^{i\theta} + \zeta(\mu)|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}} \leq \frac{k^{\mu} + |\beta|}{\left\{\int\limits_0^{2\pi} |e^{i\theta} + k^{\mu}|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}}.$$

Using these observations and the fact that the function

$$x \mapsto \frac{(1-|\beta|)mn}{1+x},\ (x \geq 0)$$

is decreasing for $|\beta| \leq 1$, we have the following result which in particular gives an extension of (7).

Corollary 2.10. If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, 1 \leq \mu < n,\ a_0 \neq 0$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \leq 1$, then for every complex number β with $|\beta| \leq 1$ and $\gamma > 0$,

$$\left\{\int\limits_0^{2\pi} \left[\left|D_\beta P(e^{i\theta})\right| + \frac{mn(1-|\beta|)}{1+k^{\mu}}\right]^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n\Big(k^{\mu} + |\beta|\Big) C_\gamma(k^{\mu}) \Big\{\int\limits_0^{2\pi} \left|P(e^{i\theta})\right|^{\gamma} d\theta\Big\}^{\frac{1}{\gamma}}, \tag{23}$$

where $C_\gamma(k^{\mu})$ is as defined in Corollary 2.3.

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality in (23) holds for $P(z) = (z^{\mu} + k^{\mu})^{\frac{n}{\mu}}$, where n is multiple of μ and β a real number with $\beta \leq 1$.

If we let $\gamma \to \infty$ in (23), noting that $C_\gamma(k^{\mu}) \to \frac{1}{1+k^{\mu}}$, we get under the same hypothesis that

$$\max_{|z|=1} |D_\beta P(z)| \leq n\left(\frac{k^{\mu} + |\beta|}{1+k^{\mu}}\right) \max_{|z|=1} |P(z)| - n\left(\frac{1-|\beta|}{1+k^{\mu}}\right) \min_{|z|=k} |P(z)|. \tag{24}$$

If we take $\beta = 0$ in (24), we get for $|z| = 1$,

$$|nP(z) - zP'(z)| \leq \frac{nk^{\mu}}{1 + k^{\mu}} \max_{|z|=1} |P(z)| - \frac{n}{1 + k^{\mu}} \min_{|z|=k} |P(z)|. \tag{25}$$

Let z_0 be on $|z| = 1$ such that $\max_{|z|=1} |P(z)| = |P(z_0)|$, then from (25), we get

$$|P'(z_0)| \geq \frac{n}{1 + k^{\mu}} \left(\max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right),$$

which by using the fact that $\max_{|z|=1} |P'(z)| \geq |P'(z_0)|$, gives a refinement of (7).
Remark 2.11. It is easy to see that Theorem 2.8 strengthens a recently proved result due to Kumar ([**13**], Theorem 3) and Corollary 2.10 sharpens a result of Mir ([**18**], Theorem 2).
Finally, in this paper, we consider the class of polynomials having a zero of order t at the origin and the rest of zeros outside or on the circle $|z| = k,\ k \geq 1$ and prove the following result which generalizes Theorem 2.1 and related polynomial inequalities for the uniform-norm as well.
Theorem 2.12. If $P(z) = z^t \left(a_0 + \sum_{j=\mu}^{n-t} a_j z^j \right), 1 \leq \mu < n - t,\ 0 \leq t < n - 1$, is a polynomial of degree n having t-fold zeros at the origin and the remaining $n - t$ zeros in $|z| \geq k,\ k \geq 1$, then for every complex number α with $|\alpha| \geq 1$ and $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left[\left| D_\alpha P(e^{i\theta}) \right| + \frac{m(n-t)(|\alpha| - 1)\xi_t(\mu)}{k^n (1 + \xi_t(\mu)} \right]^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq \left\{ (n-t) \left(|\alpha| + \xi_t(\mu) \right) C_\gamma(\xi_t(\mu)) + t|\alpha| \right\} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}, \tag{26}$$

where

$$\xi_t(\mu) = k^{\mu+1} \left\{ \frac{\binom{\mu}{n-t} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{\binom{\mu}{n-t} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1} + 1} \right\}$$

and

$$C_\gamma(\xi_t(\mu)) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \xi_t(\mu) + e^{i\theta} \right|^{\gamma} d\theta \right\}^{\frac{-1}{\gamma}}.$$

For $t = 0$, Theorem 2.12 reduces to Theorem 2.1 for $\gamma \geq 1$.
If we let $\gamma \to \infty$ in (26), we immediately get the following result.
Corollary 2.13. If $P(z) = z^t \left(a_0 + \sum_{j=\mu}^{n-t} a_j z^j \right), 1 \leq \mu < n - t,\ 0 \leq t < n - 1$, is a polynomial of degree n having t-fold zeros at the origin and the remaining $n - t$

zeros in $|z| \geq k,\ k \geq 1$, then for any complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + \xi_t(\mu)) + t(|\alpha| - 1)\xi_t(\mu)}{1 + \xi_t(\mu)} \right) \max_{|z|=1} |P(z)| - \left(\frac{(n-t)(|\alpha| - 1)\xi_t(\mu)}{k^n(1 + \xi_t(\mu))} \right) \min_{|z|=k} |P(z)|, \tag{27}$$

where $\xi_t(\mu)$ is as defined in Theorem 2.12.

Remark 2.14. It is easy to verify for example by the derivative test that for every complex number α with $|\alpha| \geq 1$, the function

$$x \mapsto \left(\frac{n(|\alpha| + x) + t(|\alpha| - 1)x}{1 + x} \right) \max_{|z|=1} |P(z)| - \left(\frac{(n-t)(|\alpha| - 1)x}{k^n(1 + x)} \right) \min_{|z|=k} |P(z)|$$

is a non-increasing function of x. If we combine this with $\xi_t(\mu) \geq k^\mu$, we get a refinement of the following result due to Mir and Dar ([**20**], Theorem 2).

Theorem 2.15. If $P(z) = z^t \left(a_0 + \sum_{j=\mu}^{n-t} a_j z^j \right), 1 \leq \mu < n - t,\ 0 \leq t < n - 1$, is a polynomial of degree n having t-fold zeros at the origin and the remaining $n - t$ zeros in $|z| \geq k,\ k \geq 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + k^\mu) + t(|\alpha| - 1)k^\mu}{1 + k^\mu} \right) \max_{|z|=1} |P(z)| - \left(\frac{(n-t)(|\alpha| - 1)}{k^{n-\mu}(1 + k^\mu)} \right) \min_{|z|=k} |P(z)|.$$

3. LEMMAS

We need the following lemmas to prove our theorems. The following lemma is due to Mir and Dar [**20**].

Lemma 3.1. If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then on $|z| = 1$,

$$|Q'(z)| \geq \xi_0(\mu)||P'(z)| + \frac{mn\xi_0(\mu)}{k^n},$$

where here and throughout $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ and $\xi_0(\mu)$ is as defined by the formula (9).

Lemma 3.2. Let v, w be any two positive real numbers such that $(w-h) \geq (v+h)x$, where $h \geq 0$ and $x \geq 1$. Then for any real number β with $0 \leq \beta < 2\pi$, it is true that

$$[(w - h) + (v + h)y]|x + e^{i\beta}| \leq (x + y)|w + ve^{i\beta}|, \tag{28}$$

for any $y \geq 1$.

Proof of Lemma 3.2. For $h = 0$, the above lemma was recently proved by Govil and Kumar ([**10**], Lemma 2.2), see also Kumar ([**13**], Lemma 2). Hence, assume that $h > 0$. Since $x \geq 1$, it can be easily seen for $0 \leq \beta < 2\pi$, that

$$\mathrm{Re}\left(\frac{1}{x + e^{i\beta}}\right) \geq \frac{1}{x+1}.$$

Also, the hypothesis $(w - h) \geq (v + h)x$, with $h > 0$ and $x \geq 1$ imply $w \geq vx$, where $v > 0$ and $w > 0$, we get

$$\begin{aligned}\left|v + \frac{w - vx}{x + e^{i\beta}}\right| &\geq \mathrm{Re}\left\{v + \frac{w - vx}{x + e^{i\beta}}\right\}\\ &= v + (w - vx)\mathrm{Re}\left(\frac{1}{x + e^{i\beta}}\right)\\ &\geq v + \frac{w - vx}{x + 1}\\ &= \frac{w + v}{x + 1},\end{aligned}$$

which implies

$$\left|\frac{x + e^{i\beta}}{w + ve^{i\beta}}\right| \leq \frac{x + 1}{w + v}.$$

Again, using $(w - h) \geq (v + h)x$ with $h > 0$ and $y \geq 1$, one can easily check that

$$\frac{x + 1}{w + v} \leq \frac{x + y}{w + vy + (y - 1)h}.$$

Thus, we have

$$\left|\frac{x + e^{i\beta}}{w + ve^{i\beta}}\right| \leq \frac{x + y}{(w - h) + (v + h)y},$$

which is (28) and this completes the proof of Lemma 3.2.
Lemma 3.3. If $a \geq 1$, $b \geq c \geq 1$ and $\gamma > 0$, then

$$\frac{a + b}{\left\{\int\limits_0^{2\pi} |e^{i\theta} + b|^\gamma d\theta\right\}^{\frac{1}{\gamma}}} \leq \frac{a + c}{\left\{\int\limits_0^{2\pi} |e^{i\theta} + c|^\gamma d\theta\right\}^{\frac{1}{\gamma}}}.$$

The above lemma is due to Govil and Kumar ([**10**], Lemma 2.6), see also Kumar [**13**]. The following lemma is due to Aziz and Shah [**3**].
Lemma 3.4. If $P(z)$ is a polynomial of degree n, then for every $\gamma > 0$ and β real,

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} \left|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})\right|^\gamma d\theta d\beta \leq 2\pi n^\gamma \int\limits_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta.$$

4. PROOFS OF THE THEOREMS

Proof of Theorem 2.1. By hypothesis $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \neq 0$ in $|z| < k,\ k \geq 1$, therefore, by Lemma 3.1, for each $\theta,\ 0 \leq \theta < 2\pi$, we have

$$\xi_0(\mu)|P'(e^{i\theta})| \leq |Q'(e^{i\theta})| - \frac{mn\xi_0(\mu)}{k^n},$$

where $\xi_0(\mu)$ is defined in the formula (9).
This gives

$$\xi_0(\mu)\left\{ \left|P'(e^{i\theta})\right| + \frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))} \right\} \leq \left|Q'(e^{i\theta})\right| - \frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))}.$$

Taking $w = |Q'(e^{i\theta})|$, $v = |P'(e^{i\theta})|$, $h = \frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))}$ and $y = |\alpha| \geq 1$ in Lemma 3.2, we get for ϕ real that

$$\left[\left(|Q'(e^{i\theta})| - \frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right) + |\alpha| \left(|P'(e^{i\theta})| + \frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right)\right] |\xi_0(\mu) + e^{i\phi}|$$
$$\leq (\xi_0(\mu) + |\alpha|) \left||Q'(e^{i\theta})| + e^{i\phi}|P'(e^{i\theta})|\right|. \tag{29}$$

Since $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, therefore, $P(z) = z^n\overline{Q\left(\frac{1}{\bar{z}}\right)}$ and it can be easily verified that, for $0 \leq \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}Q'(e^{i\theta}),$$

as well as for the polar derivative

$$D_\alpha P(e^{i\theta}) = nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}),$$

where $\alpha \in \mathbb{C}$. Using these equalities, we obtain

$$\begin{aligned} |D_\alpha P(e^{i\theta})| &\leq |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + |\alpha||P'(e^{i\theta})| \\ &= |Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|. \end{aligned} \tag{30}$$

The above inequality (30) gives for every $|\alpha|\geq 1$ and $\gamma>0$,

$$\int_0^{2\pi}\left|\xi_0(\mu)+e^{i\phi}\right|^\gamma d\phi\int_0^{2\pi}\left[\left|D_\alpha P(e^{i\theta})\right|+\frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right]^\gamma d\theta$$

$$=\int_0^{2\pi}\int_0^{2\pi}|\xi_0(\mu)+e^{i\phi}|^\gamma\left[\left|D_\alpha P(e^{i\theta})\right|+\frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right]^\gamma d\theta d\phi$$

$$\leq\int_0^{2\pi}\int_0^{2\pi}\left|\xi_0(\mu)+e^{i\phi}\right|^\gamma\left[\left|Q'(e^{i\theta})\right|+|\alpha||P'(e^{i\theta})|+\frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right]^\gamma d\theta d\phi$$

(31)

$$=\int_0^{2\pi}\int_0^{2\pi}\left|\xi_0(\mu)+e^{i\phi}\right|^\gamma\left[\left(\left|Q'(e^{i\theta})\right|-\frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right)\right.$$

$$\left.+|\alpha|\left(|P'(e^{i\theta})|+\frac{mn\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right)\right]^\gamma d\theta d\phi$$

(32)

$$\leq(\xi_0(\mu)+|\alpha|)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left||Q'(e^{i\theta})|+e^{i\phi}|P'(e^{i\theta})|\right|^\gamma d\theta d\phi.\quad\text{(by (29))}$$

Since for each $\gamma>0$ and arbitrary $a,\ b\in\mathbb{C}$ with ϕ real, the equality

$$\int_0^{2\pi}\left|a+e^{i\phi}b\right|^\gamma d\phi=\int_0^{2\pi}\left||a|+e^{i\phi}|b|\right|^\gamma d\phi$$

holds, using this and Lemma 3.4 in (31), we obtain for each $\gamma>0$ and $|\alpha|\geq 1$,

$$\int_0^{2\pi}\left|\xi_0(\mu)+e^{i\phi}\right|^\gamma d\phi\int_0^{2\pi}\left[\left|D_\alpha P(e^{i\theta})\right|+\frac{mn(|\alpha|-1)\xi_0(\mu)}{k^n(1+\xi_0(\mu))}\right]^\gamma d\theta$$

$$\leq(\xi_0(\mu)+|\alpha|)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta})+e^{i\phi}P'(e^{i\theta})\right|^\gamma d\phi d\theta$$

$$\leq(\xi_0(\mu)+|\alpha|)^\gamma\,2\pi n^\gamma\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta,$$

which on raising the power $\frac{1}{\gamma}$ on both sides gives inequality (16). This completes the proof of Theorem 2.1.

Proof of Theorem 2.8. By hypothesis $P(z)=a_nz^n+\sum\limits_{j=\mu}^{n}a_{n-j}z^{n-j},\ 1\leq\mu<n,$

has all its zeros in $|z| \leq k,\ k \leq 1$, and $a_0 \neq 0$. Therefore, the polynomial $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ of degree n has no zeros in $|z| < \frac{1}{k},\ \frac{1}{k} \geq 1$. Applying the inequality (16) to $Q(z)$, we get for every complex number α with $|\alpha| \geq 1$ and $\gamma > 0$,

$$\left\{\int_0^{2\pi}\left[\left|D_\alpha Q(e^{i\theta})\right| + \frac{m'n(|\alpha|-1)\xi_0'(\mu)}{\frac{1}{k^n}(1+\xi_0'(\mu))}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$
$$\leq n\left(|\alpha| + \xi_0'(\mu)\right)C_\gamma\left(\xi_0'(\mu)\right)\left\{\int_0^{2\pi}\left|Q(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{33}$$

where

$$m' = \min_{|z|=\frac{1}{k}}|Q(z)| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m}{k^n},$$

$$\begin{aligned}\xi_0'(\mu) &= \frac{1}{k^{\mu+1}}\left(\frac{\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right|\frac{1}{k^{\mu-1}}+1}{\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right|\frac{1}{k^{\mu+1}}+1}\right)\\ &= \frac{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}{\mu|a_{n-\mu}|k^{\mu-1} + n|a_n|k^{2\mu}}\\ &= \frac{1}{\zeta(\mu)}\end{aligned}$$

$$\begin{aligned}\text{and}\quad C_\gamma\left(\xi_0'(\mu)\right) &= C_\gamma\left(\frac{1}{\zeta(\mu)}\right)\\ &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{\zeta(\mu)} + e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{-1}{\gamma}}\\ &= \zeta(\mu)\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1 + \zeta(\mu)e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{-1}{\gamma}}\\ &= \zeta(\mu)\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\zeta(\mu) + e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{-1}{\gamma}}\\ &= \zeta(\mu)C_\gamma(\zeta(\mu)).\end{aligned}$$

Using these observations and $|Q(e^{i\theta})| = |P(e^{i\theta})|$ for $0 \leq \theta < 2\pi$ in (33), we obtain for $|\alpha| \geq 1$ and $\gamma > 0$, that

$$\left\{\int_0^{2\pi}\left[\left|D_\alpha Q(e^{i\theta})\right| + \frac{mn(|\alpha|-1)}{1+\zeta(\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\alpha|\zeta(\mu)+1)C_\gamma(\zeta(\mu))\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \tag{34}$$

It is easy to verify (for example, see [**2**]), that for $|\alpha| \neq 0$,

$$|D_\alpha Q(e^{i\theta})| = |\overline{\alpha}|\left|D_{\frac{1}{\overline{\alpha}}}P(e^{i\theta})\right|,$$

which on using in (34), gives for $|\alpha| \geq 1$ and $\gamma > 0$,

$$|\alpha|\left\{\int_0^{2\pi}\left[\left|D_{\frac{1}{\overline{\alpha}}}P(e^{i\theta})\right| + \frac{mn\left(1-\frac{1}{|\alpha|}\right)}{1+\zeta(\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\alpha|\zeta(\mu)+1)C_\gamma(\zeta(\mu))\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \tag{35}$$

Replacing $\frac{1}{\overline{\alpha}}$ by β so that $|\beta| \leq 1$, we obtain from (35) that for each $\gamma > 0$,

$$\left\{\int_0^{2\pi}\left[\left|D_\beta P(e^{i\theta})\right| + \frac{(1-|\beta|)mn}{1+\zeta(\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\beta|+\zeta(\mu))C_\gamma(\zeta(\mu))\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}},$$

which is (22) and this completes the proof of Theorem 2.8.

Proof of Theorem 2.12. If $P(z) = z^t\rho(z)$, where $\rho(z) = a_0 + \sum_{j=\mu}^{n-t} a_j z^j, 1 \leq \mu < n-t$, is a polynomial of degree $n-t$ having no zeros in $|z| \leq k,\ k \geq 1$. Applying the inequality (16) to the polynomial $\rho(z)$, we get for $|\alpha| \geq 1$ and $\gamma \geq 1$,

$$\left\{\int_0^{2\pi}\left[\left|D_\alpha \rho(z)\right| + \frac{m'n(|\alpha|-1)\xi_t(\mu)}{k^{n-t}(1+\xi_t(\mu))}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq \frac{(n-t)(|\alpha|+\xi_t(\mu))}{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\xi_t(\mu)+e^{i\beta}\right|^\gamma d\beta\right\}^{\frac{1}{\gamma}}}\left\{\int_0^{2\pi}\left|\rho(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{36}$$

where $m' = \min_{|z|=k}|\rho(z)| = \frac{1}{k^t}\min_{|z|=k}|P(z)| = \frac{m}{k^t}$
and $\xi_t(\mu)$ is as defined in Theorem 2.12.
Now

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\ &= nz^t\rho(z) + (\alpha - z)\Big(tz^{t-1}\rho(z) + z^t\rho'(z)\Big) \\ &= z^t\Big((n-t)\rho(z) + (\alpha - z)\rho'(z)\Big) + \alpha t z^{t-1}\rho(z) \\ &= z^t D_\alpha\rho(z) + \alpha t z^{t-1}\rho(z), \end{aligned}$$

which implies

$$zD_\alpha P(z) = z^{t+1}D_\alpha\rho(z) + \alpha t P(z). \tag{37}$$

Hence for $0 \le \theta < 2\pi$, we get from (37) that

$$\begin{aligned} |D_\alpha P(e^{i\theta})| &= |e^{i(t+1)\theta}D_\alpha\rho(e^{i\theta}) + \alpha t P(e^{i\theta})| \\ &\le |D_\alpha\rho(e^{i\theta})| + t|\alpha||P(e^{i\theta})|, \end{aligned}$$

which gives, by using Minkowski's inequality for $\gamma \ge 1$,

$$\begin{aligned} &\left\{\int_0^{2\pi}\left[\left|D_\alpha P(e^{i\theta})\right| + \frac{mn(|\alpha|-1)\xi_t(\mu)}{k^n(1+\xi_t(\mu))}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\ &\le \left\{\int_0^{2\pi}\left[\left|D_\alpha\rho(e^{i\theta})\right| + \frac{mn(|\alpha|-1)\xi_t(\mu)}{k^n(1+\xi_t(\mu))} + t|\alpha||P(e^{i\theta})|\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\ &\le \left\{\int_0^{2\pi}\left[\left|D_\alpha\rho(e^{i\theta})\right| + \frac{mn(|\alpha|-1)\xi_t(\mu)}{k^n(1+\xi_t(\mu))}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} + t|\alpha|\left\{\int_0^{2\pi}|P(e^{i\theta})|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \end{aligned} \tag{38}$$

Using (36) in (38) and noting that $|\rho(e^{i\theta})| = |e^{it\theta}\rho(e^{i\theta})| = |P(e^{i\theta})|$, it follows that for every $|\alpha| \ge 1$ and $\gamma \ge 1$,

$$\begin{aligned} &\left\{\int_0^{2\pi}\left[\left|D_\alpha P(e^{i\theta})\right| + \frac{mn(|\alpha|-1)\xi_t(\mu)}{k^n(1+\xi_t(\mu))}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\ &\le \left\{\frac{(n-t)(|\alpha|+\xi_t(\mu))}{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\xi_t(\mu)+e^{i\beta}\right|^\gamma d\beta\right\}^{\frac{1}{\gamma}}} + t|\alpha|\right\}\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \end{aligned}$$

which is (26) and this completes the proof of Theorem 2.12.

REFERENCES

1. V. V. Arestov: *On integral inequalities for trigonometric polynomials and their derivatives.* Izv. Akad. Nauk SSSR, Ser. Mat., **45** (1981), 3–22.

2. A. Aziz: *Inequalities for the polar derivative of a polynomial.* J. Approx. Theory, **55** (1988), 183–193.

3. A. Aziz and W. M. Shah: *L^q inequalities for polynomials with restricted zeros.* Glasnik Mate., **32** (1997), 247–258.

4. A. Aziz and W. M. Shah: *An integral mean estimate for polynomials.* Indian J. Pure Appl. Math., **28** (1997), 1413–1419.

5. S. Bernstein: *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné.* Mem. Acad. R. Belg., **4** (1912), 1–103.

6. N. G. de-Bruijn: *Inequalities concerning polynomials in the complex domain.* Nederl. Akad. Wetnesch Proc., **50** (1947), 1265–1272.

7. T. N. Chan and M. A. Malik: *On Erdös-Lax theorem.* Proc. Indian Acad. Sci., **92** (1983), 191–193.

8. K. K. Dewan, N. Singh and R. S. Yadav: *Inequalities concerning polynomials having zeros in closed exterior or closed interior of a circle.* Southeast Asian Bull. Math., **27** (2003), 591–597.

9. R. B. Gardner and A. Weems: *A Bernstein type L^p inequality for a certain class of polynomials.* J. Math. Anal. Appl., **219** (1998), 472–478.

10. N. K. Govil and P. Kumar: *On L^p inequalities involving polar derivative of a polynomial.* Acta Math. Hung., **152** (2017), 130–139.

11. N. K. Govil and Q. I. Rahman: *Functions of exponential type not vanishing in a half-plane and related polynomials.* Trans. Amer. Math. Soc., **137** (1969), 501–517.

12. V. K. Jain: *On polynomials having zeros in closed exterior or closed interior of a circle.* Indian J. Pure Appl. Math., **30** (1999), 153–159.

13. P. Kumar: *On Zygmund-type inequalities involving polar derivative of a lacunary-type polynomial.* Bull. Math. Soc. Sci. Math. Roumanie Tome, **62** (2019), 163–172.

14. P. D. Lax: *Proof a a conjecture of P. Erdös on the derivative of a polynomial.* Bull. Amer. Math. Soc., **50** (1944), 509–513.

15. M. A. Malik: *On the derivative of a polynomial.* J. Lond. Math. Soc., **1** (1969), 57–60.

16. M. Marden: *Geometry of Polynomials.* Math. Surveys, No. 3, Amer. Math. Soc., Providence R.I.,1966.

17. G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias: *Topics in Polynomials, Extremal Problems, Inequalities, Zeros.* World Scientific, Singapore, 1994.

18. A. Mir: *Bernstein type integral inequalities for a certain class of polynomials.* Mediterranean J. Math., **16** (2019), (Art. 143) pp. 1-11.

19. A. Mir and I. Hussain: *On the Erdös-Lax inequality concerning polynomials.* C. R. Acad. Sci. Paris Ser. I, **355** (2017), 1055–1062.

20. A. Mir and B. Dar: *On the polar derivative of a polynomial.* J. Ramanujan Math. Soc., **29** (2014), 403–412.

21. M. A. Qazi: *On the maximum modulus of polynomials.* Proc. Amer. Math. Soc., **115** (1992), 337–343.

22. Q. I. Rahman and G. Schmeisser: *Analytic Theory of Polynomials.* Oxford University Press, 2002.

23. W. M. Shah and A. Liman: *Integral mean estimates for polynomials with restricted zeros.* Tamsui Oxford Jour. Inform. Math. Sci., **27** (2011), 165–181.

24. P. Turán: *Über die Ableitung von Polynomen.* Compositio Math., **7** (1939), 89–95.

25. A. Zygmund: *A remark on conjugate series.* Proc. London Math. Soc., **34** (1932), 392–400.

Abdullah Mir
Department of Mathematics,
University of Kashmir
Hazaratbal,
Srinagar, 190006
E-mail: *drabmir@yahoo.com*

(Received 12.04.2020)
(Revised 11.10.2020)