DIFFERENTIAL TRANSCENDENCE OF SOLUTIONS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS BASED ON TOTAL REDUCTION OF THE SYSTEM

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In this paper we consider total reduction of the nonhomogeneous linear system of operator equations with constant coefficients and commuting operators. The totally reduced system obtained in this manner is completely decoupled. All equations of the system differ only in the variables and in the nonhomogeneous terms. The homogeneous parts are obtained using the generalized characteristic polynomial of the system matrix. We also indicate how this technique may be used to examine differential transcendence of the solution of the linear system of the differential equations with constant coefficients over the complex field and meromorphic free terms.

1. INTRODUCTION

One way to solve a linear system of operator equations with constant coefficients is to decompose it into several subsystems using Jordan canonical form. Every subsystem of the reduced system corresponds to one block of Jordan canonical form of the system matrix. These subsystems are uncoupled, so we may solve each of them separately, and then simply assemble these individual solutions together to obtain a solution of the general system.

If the field of coefficients is not algebraically closed, Jordan canonical form of a matrix can only be obtained by adding a field extension. The rational canonical form of a matrix is the best diagonal block form that can be obtained over the field of coefficients and it corresponds to the factorization of the characteristic polynomial into invariant factors without adding any field extension. In [**9**] the idea to use the rational instead of Jordan canonical form to reduce a linear system of first-order operator equations to an equivalent partially reduced system was introduced. The reduced system consists of higher-order linear operator equations in one variable and first-order linear operator equations in two variables.

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Another method for solving a linear system of first-order operator equations which does not require a change of basis is discussed in [**10**]. The system is reduced to a totally reduced system, i.e., to a system with separated variables, by using the characteristic polynomial $\Delta_B(\lambda) = \det(\lambda I - B)$ of the system matrix B. This system consists of higher-order operator equations which differ in variables and nonhomogeneous terms. In order to obtain the totally reduced system we need the recurrence for calculating coefficients of the adjugate matrix of the matrix $\lambda I - B$.

In [**3**] we consider a total reduction of a linear system of operator equations with the system matrix in the companion form, by finding the adjugate matrix of the characteristic matrix of the system matrix. We also indicated how this technique may be used to connect differential transcendence of the solution with the coefficients of the system.

Here we expand our research to nonhomogeneous linear systems of operator equations with constant coefficients involving more than one operator.

The paper [**2**] deals with the total reduction of the systems of this form in two or three variables. Some applications of the total reduction method for solving systems of differential equations also can be found there.

Partial reduction of nonhomogeneous linear systems of operator equations involving more than one operator with the system matrix in the companion form is considered in [**4**]. The obtained equivalent system consists of linear operator equations having only one or two variables. Homogeneous part of the equation in one unknown is obtained using generalized characteristic polynomial of the system matrix.

The paper is organized as follows. In Section 2 total reduction of linear systems with commuting operators will be presented. As a consequence of the method, in Section 3 we will obtain a connection between differential transcendence of the solution of a linear system of first-order differential equations where exactly one nonhomogeneous part is a differentially transcendental meromorphic function, and the values of the sums of the principal minors of submatrices of the system matrix.

2. THE TOTAL REDUCTION OF A LINEAR SYSTEM WITH COMMUTING OPERATORS

2.1. The generalized characteristic polynomial

The matrix

$$B(\vec{\lambda}) = \begin{bmatrix} \lambda_1 - b_{11} & -b_{12} & \ldots & -b_{1n} \\ -b_{21} & \lambda_2 - b_{22} & \ldots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \ldots & \lambda_n - b_{nn} \end{bmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) - B$$

is called the **generalized characteristic matrix** of $B \in K^{n \times n}$. The determinant $\Delta_B(\vec{\lambda}) = \Delta_B(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the matrix $B(\vec{\lambda})$ is called the **generalized**

characteristic polynomial. This construction generalizes the bivariate case of generalized characteristic polynomial from [**16**]. If the generalized characteristic polynomial is considered as a polynomial in variables $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the coefficient of the term $\lambda_{\psi_1} \lambda_{\psi_2} \ldots \lambda_{\psi_r}$, $1 \leq r \leq n$, $\psi_1 < \psi_2 < \cdots < \psi_r$, is equal to the principal minor of B corresponding to the set $\{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n\}$ multiplied by $(-1)^{n-r}$. Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$ and let $p(\phi)$ denote the number of inversions in the permutation $\phi \in S_n$. Then we have $\Delta_B(\vec{\lambda}) = \sum_{\phi \in S_n} (-1)^{p(\phi)} (\lambda_1 \delta_{1\phi_1} - b_{1\phi_1})(\lambda_2 \delta_{2\phi_2} - b_{2\phi_2}) \ldots (\lambda_n \delta_{n\phi_n} - b_{n\phi_n})$, for

$$\delta_{i\phi_i} = \begin{cases} 1, & i = \phi_i \\ 0, & i \neq \phi_i \end{cases}.$$ The coefficient of the term $\lambda_{\psi_1} \lambda_{\psi_2} \ldots \lambda_{\psi_r}$ in the polynomial $\Delta_B(\vec{\lambda})$ is the expression

$$(-1)^{n-r} \sum_{\phi \in S_n} (-1)^{p(\phi)} \delta_{\psi_1 \phi_{\psi_1}} \delta_{\psi_2 \phi_{\psi_2}} \ldots \delta_{\psi_r \phi_{\psi_r}} b_{\psi_{r+1} \phi_{\psi_{r+1}}} b_{\psi_{r+2} \phi_{\psi_{r+2}}} \ldots b_{\psi_n \phi_{\psi_n}}.$$

The sum in the above expression is the determinant of a matrix obtained from the matrix B by replacing rows $\psi_1, \psi_2, \ldots, \psi_r$ with rows $\psi_1, \psi_2 \ldots, \psi_r$ of the $n \times n$ identity matrix. Using the row operation which involves replacing a row with a multiple of another row added to itself, we can use the ones to annihilate everything above them and below them, obtaining a modified matrix which has determinant equal to the principal minor of the order $n - r$ of B corresponding to the set $\{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n\}$.

We write $\text{adj}(B(\vec{\lambda}))$ for the adjugate matrix of the generalized characteristic matrix $B(\vec{\lambda})$ of B and $B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n} \in K^{n \times n}$ for the coefficient of the term $\lambda_{\psi_1} \lambda_{\psi_2} \ldots \lambda_{\psi_r}$ in $\text{adj}(B(\vec{\lambda}))$, where $\psi_1 < \psi_2 < \cdots < \psi_r$, $\psi_{r+1} < \psi_{r+2} < \cdots < \psi_n$. For the matrix $B(\vec{\lambda})$ it holds $B(\vec{\lambda}) = \lambda_1 E_{11} + \lambda_2 E_{22} + \cdots + \lambda_n E_{nn} - B$, where E_{ii} is the $n \times n$ binary matrix containing zeros in all positions except for the one in the (i, i)-th position. Exponents of the variables in each term of the generalized characteristic polynomial are zeros or ones. Comparing the coefficients on the both sides of the equations $B(\vec{\lambda}) \cdot \text{adj}(B(\vec{\lambda})) = \Delta_B(\vec{\lambda})I$ and $\text{adj}(B(\vec{\lambda})) \cdot B(\vec{\lambda}) = \Delta_B(\vec{\lambda})I$, we conclude that $E_{\psi_i \psi_i} \cdot B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n} = 0$ and $B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n} \cdot E_{\psi_i \psi_i} = 0$ for $1 \leq i \leq r \leq n$. Hence, we get that all entries of rows and columns $\psi_1, \psi_2, \ldots, \psi_r$ of the matrix $B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n}$ are equal to zero. The remaining coefficients of the matrix $B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n}$ we obtain by determining the coefficient of the term $\lambda_{\psi_1} \lambda_{\psi_2} \ldots \lambda_{\psi_r}$ in the cofactors of the matrix $B(\vec{\lambda})$. The cofactors of matrix $B(\vec{\lambda})$ are linear in each row. Therefore, the (i, j)-th entry of the matrix $B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n}$ is equal to the (j, i)-th cofactor of the matrix obtained from the matrix $-B$ by replacing columns $\psi_1, \psi_2, \ldots, \psi_r$ with columns $\psi_1, \psi_2, \ldots, \psi_r$ of the $n \times n$ identity matrix. The cofactor of the (j, i)-th entry of the given matrix containing rows and columns $\psi_1, \psi_2, \ldots, \psi_r$ is equal to the cofactor of the $(j - l, i - k)$-th entry of matrix obtained from the matrix $-B$ by deleting rows and columns $\psi_1, \psi_2, \ldots, \psi_r$. The numbers k and l are the number of rows, respectively columns from the set $\{\psi_1, \psi_2, \ldots, \psi_r\}$ for which $\psi_s < i$ and $\psi_t < j$ hold, $1 \leq s, t \leq r$. Thus, if the matrix $\text{adj}(B(\vec{\lambda}))$ is considered as a polynomial with matrix coefficients in variables $\lambda_1, \ldots, \lambda_n$, then the coefficient of

the term $\lambda_{\psi_1} \lambda_{\psi_2} \ldots \lambda_{\psi_r}$ is formed by expanding the adjugate matrix of the submatrix of $-B$ corresponding to rows and columns $\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n$, with zero rows and columns $\psi_1, \psi_2, \ldots, \psi_r$.

2.2. The main result

In this subsection we will derive explicit formulas for transformation of a linear system of operator equations with distinct commuting operators into a totally reduced system.

Let K be a field, V a vector space over K and $A_i : V \to V$ a linear operator on the vector space V, $1 \le i \le n$. We will consider nonhomogeneous linear system of operator equations of the form

(1)
$$
\begin{array}{rcl}
A_1(x_1) & = & b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n + \varphi_1 \\
A_2(x_2) & = & b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n + \varphi_2 \\
& \vdots & \\
A_n(x_n) & = & b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n + \varphi_n,
\end{array}
$$

under the assumption that $A_1, A_2, \ldots, A_n$ are commuting linear operators, i.e., $A_i \circ A_j = A_j \circ A_i$, where $b_{ij} \in K$ and $\varphi_i \in V$, for $1 \le i, j \le n$. The system (1) can be rewritten in the following matrix form

(2)
$$
\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi},
$$

where $B = [b_{ij}]_{n \times n}$ is the system matrix, $\vec{x} = [x_1 \ x_2 \ldots x_n]^T$ is the column of unknowns, $\vec{\varphi} = [\varphi_1 \ \varphi_2 \ldots \varphi_n]^T$ is the nonhomogeneous term and $\vec{A}$ is the vector operator defined componentwise $\vec{A}(\vec{x}) = [A_1(x_1) \ A_2(x_2) \ldots A_n(x_n)]^T$.

Theorem 1. *Suppose that the linear system of operator equations is given in the matrix form*

$$
\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi},
$$

and that the matrix $B_{\psi_{r+1} \psi_{r+2} \ldots \psi_n}$ is the coefficient of the term $\lambda_{\psi_1} \lambda_{\psi_2} \ldots \lambda_{\psi_r}$ in the adjugate matrix $adj(B(\vec{\lambda}))$ of the generalized characteristic matrix $B(\vec{\lambda})$ of the system matrix B viewed as a polynomial in variables $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$
\Delta_B(\vec{A})(\vec{x}) = \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} B_{\psi_{r+1} \psi_{r+2} \ldots \psi_n} A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi}).
$$

Proof. Replacing $\vec{\lambda}$ by $\vec{A}$ in the equation $\Delta_B(\vec{\lambda})I = adj(B(\vec{\lambda})) \cdot B(\vec{\lambda})$ we obtain that

$$
\begin{aligned}
\Delta_B(\vec{A})I &= \mathrm{adj}(B(\vec{A})) \cdot B(\vec{A}) \\
\Delta_B(\vec{A})(\vec{x}) &= \mathrm{adj}(B(\vec{A})) \cdot \left(B(\vec{A}) \cdot \vec{x} \right) = \mathrm{adj}(B(\vec{A})) \cdot \left(\vec{A}(\vec{x}) - B \cdot \vec{x} \right) \\
&= \mathrm{adj}(B(\vec{A})) \cdot \vec{\varphi} = \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} B_{\psi_{r+1} \psi_{r+2} \ldots \psi_n} A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi}).
\end{aligned}
$$

The matrix $\mathrm{adj}(B(\vec{A})) \cdot B(\vec{A})$ has combined scalar and operator entries, so we can write either $\big(\mathrm{adj}(B(\vec{A})) \cdot B(\vec{A})\big)(\vec{x})$ or $\big(\mathrm{adj}(B(\vec{A})) \cdot B(\vec{A})\big) \cdot \vec{x}$. $\qquad\square$

This theorem is a generalization of Theorem 4.1. from [**10**].

We write $B^i(\vec{v})$ for the matrix obtained from B by replacing its i-th column by the column $\vec{v} = [v_1 \ v_2 \ldots v_n]^T$. We denote by $\delta^i_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n}(B; \vec{v})$, $\psi_{r+1} < \psi_{r+2} < \cdots < \psi_n$, the principal minor of the matrix $B^i(\vec{v})$ containing columns $\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n$ and for which there is j, $1 \le j \le n - r$, such that $i = \psi_{r+j}$. If $i \ne \psi_{r+j}$, for each j, $1 \le j \le n - r$, then we define $\delta^i_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n}(B; \vec{v})$ to be zero.

In the following lemma we give a correspondence between the coefficients $B_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n}$ of the matrix polynomial $\mathrm{adj}(B(\vec{\lambda}))$ and the principal minors $\delta^i_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n}(B; \vec{v})$.

Lemma 2. *Given an arbitrary column $\vec{v} = [v_1 \, v_2 \, \ldots \, v_n]^T \in K^{n \times 1}$, it holds*

$$B_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n} \cdot \vec{v} = (-1)^{n-r-1} \begin{bmatrix} \delta^1_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n}(B; \vec{v}) \\ \delta^2_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n}(B; \vec{v}) \\ \vdots \\ \delta^n_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n}(B; \vec{v}) \end{bmatrix}.$$

Proof. The adjugate matrix of the $(n-r) \times (n-r)$ submatrix of $-B$ is equal to the adjugate matrix of the corresponding submatrix of B multiplied by $(-1)^{n-r-1}$. By using Laplace expansion along the i-th column we can conclude that the product of the adjugate matrix of B and an arbitrary column $\vec{v}$ is equal to the column whose i-th entry is the determinant of the matrix obtained from B by replacing its i-th column by the column $\vec{v}$. The matrix $B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n}$ is equal to the adjugate matrix of the matrix obtained from $-B$ by deleting columns and rows $\psi_1, \psi_2, \ldots, \psi_r$. Therefore, the i-th entry of the product $B_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n} \cdot \vec{v}$ is equal to the determinant of the submatrix of B which contains columns and rows $\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n$ and where the i-th column is replaced with the column obtained from column $\vec{v}$ by deleting rows $\psi_1, \psi_2, \ldots, \psi_r$, multiplied by $(-1)^{n-r-1}$. Hence, the i-th component of the product $B_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n} \cdot \vec{v}$ is the product of the minor $\delta^i_{\psi_{r+1},\psi_{r+2},\ldots,\psi_n}(B; \vec{v})$ and $(-1)^{n-r-1}$. $\qquad\square$

Remark 1. *Taking $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ we have that the sum of the coefficients of the monomials of the total degree r of the adjugate matrix $\mathrm{adj}(B(\vec{\lambda}))$ is equal to the coefficient B_{n-r-1} of λ^r of the adjugate matrix of the characteristic matrix $\lambda I - B$ of the system matrix B. Thus, we can conclude that Theorem 3.3. from [**10**] is a special case of this lemma.*

Theorem 3. *For the linear system of operator equations given in the matrix form*

$$\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi},$$

it holds

$$\Delta_B(\vec{A})(\vec{x}) = \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} (-1)^{n-r-1} \vec{\delta}_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} \big(B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})\big),$$

where

$$\vec{\delta}_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} \big(B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})\big) =$$

$$\begin{bmatrix} \delta^1_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} (B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})) \\ \delta^2_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} (B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})) \\ \vdots \\ \delta^n_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} (B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})) \end{bmatrix}.$$

Proof. It is an immediate consequence of Theorem 1 and Lemma 2 as follows

$$\Delta_B(\vec{A})(\vec{x})$$

$$\underset{\text{Theorem 1}}{=} \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} B_{\psi_{r+1}\psi_{r+2}\ldots\psi_n} A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})$$

$$\underset{\text{Lemma 2}}{=} \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} (-1)^{n-r-1} \vec{\delta}_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} \big(B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})\big).$$

$\square$

We can now rephrase the previous theorem.

Theorem 4. *The linear system of operator equations*

$$\begin{aligned} A_1(x_1) &= b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n + \varphi_1 \\ A_2(x_2) &= b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n + \varphi_2 \\ &\vdots \\ A_n(x_n) &= b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n + \varphi_n, \end{aligned}$$

can be reduced to the totally reduced system

$$\Delta_B(\vec{A})(x_1) = \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} (-1)^{n-r-1} \delta^1_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} \big(B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})\big)$$

$$\Delta_B(\vec{A})(x_2) = \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} (-1)^{n-r-1} \delta^2_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} \big(B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})\big)$$

$$\vdots$$

$$\Delta_B(\vec{A})(x_n) = \sum_{r=0}^{n-1} \sum_{1 \le \psi_1 < \cdots < \psi_r \le n} (-1)^{n-r-1} \delta^n_{\psi_{r+1}, \psi_{r+2}, \ldots, \psi_n} \big(B; A_{\psi_1} \circ A_{\psi_2} \circ \cdots \circ A_{\psi_r}(\vec{\varphi})\big).$$

Next we illustrate Theorem 4 by means of an explicit example.

Example 1. *Consider the system of second-order linear differential equations*

$$x_1''(t) = \cos t$$
$$x_2''(t) + 2x_2'(t) = -4x_1(t) + e^t,$$

with the initial conditions $x_1(0) = 1$, $x_1'(0) = 2$, $x_2(0) = 0$, $x_2'(0) = -1$, examined in [**18**]. *The system matrix is $B = \begin{bmatrix} 0 & 0 \\ -4 & 0 \end{bmatrix}$. The generalized characteristic polynomial is $\Delta_{\mathbf{B}}(\lambda_1, \lambda_2) = \begin{vmatrix} \lambda_1 & 0 \\ 4 & \lambda_2 \end{vmatrix} = \lambda_1 \lambda_2$. The operator $\Delta_B(A_1, A_2)(u) = (A_1 \circ A_2)(u)$ is equal to the composition of the differential operators $A_1(u) = u''$ and $A_2(u) = u'' + 2u'$, so we have $\Delta_B(A_1, A_2)(u) = u^{(4)} + 2u'''$. By applying Theorem 4, the totally reduced system of the previous system is*

$$\Delta_B(A_1, A_2)(x_1(t)) = x_1^{(4)}(t) + 2x_1'''(t)$$

$$= \left[\begin{array}{cc} \boxed{A_2(\varphi_1(t))} & 0 \\ A_2(\varphi_2(t)) & 0 \end{array} \right] - \left[\begin{array}{cc} \varphi_1(t) & 0 \\ \varphi_2(t) & 0 \end{array} \right]$$

$$= A_2(\varphi_1(t)) = \varphi_1''(t) - 2\varphi_1'(t) = -\cos t - 2\sin t$$

$$\Delta_B(A_1, A_2)(x_2(t)) = x_2^{(4)}(t) + 2x_2'''(t)$$

$$= \left[\begin{array}{cc} 0 & A_1(\varphi_1(t)) \\ -4 & \boxed{A_1(\varphi_2(t))} \end{array} \right] - \left[\begin{array}{cc} 0 & \varphi_1(t) \\ -4 & \varphi_2(t) \end{array} \right]$$

$$= A_1(\varphi_2(t)) - 4\varphi_1(t) = \varphi_2''(t) - 4\varphi_1(t) = e^t - 4\cos t.$$

Only the framed minors are taken into consideration. Furthermore, the adjugate matrix of the generalized characteristic matrix is $adj(B(\vec{A})) = \begin{bmatrix} A_2 & 0 \\ 4 & A_1 \end{bmatrix}$ and the nonhomogeneous term is $\vec{\varphi} = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ e^t \end{bmatrix}$. Another approach for calculating the nonhomogeneous term of the totaly reduced system is to multiply $adj(B(\vec{A}))$ by $\vec{\varphi}$. In this fashion we obtain two higher-order nonhomogeneous differential equations which differ only in nonhomogeneous terms. We need to add two more initial conditions for each of the equations $x_1''(0) = 1$, $x_1'''(0) = 0$, $x_2''(0) = -1$ and $x_2'''(0) = -5$. After a short computation we obtain the solution

$$x_1(t) = 2 + 2t - \cos t$$
$$x_2(t) = -2t - 2t^2 + \tfrac{7}{15}e^{-2t} + \tfrac{1}{3}e^t - \tfrac{4}{5}\cos t + \tfrac{8}{5}\sin t.$$

The system could also be solved by a method proposed in [**10**], *but this method requires the substitution of the derivatives x_1' and x_2' by the new functions x_3 and x_4,*

and the transformation of the system into a linear system of first-order differential equations

$$
\begin{aligned}
x_1'(t) &= x_3(t) \\
x_2'(t) &= x_4(t) \\
x_3'(t) &= \cos t \\
x_4'(t) &= -4x_1(t) - 2x_4(t) + e^t.
\end{aligned}
$$

The totally reduced system of the obtained system consists of four higher-order nonhomogeneous differential equations, so the calculation is far more complex.

3. THE DIFFERENTIAL TRANSCENDENCE AND THE TOTAL REDUCTION OF A LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

In this section we are dealing with linear systems of first-order differential equations in variable z with constant coefficients in the complex field $\mathbb{C}$ over the vector space of meromorphic functions $\mathcal{M}$ of the form

$$
\begin{aligned}
\frac{dx_1}{dz} &= b_{11}x_1(z) + b_{12}x_2(z) + \cdots + b_{1n}x_n(z) + \varphi_1(z) \\
\frac{dx_2}{dz} &= b_{21}x_1(z) + b_{22}x_2(z) + \cdots + b_{2n}x_n(z) + \varphi_2(z) \\
&\vdots \\
\frac{dx_n}{dz} &= b_{n1}x_1(z) + b_{n2}x_2(z) + \cdots + b_{nn}x_n(z) + \varphi_n(z),
\end{aligned}
$$

(3)

where exactly one coordinate of the nonhomogeneous term $\vec{\varphi}(z) = [\varphi_1(z) \ldots \varphi_n(z)]^T$ is a differentially transcendental function over $\mathbb{C}$. According to Theorem 4 the system can be reduced to the totally reduced system

$$
\begin{aligned}
\Delta_B(\tfrac{d}{dz})(x_1(z)) &= \sum_{r=0}^{n-1} (-1)^{n-r-1} \delta_{n-r}^1\big(B; \vec{\varphi}^{(r)}(z)\big) \\
\Delta_B(\tfrac{d}{dz})(x_2(z)) &= \sum_{r=0}^{n-1} (-1)^{n-r-1} \delta_{n-r}^2\big(B; \vec{\varphi}^{(r)}(z)\big) \\
&\vdots \\
\Delta_B(\tfrac{d}{dz})(x_n(z)) &= \sum_{r=0}^{n-1} (-1)^{n-r-1} \delta_{n-r}^n\big(B; \vec{\varphi}^{(r)}(z)\big),
\end{aligned}
$$

(4)

where $\Delta_B\left(\tfrac{d}{dz}\right)(x(z)) = x^{(n)}(z) - \delta_1(B)x^{(n-1)}(z) + \cdots + (-1)^n \delta_n(B)x(z)$, $\delta_k(B)$ is the sum of the principal minors of the order k of the matrix B, $\delta_{n-r}^i\big(B; \vec{\varphi}^{(r)}(z)\big)$

is the sum of the principal minors of order $n - r$ of the matrix obtained by substituting $\vec{\varphi}^{(r)}(z) = [\varphi_1^{(r)}(z)\ \varphi_2^{(r)}(z)\ \ldots\ \varphi_n^{(r)}(z)]^T$ for the i-th column of B.

3.1. Transcendental and differentially algebraic extensions

This subsection is concerned with the standard concept of algebraic independence, preliminary results in differential algebra, such as differential rings, differentially algebraic independence, differential polynomials, as well as certain properties of extensions of differential fields. For the proofs and more details please refer to [**5, 17**].

Let F be an extension of a field K and let S be a subset of F. A set S is called **algebraically dependent** over K if there exists a nonzero polynomial $p(x_1, x_2, \ldots, x_n)$ with coefficients in K such that $p(s_1, s_2, \ldots, s_n) = 0$ for some distinct elements $s_1, s_2, \ldots, s_n \in S$. We call a set S **algebraically independent** over K if it is not algebraically dependent over K.

A **differential ring** R is a commutative ring with identity equipped with a map $D : R \to R$ such that

1. $(\forall x, y \in R)\, D(x + y) = D(x) + D(y)$;

2. $(\forall x, y \in R)\, D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$.

A map $D : R \to R$ satisfying these properties is called a **derivation**. If the ring R with the derivation D is a field we say that R is a **differential field**.

If R_0 is a subring of the ring R with the property that $(\forall x \in R_0)\, D(x) \in R_0$, then D becomes the derivation of R_0 and R_0 is a differential ring. We say in this case that R_0 is a **differential subring** of R, and that R is a **differential overring** of R_0. In the case of fields we use the terms **differential subfield** and **differential extension**, respectively.

If S is any set of elements of R, and R_0 is a differential subring of R, there exists the smallest (with respect to set inclusion) differential subring of R containing all the elements of R_0 and S. This ring is called the **differential ring generated by S over R_0** and is denoted by $R_0\{S\}$. If ΘS denotes the set of all elements $D^n(s)$ with $s \in S$ and $n \in \mathbb{N}_0$, then $R_0\{S\}$ coincides, as a ring, with the ring $R_0[\Theta S]$ generated by ΘS over R_0.

From now on, we assume that K is a differential field. We say that a set S of elements of a differential extension of K is **differentially algebraically dependent** over K if the set $\{D^n(s) \mid n \in \mathbb{N}_0, s \in S\}$ is algebraically dependent over K. Otherwise, the set S is **differentially algebraically independent** over K. In the special case when the set S consists of a single element x, we say that x is, respectively, **differentially algebraic** or **differentially transcendental** over K. The smallest (with respect to set inclusion) differential field containing x and all the elements of K, is denoted by $K\langle x \rangle$. An extension F of K is said to be **differentially algebraic** over K if each element of F is differentially algebraic over K.

Let x be differentially transcendental over K. The derivation $D : K \to K$ can be extended to a unique derivation of the ring $K[\{D^n(x)\}_{n \in \mathbb{N}_0}]$ mapping $D^n(x)$ onto $D^{n+1}(x)$. Therefore, the ring $K[\{D^n(x)\}_{n \in \mathbb{N}_0}]$ becomes a differential ring. The elements of $K\{x\} = K[\{D^n(x)\}_{n \in \mathbb{N}_0}]$ are called **differential polynomials over** R **in** x.

Let K be a differential field of characteristic zero. In this case the notions "differentially separable" and "differentially algebraic" coincide. Therefore, we can rephrase Proposition 8 and its Corollary from [**5**, pp. 101-–102], as follows.

Proposition 5. *Let F be a differential extension of a differential field K and $\alpha, \beta \in F \backslash K$. If β is differentially algebraic over $K\langle\alpha\rangle$ and α is differentially algebraic over K, then β is differentially algebraic over K.*

Corollary 6. *Let F be a differential extension of a differential field K. The set F_0 of all the elements of F that are differentially algebraic over K is a differential field.*

3.2. Differential transcendence and total reduction of a linear system of differential equations

After reviewing some of the standard facts on differentially algebraic extensions, we turn back to an examination of differential transcendence of the solution of a linear system of first-order differential equations in variable z with constant coefficients in the complex field $\mathbb{C}$ over the vector space of meromorphic functions $\mathcal{M}$ where exactly one coordinate of the nonhomogeneous term $\vec{\varphi}(z) = [\varphi_1(z) \ldots \varphi_n(z)]^T$ is a differentially transcendental function over $\mathbb{C}$.

Lemma 7. *Let $x_0(z) \in \mathcal{M}$ be a solution of the differential equation*

$$x^{(n)}(z) + d_1 x^{(n-1)}(z) + \cdots + d_{n-1} x'(z) + d_n x(z) = \varphi(z),$$

for $d_1, d_2, \ldots, d_n \in \mathbb{C}$ and $\varphi(z) \in \mathcal{M}$. Then $x_0(z)$ is differentially transcendental over $\mathbb{C}$ if and only if $\varphi(z)$ is differentially transcendental over $\mathbb{C}$.

Proof. This follows easily from Corollary 6. $\square$

Remark 2. *An interesting generalization of the previous lemma is Theorem 2.8 from [**14**]. Various applications of this theorem can be found in [**6, 7, 8, 13, 15**].*

Lemma 8. *Let $\psi_1(z), \psi_2(z), \ldots, \psi_n(z) \in \mathcal{M}$ be differentially algebraic over $\mathbb{C}$. Then $\varphi(z) \in \mathcal{M}$ is differentially transcendental over $\mathbb{C}$ if and only if the value of some differential polynomial in $n + 1$ variables with coefficients in the field $\mathbb{C}$ at $(\varphi(z), \psi_1(z), \psi_2(z), \ldots, \psi_n(z))$ is differentially transcendental over $\mathbb{C}$.*

Proof. This follows from Proposition 5 and Corollary 6. $\square$

Theorem 9. *Let the entries of the nonhomogeneous term of the system* (3)

$$\varphi_1(z), \ldots, \varphi_{i-1}(z), \varphi_{i+1}(z), \ldots \varphi_n(z) \in \mathcal{M}$$

be differentially algebraic over $\mathbb{C}$ *and let* $\varphi_i(z) \in \mathcal{M}$ *be differentially transcendental over* $\mathbb{C}$. *Then the j-th coordinate of the solution* $\vec{x}_0(z)$ *of the totally reduced system* (4) *is differentially algebraic over* $\mathbb{C}$ *if and only if the function* $\varphi_i(z)$ *does not appear in the sum* $\displaystyle\sum_{k=1}^{n} (-1)^{k-1} \delta_k^j \big(B; \vec{\varphi}^{(n-k)}(z)\big).$

Proof. This is an immediate consequence of Theorem 4 and Lemmas 7 and 8. □

We will now consider assumptions under which the function $\varphi_i(z) \in \mathcal{M}$ does not appear in the sum

$$\sum_{k=1}^{n} (-1)^{k-1} \delta_k^j \big(B; \vec{\varphi}^{(n-k)}(z)\big), \text{ for } n \in \mathbb{N}.$$

By applying linearity of $\delta_k^j\big(B; \vec{\varphi}^{(n-k)}(z)\big)$ with respect to the j-th column $\vec{\varphi}^{(n-k)}(z)$ we obtain $\delta_k^j\big(B; \vec{\varphi}^{(n-k)}(z)\big) = \sum_{s=1}^{n} \varphi_s^{(n-k)}(z) \delta_k^j\big(B; \vec{e_s}\big)$, where $\vec{e_s}$ denotes the column whose only nonzero entry is 1 in the s-th position. Each nonzero minor in the sum $\delta_k^j\big(B; \vec{e_s}\big)$ necessarily contains the j-th and s-th row and column. Hence, it holds $\delta_k^j\big(B; \vec{e_s}\big) = -\delta_{k-1}^{j'}\big([B^j(B_{\downarrow s})]_{\widehat{s}}\big)$, where $[B^j(B_{\downarrow s})]_{\widehat{s}}$ is the $(n-1) \times (n-1)$ matrix obtained from B by interchanging the j-th and s-th column and by deleting the s-th row and column, $j' = \begin{cases} j, & j < s \\ j-1, & j > s \end{cases}$. So we have the following theorem.

Theorem 10. *The function* $\varphi_i(z) \in \mathcal{M}$ *does not appear in the sum*

$$\sum_{k=1}^{n} (-1)^{k-1} \delta_k^j \big(B; \vec{\varphi}^{(n-k)}(z)\big), \ i \neq j,$$

if and only if the sums of all principal minors of order k, $1 \leq k \leq n-1$, containing the j-th column for $j < i$, respectively the $(j-1)$-st column for $j > i$, of the matrix obtained from B by interchanging the j-th and i-th column and by deleting the i-th row and column, are equal to zero.

An important point to note here is that if more than one component of the nonhomogeneous term of the system (3) is differentially transcendental over $\mathbb{C}$ we need to examine their differential independence, see [**11**].

An interesting consideration of differential transcendence of the coordinates of the state vector of a time-invariant linear control system depending on differential transcendence of the input matrix can be found in [**1, 12**].

We illustrate Theorem 10 using the following two examples.

Example 2. *Consider the system of first-order linear differential equations*

$$\frac{dx_1}{dz} = x_1(z) + x_2(z) + 2x_3(z)$$
$$\frac{dx_2}{dz} = x_2(z) + x_3(z) + u(z)$$
$$\frac{dx_3}{dz} = x_1(z),$$

where u is differentially transcendental over $\mathbb{C}$, discussed in [1]. The characteristic polynomial of the system matrix is $\Delta_{\mathbf{B}}(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 1$. Applying Theorem 4 we obtain the totally reduced system of the given system

$$\Delta_B(\tfrac{d}{dz})(x_1) = \frac{d^3x_1}{dz^3} - 2\frac{d^2x_1}{dz^2} - \frac{dx_1}{dz} + x_1$$

$$= \begin{vmatrix} \boxed{0} & 1 & 2 \\ \frac{d^2u}{dz^2} & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 2 \\ \frac{du}{dz} & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 \\ u & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \frac{du}{dz}$$

$$\Delta_B(\tfrac{d}{dz})(x_2) = \frac{d^3x_2}{dz^3} - 2\frac{d^2x_2}{dz^2} - \frac{dx_2}{dz} + x_2$$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 0 & \boxed{\frac{d^2u}{dz^2}} & 1 \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 2 \\ 0 & \frac{du}{dz} & 1 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ 0 & u & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{d^2u}{dz^2} - \frac{du}{dz} - 2u$$

$$\Delta_B(\tfrac{d}{dz})(x_3) = \frac{d^3x_3}{dz^3} - 2\frac{d^2x_3}{dz^2} - \frac{dx_3}{dz} + x_3$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{d^2u}{dz^2} \\ 1 & 0 & \boxed{0} \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{du}{dz} \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & u \\ 1 & 0 & 0 \end{vmatrix} = u.$$

Only framed minors are taken into consideration. According to Theorem 10 all x_1, x_2 and x_3 are differentially transcendental over $\mathbb{C}$.

Example 3. *Let us now deal with the system of first-order linear differential equations*

$$\frac{dx_1}{dz} = x_1(z) + 2x_3(z)$$
$$\frac{dx_2}{dz} = x_2(z) + x_3(z) + u(z)$$
$$\frac{dx_3}{dz} = x_1(z),$$

where u is again differentially transcendental over $\mathbb{C}$. Here, the characteristic polynomial of the system matrix is $\Delta_{\mathbf{B}}(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$. By Theorem 4 the initial system can be transformed into totally reduced system consisting of three higher-order linear differential equations

$$
\begin{aligned}
\Delta_B(\tfrac{d}{dz})(x_1) \quad &= \frac{d^3 x_1}{dz^3} - 2\frac{d^2 x_1}{dz^2} - \frac{dx_1}{dz} + 2x_1 = 0 \\
\Delta_B(\tfrac{d}{dz})(x_2) \quad &= \frac{d^3 x_2}{dz^3} - 2\frac{d^2 x_2}{dz^2} - \frac{dx_2}{dz} + 2x_2 = \frac{d^2 u}{dz^2} - \frac{du}{dz} - 2u \\
\Delta_B(\tfrac{d}{dz})(x_3) \quad &= \frac{d^3 x_3}{dz^3} - 2\frac{d^2 x_3}{dz^2} - \frac{dx_3}{dz} + 2x_3 = 0.
\end{aligned}
$$

By Theorem 10 only the second coordinate x_2 is a differentially transcendental function over $\mathbb{C}$.

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