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SOME LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS AND INEQUALITIES FOR MULTINOMIAL COEFFICIENTS AND MULTIVARIATE BETA FUNCTIONS

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Dedicated to people facing and battling COVID-19

In the paper, the authors extend a function arising from the Bernoulli trials in probability and involving the gamma function to its largest ranges, find logarithmically complete monotonicity of these extended functions, and, in light of logarithmically complete monotonicity of these extended functions, derive some inequalities for multinomial coefficients and multivariate beta functions. These results recover, extend, and generalize some known conclusions.

1. BACKGROUND AND MOTIVATION

Let us denote by $P_{n,k}(p)$ the probability of achieving exactly k successes in n Bernoulli trials with success probability p. Then

$$P_{n,k}(p) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \tag{1.1}$$

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where $\Gamma(z)$ denotes the classical Euler gamma function which can be defined [**1, 19, 24**] by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t,\quad \Re(z)>0$$

or by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^n(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

In the technical report [**16**], Leblanc and Johnson considered a problem: which is more likely to happen: k successes in n trials or $2k$ successes in $2n$ trials? They proved that

$$P_{2n,2k}(p)\le P_{n,k}(p),\quad 0\le k\le n,\quad p\in(0,1). \tag{1.2}$$

This means that k successes in n trials is more likely to happen than $2k$ successes in $2n$ trials. One year later, the same authors generalized the inequality (1.2) in [**17**, Corollary 2.4] by

$$P_{(j+1)n,(j+1)k}(p)\le P_{jn,jk}(p),\quad j\in\mathbb{N}. \tag{1.3}$$

About ten years later, Alzer extended the inequality (1.3) in the note [**2**] by considering the function

$$G(x)=G_{k,n;p}(x)=\frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((n-k)x+1)}p^{kx}(1-p)^{(n-k)x} \tag{1.4}$$

and essentially proving that the function $G(x)$ is logarithmically completely monotonic on $(0,\infty)$, where k and n are integers with $0\le k\le n$, $p\in(0,1)$, and an infinitely differentiable and positive function $F(x)$ is said [**4, 5, 27, 29, 37**] to be logarithmically completely monotonic on an interval I if and only if $(-1)^m[\ln F(x)]^{(m)}\ge 0$ for all $m\ge 2$ and $x\in I$. We observe that we can write

$$G(x)=G_{k,n;p}(x)=\binom{nx}{kx}p^{kx}(1-p)^{(n-k)x}=\frac{n}{k(n-k)}\frac{p^{kx}(1-p)^{(n-k)x}}{x\,\mathrm{B}(kx,(n-k)x)},$$

where

$$\binom{x}{y}=\frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}=\frac{1}{(x+1)\,\mathrm{B}(x-y+1,y+1)},\quad x,y\in\mathbb{C} \tag{1.5}$$

and

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t,\quad \Re(x),\Re(y)>0$$

denotes the classical Euler beta function and $\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Recall from [**18**, Chapter XIII], [**37**, Chapter 1], and [**38**, Chapter IV] that an infinitely differentiable and nonnegative function $f(x)$ is said to be completely monotonic on an interval I if and only if

$$0 \le (-1)^{m-1} f^{(m-1)}(x) < \infty, \quad m \ge 2, \quad x \in I.$$

The Bernstein–Widder theorem [**38**, p. 161, Theorem 12b] characterizes that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0,\infty)$ is that

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}\mu(t), \quad x \in (0,\infty),$$

where $\mu(t)$ is non-decreasing and the above integral converges for $x \in (0,\infty)$. In other words or simply speaking, a function is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform. Recall from [**5, 11, 27, 37**] that a logarithmically completely monotonic function must be completely monotonic on the same defined interval, but not conversely. This is why we restate here the main result in [**2**] in terms of the logarithmically complete monotonicity. For more information on new developments of this topic, please refer to [**10, 12, 25, 30, 34, 35, 37**] and closely related references therein.

In this paper, we first consider the function

$$\begin{aligned}(1.6)\quad Q(x) = Q_{\alpha,\beta;p}(x) &= \frac{\Gamma((\alpha+\beta)x+1)}{\Gamma(\alpha x+1)\Gamma(\beta x+1)} p^{\alpha x}(1-p)^{\beta x} \\ &= \binom{(\alpha+\beta)x}{\alpha x} p^{\alpha x}(1-p)^{\beta x} = \frac{\alpha+\beta}{\alpha\beta} \frac{p^{kx}(1-p)^{(n-k)x}}{x\,\mathrm{B}(\alpha x,\beta x)}\end{aligned}$$

for $x \in (0,\infty)$, where $\alpha,\beta > 0$ and $p \in (0,1)$. It is easy to see that the function $Q(x)$ is an extension of $G(x)$ and $P_{n,k}(p)$ and satisfies

$$Q_{\alpha,\beta;p}(x) = Q_{\beta,\alpha;1-p}(x), \quad Q_{k,n-k;p}(x) = G_{k,n;p}(x), \quad Q_{k,n-k;p}(1) = P_{n,k}(p).$$

In Section , we will verify that the function $Q(x)$ is logarithmically completely monotonic on $(0,\infty)$.

More generally, we can consider the function

$$\begin{aligned}(1.7)\quad \mathcal{Q}(x) = \mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x) &= \frac{\Gamma\bigl(1 + x\sum_{i=1}^m a_i\bigr)}{\prod_{i=1}^m \Gamma(1+xa_i)} \prod_{i=1}^m p_i^{xa_i} \\ &= \binom{x\sum_{i=1}^m a_i}{xa_1, xa_2, \dots, xa_m} \prod_{i=1}^m p_i^{xa_i} = \frac{\sum_{i=1}^m a_i}{\prod_{i=1}^m a_i} \frac{\prod_{i=1}^m p_i^{xa_i}}{x^{m-1}\,\mathrm{B}(xa_1, xa_2, \dots, xa_m)}\end{aligned}$$

for $x \in (0,\infty)$ and $m \ge 2$, where $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$, $\boldsymbol{p} = (p_1, p_2, \dots, p_m)$ with $p_i \in (0,1)$ for $1 \le i \le m$ and $\sum_{i=1}^m p_i = 1$, the notation

$$\binom{\sum_{i=1}^m a_i}{a_1, a_2, \dots, a_m} = \prod_{i=1}^m \binom{\sum_{\ell=1}^i a_\ell}{a_i}$$

in terms of the notation in (1.5) is called the multinomial coefficient, and

$$\mathrm{B}(a_1, a_2, \dots, a_m) = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_m)}{\Gamma(a_1 + a_2 + \dots + a_m)}$$

is called the multivariate beta function. It is obvious that the function $\mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x)$ is a generalization and an extension of the functions $Q_{\alpha,\beta;p}(x)$, $G_{k,n;p}(x)$, and $P_{n,k}(p)$ defined in (1.1), (1.4), and (1.6) respectively. Concretely speaking,

$$\begin{aligned}\mathcal{Q}_{(\alpha,\beta),(p,1-p);2}(x) &= Q_{\alpha,\beta;p}(x),\\ \mathcal{Q}_{(k,n-k),(p,1-p);2}(x) &= G_{k,n;p}(x),\\ \mathcal{Q}_{(k,n-k),(p,1-p);2}(1) &= P_{n,k}(p).\end{aligned}$$

In Section , we will show that the function $\mathcal{Q}(x)$ is logarithmically completely monotonic on $(0, \infty)$.

In Section , in light of logarithmically complete monotonicity of $Q(x)$ and $\mathcal{Q}(x)$, we will offer some inequalities for multinomial coefficients. In Section , we will reformulate combinatorial inequalities obtained in Section in terms of multivariate beta functions, that is, we will present some inequalities for multivariate beta functions. In Section 4.3, the last section of this paper, we will recover some known results in [**2, 17**] from those inequalities obtained in Section for multinomial coefficients.

2. COMPLETELY MONOTONIC FUNCTIONS

We now start off to prove our first main result in this paper: the function $Q(x)$ is logarithmically completely monotonic on $(0, \infty)$.

Theorem 2.1. *For $\alpha, \beta > 0$ and $p \in (0, 1)$, the function $Q(x) = Q_{\alpha,\beta;p}(x)$ defined in* (1.6) *is logarithmically completely monotonic on* $(0, \infty)$.

Proof. Straightforward computation yields

$$\begin{aligned}\ln Q(x) &= \ln\Gamma((\alpha+\beta)x + 1) - \ln\Gamma(\alpha x + 1)\\ &\quad - \ln\Gamma(\beta x + 1) + \alpha x \ln p + \beta x \ln(1-p),\\ [\ln Q(x)]' &= (\alpha+\beta)\psi((\alpha+\beta)x + 1) - \alpha\psi(\alpha x + 1)\\ &\quad - \beta\psi(\beta x + 1) + \alpha \ln p + \beta \ln(1-p),\\ [\ln Q(x)]'' &= (\alpha+\beta)^2\psi'((\alpha+\beta)x + 1) - \alpha^2\psi'(\alpha x + 1) - \beta^2\psi'(\beta x + 1).\end{aligned}$$

From

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt}\,\mathrm{d}t, \quad \Re(z) > 0$$

in [**1**, p. 260, 6.4.1], it follows that

$$\psi'(\tau z+1)=\int_0^\infty \frac{t}{1-e^{-t}}e^{-(\tau z+1)t}\mathrm{d}t$$
$$=\int_0^\infty \frac{t}{e^t-1}e^{-\tau z t}\mathrm{d}t=\frac{1}{\tau}\int_0^\infty h\biggl(\frac{v}{\tau}\biggr)e^{-vz}\mathrm{d}v,$$

where $\tau>0$ and $h(t)=\frac{t}{e^t-1}$. Accordingly, we have

$$[\ln Q(x)]''=\int_0^\infty \biggl[(\alpha+\beta)h\biggl(\frac{v}{\alpha+\beta}\biggr)-\alpha h\biggl(\frac{v}{\alpha}\biggr)-\beta h\biggl(\frac{v}{\beta}\biggr)\biggr]e^{-xv}\mathrm{d}v. \tag{2.8}$$

Since the function $h(t)$ is decreasing on $(-\infty,\infty)$, we obtain

$$(\alpha+\beta)h\biggl(\frac{v}{\alpha+\beta}\biggr)=\alpha h\biggl(\frac{v}{\alpha+\beta}\biggr)+\beta h\biggl(\frac{v}{\alpha+\beta}\biggr)\ge \alpha h\biggl(\frac{v}{\alpha}\biggr)+\beta h\biggl(\frac{v}{\beta}\biggr).$$

Substituting this inequality into the equation (2.8) reveals that the second derivative $[\ln Q(x)]''$ is completely monotonic on $(0,\infty)$.

By the complete monotonicity of $[\ln Q(x)]''$, we see that the first derivative $[\ln Q(x)]'$ is strictly increasing on $(0,\infty)$, hence,

$$\begin{aligned}
[\ln Q(x)]' &\le \lim_{x\to\infty}[(\alpha+\beta)\psi((\alpha+\beta)x+1)-\alpha\psi(\alpha x+1)-\beta\psi(\beta x+1)]\\
&\quad+\alpha\ln p+\beta\ln(1-p)\\
&=\lim_{x\to\infty}\biggl[(\alpha+\beta)\psi((\alpha+\beta)x)-\alpha\psi(\alpha x)-\beta\psi(\beta x)-\frac{1}{x}\biggr]\\
&\quad+\alpha\ln p+\beta\ln(1-p)\\
&=\lim_{x\to\infty}\Biggl((\alpha+\beta)[\psi((\alpha+\beta)x)-\ln((\alpha+\beta)x)]-\alpha[\psi(\alpha x)-\ln(\alpha x)]\\
&\quad-\beta[\psi(\beta x)-\ln(\beta x)]-\frac{1}{x}+(\alpha+\beta)\ln((\alpha+\beta)x)\\
&\quad-\alpha\ln(\alpha x)-\beta\ln(\beta x)\Biggr)+\alpha\ln p+\beta\ln(1-p)\\
&=(\alpha+\beta)\ln(\alpha+\beta)-\alpha\ln\frac{\alpha}{p}-\beta\ln\frac{\beta}{1-p}\\
&=\biggl[p\frac{\alpha}{p}+(1-p)\frac{\beta}{1-p}\biggr]\ln\biggl[p\frac{\alpha}{p}+(1-p)\frac{\beta}{1-p}\biggr]\\
&\quad-p\biggl(\frac{\alpha}{p}\ln\frac{\alpha}{p}\biggr)-(1-p)\biggl(\frac{\beta}{1-p}\ln\frac{\beta}{1-p}\biggr)\\
&<0,
\end{aligned}$$

where we used the facts that the function $x\ln x$ is convex on $(0,\infty)$ and that

$$\lim_{x\to\infty}[\ln x-\psi(x)]=0, \tag{2.9}$$

see [**13**, Theorem 1] and [**14**]. In conclusion, the function $Q(x)$ is logarithmically completely monotonic on $(0,\infty)$. The proof of Theorem 2.1 is complete. □

We now prove our second main result in this paper: the function $\mathcal{Q}(x)$ is logarithmically completely monotonic on $(0,\infty)$.

Theorem 2.2. *Let $m \ge 2$, $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$, and $\boldsymbol{p} = (p_1, p_2, \dots, p_m)$ with $p_i \in (0,1)$ for $1 \le i \le m$ and $\sum_{i=1}^m p_i = 1$. Then the function $\mathcal{Q}(x) = \mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x)$ defined in (1.7) is logarithmically completely monotonic on $(0,\infty)$.*

Proof. Direct calculation gives

$$\ln \mathcal{Q}(x) = \ln\Gamma\left(1 + x\sum_{i=1}^m a_i\right) - \sum_{i=1}^m \ln\Gamma(1+a_i x) + x\sum_{i=1}^m a_i \ln p_i,$$

$$[\ln \mathcal{Q}(x)]' = \left(\sum_{i=1}^m a_i\right)\psi\left(1 + x\sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i\psi(1+a_i x) + \sum_{i=1}^m a_i \ln p_i,$$

and

$$[\ln \mathcal{Q}(x)]'' = \left(\sum_{i=1}^m a_i\right)^2 \psi'\left(1 + x\sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i^2\psi'(1+a_i x).$$

As did in the proof of Theorem 2.1, we can obtain

$$[\ln \mathcal{Q}(x)]'' = \int_0^\infty \left[\left(\sum_{i=1}^m a_i\right) h\left(\frac{v}{\sum_{i=1}^m a_i}\right) - \sum_{i=1}^m a_i h\left(\frac{v}{a_i}\right)\right] e^{-xv}\,\mathrm{d}v. \tag{2.10}$$

Since the function $h(t)$ is decreasing on $(-\infty,\infty)$, we obtain

$$\left(\sum_{i=1}^m a_i\right) h\left(\frac{v}{\sum_{i=1}^m a_i}\right) = \sum_{i=1}^m a_i h\left(\frac{v}{\sum_{i=1}^m a_i}\right) \ge \sum_{i=1}^m a_i h\left(\frac{v}{a_i}\right).$$

Combining this with (2.10) yields that the second derivative $[\ln \mathcal{Q}(x)]''$ is completely monotonic on $(0,\infty)$.

Complete monotonicity of $[\ln \mathcal{Q}(x)]''$ implies that the first derivative $[\ln \mathcal{Q}(x)]'$ is strictly increasing on $(0,\infty)$, therefore,

$$\begin{aligned}
[\ln \mathcal{Q}(x)]' &\le \lim_{x\to\infty}\left[\left(\sum_{i=1}^m a_i\right)\psi\left(1 + x\sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i\psi(1+a_i x)\right] + \sum_{i=1}^m a_i \ln p_i \\
&= \lim_{x\to\infty}\left[\left(\sum_{i=1}^m a_i\right)\psi\left(x\sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i\psi(a_i x) - \frac{m-1}{x}\right] + \sum_{i=1}^m a_i \ln p_i \\
&= \lim_{x\to\infty}\left\{\left(\sum_{i=1}^m a_i\right)\left[\psi\left(x\sum_{i=1}^m a_i\right) - \ln\left(x\sum_{i=1}^m a_i\right)\right]\right. \\
&\quad - \sum_{i=1}^m a_i[\psi(a_i x) - \ln(a_i x)] + \left(\sum_{i=1}^m a_i\right)\ln\left(x\sum_{i=1}^m a_i\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{i=1}^{m} a_i \ln(a_i x) \Bigg\} + \sum_{i=1}^{m} a_i \ln p_i \\
&= \left(\sum_{i=1}^{m} a_i\right) \ln\left(\sum_{i=1}^{m} a_i\right) - \sum_{i=1}^{m} a_i \ln a_i + \sum_{i=1}^{m} a_i \ln p_i \\
&= \left(\sum_{i=1}^{m} p_i \frac{a_i}{p_i}\right) \ln\left(\sum_{i=1}^{m} p_i \frac{a_i}{p_i}\right) - \sum_{i=1}^{m} p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i} \\
&\le 0,
\end{aligned}
$$

where, as did in the proof of Theorem 2.1, we used the limit (2.9) and convexity of the function $x \ln x$ on $(0, \infty)$. The proof of Theorem 2.2 is complete. □

3. THREE INEQUALITIES FOR MULTINOMIAL COEFFICIENTS

In light of logarithmically complete monotonicity of $Q(x)$ and $\mathcal{Q}(x)$, we now offer some inequalities for multinomial coefficients.

Theorem 3.3. *For $\ell, m \ge 2$, let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$, $x_j > 0$ for $1 \le j \le \ell$, and $\lambda_j \in (0,1)$ with $\sum_{j=1}^{\ell} \lambda_j = 1$. Then*

$$
\binom{\sum_{j=1}^{\ell} \lambda_j x_j \sum_{i=1}^{m} a_i}{a_1 \sum_{j=1}^{\ell} \lambda_j x_j, a_2 \sum_{j=1}^{\ell} \lambda_j x_j, \dots, a_m \sum_{j=1}^{\ell} \lambda_j x_j} \le \prod_{j=1}^{\ell} \binom{x_j \sum_{i=1}^{m} a_i}{x_j a_1, x_j a_2, \dots, x_j a_m}^{\lambda_j} \tag{3.11}
$$

and the equality in (3.11) *holds if and only if $x_1 = x_2 = \dots = x_\ell$. In particular, when $\ell = m = 2$,*

$$
\binom{(a_1 + a_2)(\lambda_1 x_1 + \lambda_2 x_2)}{a_1(\lambda_1 x_1 + \lambda_2 x_2)} \le \binom{(a_1 + a_2) x_1}{a_1 x_1}^{\lambda_1} \binom{(a_1 + a_2) x_2}{a_1 x_2}^{\lambda_2} \tag{3.12}
$$

and the equality in (3.12) *is valid if and only if $x_1 = x_2$.*

Proof. Logarithmically complete monotonicity in Theorem 2.2 implies that the function $\mathcal{Q}(x)$ is logarithmically convex on $(0, \infty)$. Hence, we acquire

$$
\mathcal{Q}\left(\sum_{j=1}^{\ell} \lambda_j x_j\right) \le \prod_{j=1}^{\ell} \mathcal{Q}^{\lambda_j}(x_j).
$$

Making use of the expression

$$
\mathcal{Q}(x) = \binom{x \sum_{i=1}^{m} a_i}{x a_1, x a_2, \dots, x a_m} \prod_{i=1}^{m} p_i^{x a_i}
$$

arrives at

$$\binom{\sum_{j=1}^{\ell}\lambda_j x_j\sum_{i=1}^{m}a_i}{a_1\sum_{j=1}^{\ell}\lambda_j x_j, a_2\sum_{j=1}^{\ell}\lambda_j x_j,\dots,a_m\sum_{j=1}^{\ell}\lambda_j x_j}\prod_{i=1}^{m}p_i^{a_i\sum_{j=1}^{\ell}\lambda_j x_j} \le \prod_{j=1}^{\ell}\left[\binom{x_j\sum_{i=1}^{m}a_i}{x_ja_1,x_ja_2,\dots,x_ja_m}\prod_{i=1}^{m}p_i^{a_ix_j}\right]^{\lambda_j}$$

which can be rearranged as (3.11).

The inequality (3.12) can also be independently derived from logarithmically complete monotonicity of $Q(x)$. The proof of Theorem 3.3 is complete. □

Theorem 3.4. *For $\ell, m \ge 2$, let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$ and let $x_j > 0$ for $1 \le j \le \ell$. Then*

$$\prod_{j=1}^{\ell}\binom{x_j\sum_{i=1}^{m}a_i}{x_ja_1,x_ja_2,\dots,x_ja_m} < \binom{\sum_{j=1}^{\ell}x_j\sum_{i=1}^{m}a_i}{a_1\sum_{j=1}^{\ell}x_j, a_2\sum_{j=1}^{\ell}x_j,\dots,a_m\sum_{j=1}^{\ell}x_j}. \tag{3.13}$$

In particular, when $\ell = m = 2$,

$$\binom{(a_1+a_2)x_1}{a_1x_1}\binom{(a_1+a_2)x_2}{a_1x_2} < \binom{(a_1+a_2)(x_1+x_2)}{a_1(x_1+x_2)}. \tag{3.14}$$

Proof. In [**2**, Lemma 3], it was established that, if $g : [0,\infty) \to (0,1]$ is differentiable and $\frac{g'}{g}$ is strictly increasing on $(0,\infty)$, then $g(x)g(y) < g(x+y)$ for $x, y \in (0,\infty)$. From this, we can inductively derive

$$\prod_{j=1}^{\ell}g(x_j) < g\left(\sum_{j=1}^{\ell}x_j\right).$$

Applying this inequality to the function $\mathcal{Q}(x)$ yields

$$\prod_{j=1}^{\ell}\left[\binom{x_j\sum_{i=1}^{m}a_i}{x_ja_1,x_ja_2,\dots,x_ja_m}\prod_{i=1}^{m}p_i^{a_ix_j}\right] < \binom{\sum_{j=1}^{\ell}x_j\sum_{i=1}^{m}a_i}{a_1\sum_{j=1}^{\ell}x_j, a_2\sum_{j=1}^{\ell}x_j,\dots,a_m\sum_{j=1}^{\ell}x_j}\prod_{i=1}^{m}p_i^{a_i\sum_{j=1}^{\ell}x_j}$$

which can be rewritten as (3.13).

The inequality (3.14) can also be independently derived from logarithmically complete monotonicity of $Q(x)$. The proof of Theorem 3.4 is complete. □

Theorem 3.5. *For $m \ge 2$, let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$. If $0 < a \le c$ and $x > 0$, then*

$$\binom{(a+x)\sum_{i=1}^m a_i}{(a+x)a_1,(a+x)a_2,\dots,(a+x)a_m}\binom{c\sum_{i=1}^m a_i}{ca_1,ca_2,\dots,ca_m} \le \binom{a\sum_{i=1}^m a_i}{aa_1,aa_2,\dots,aa_m}\binom{(c+x)\sum_{i=1}^m a_i}{(c+x)a_1,(c+x)a_2,\dots,(c+x)a_m} \tag{3.15}$$

and the equality in (3.15) *holds if and only if* $a=c$. *In particular, when* $m=2$,

$$\binom{(a+x)(a_1+a_2)}{(a+x)a_1}\binom{c(a_1+a_2)}{ca_1} \le \binom{a(a_1+a_2)}{aa_1}\binom{(c+x)(a_1+a_2)}{(c+x)a_1} \tag{3.16}$$

and the equality in (3.16) *holds if and only if* $a=c$.

Proof. For $0<a<c$, define

$$V(x)=\ln\mathcal{Q}(a+x)+\ln\mathcal{Q}(c)-\ln\mathcal{Q}(a)-\ln\mathcal{Q}(c+x).$$

Since

$$V'(x)=\frac{\mathcal{Q}'(a+x)}{\mathcal{Q}(a+x)}-\frac{\mathcal{Q}'(c+x)}{\mathcal{Q}(c+x)}$$

and logarithmically complete monotonicity of $\mathcal{Q}(x)$ implies that $\frac{\mathcal{Q}'(x)}{\mathcal{Q}(x)}$ is strictly increasing on $(0,\infty)$, we conclude that $V'(x)<0$ and $V(x)<V(0)=0$. Therefore,

$$\ln\mathcal{Q}(a+x)+\ln\mathcal{Q}(c)\le\ln\mathcal{Q}(a)+\ln\mathcal{Q}(c+x),$$

which is equivalent to

$$\ln\left[\binom{(a+x)\sum_{i=1}^m a_i}{(a+x)a_1,(a+x)a_2,\dots,(a+x)a_m}\prod_{i=1}^m p_i^{a_i(a+x)}\right] + \ln\left[\binom{c\sum_{i=1}^m a_i}{ca_1,ca_2,\dots,ca_m}\prod_{i=1}^m p_i^{a_ic}\right] \le \ln\left[\binom{a\sum_{i=1}^m a_i}{aa_1,aa_2,\dots,aa_m}\prod_{i=1}^m p_i^{a_ia}\right] + \ln\left[\binom{(c+x)\sum_{i=1}^m a_i}{(c+x)a_1,(c+x)a_2,\dots,(c+x)a_m}\prod_{i=1}^m p_i^{a_i(c+x)}\right].$$

This can be simplified as (3.15).

The inequality (3.16) can also be independently derived from logarithmically complete monotonicity of $Q(x)$. The proof of Theorem 3.5 is complete. □

4. INEQUALITIES FOR MULTIVARIATE BETA FUNCTIONS

For $a_i>0$ and $i\in\mathbb{N}$, the multinomial coefficient and the multivariate beta function have the relation

$$\binom{\sum_{i=1}^m a_i}{a_1,a_2,\dots,a_m}=\frac{\sum_{i=1}^m a_i}{\prod_{i=1}^m a_i}\frac{1}{\mathrm{B}(a_1,a_2,\dots,a_m)}.$$

Therefore, from those inequalities for multinomial coefficients in Section , we can derive some inequalities for the multivariate beta function $\mathrm{B}(a_1, a_2, \dots, a_m)$. In other words, Theorems (3.3) to (3.5) can be respectively reformulated as the following forms.

4.1 First inequality for multivariate beta function

For $\ell, m \ge 2$, let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$, $x_j > 0$ for $1 \le j \le \ell$, and $\lambda_j \in (0,1)$ with $\sum_{j=1}^{\ell} \lambda_j = 1$. Then

$$\frac{\mathrm{B}\Bigl(a_1 \sum_{j=1}^{\ell} \lambda_j x_j, a_2 \sum_{j=1}^{\ell} \lambda_j x_j, \dots, a_m \sum_{j=1}^{\ell} \lambda_j x_j\Bigr)}{\prod_{j=1}^{\ell} \mathrm{B}^{\lambda_j}(a_1 x_j, a_2 x_j, \dots, a_m x_j)} \ge \left(\frac{\prod_{j=1}^{\ell} x_j^{\lambda_j}}{\sum_{j=1}^{\ell} \lambda_j x_j}\right)^{m-1} \tag{4.17}$$

and the equality in (4.17) holds if and only if $x_1 = x_2 = \dots = x_\ell$. In particular, when $\ell = m = 2$,

$$\frac{\mathrm{B}(a_1(\lambda_1 x_1 + \lambda_2 x_2), a_2(\lambda_1 x_1 + \lambda_2 x_2))}{\mathrm{B}^{\lambda_1}(a_1 x_1, a_2 x_1)\,\mathrm{B}^{\lambda_2}(a_1 x_2, a_2 x_2)} \ge \frac{x_1^{\lambda_1} x_2^{\lambda_2}}{\lambda_1 x_1 + \lambda_2 x_2} \tag{4.18}$$

and the equality in (4.18) is valid if and only if $x_1 = x_2$.

The inequality (4.17) or (4.18) implies that $x^{m-1}\,\mathrm{B}(xa_1, xa_2, \dots, xa_m)$ for $m \ge 2$ is logarithmically concave with respect to $x \in (0, \infty)$. More generally, we claim that the reciprocal $\frac{1}{x^{m-1}\,\mathrm{B}(xa_1, xa_2, \dots, xa_m)}$ for $m \ge 2$ is a logarithmically completely monotonic function of $x \in (0, \infty)$.

4.2 Second inequality for multivariate beta function

For $\ell, m \ge 2$, let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$ and let $x_j > 0$ for $1 \le j \le \ell$. Then

$$\frac{\prod_{j=1}^{\ell} \mathrm{B}(a_1 x_j, a_2 x_j, \dots, a_m x_j)}{\mathrm{B}\Bigl(a_1 \sum_{j=1}^{\ell} x_j, a_2 \sum_{j=1}^{\ell} x_j, \dots, a_m \sum_{j=1}^{\ell} x_j\Bigr)} > \left(\frac{\sum_{i=1}^{m} a_i}{\prod_{i=1}^{m} a_i}\right)^{\ell-1} \left(\frac{\sum_{j=1}^{\ell} x_j}{\prod_{j=1}^{\ell} x_j}\right)^{m-1}.$$

In particular, when $\ell = m = 2$,

$$\frac{\mathrm{B}(a_1 x_1, a_2 x_1)\,\mathrm{B}(a_1 x_2, a_2 x_2)}{\mathrm{B}(a_1(x_1 + x_2), a_2(x_1 + x_2))} > \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\left(\frac{1}{x_1} + \frac{1}{x_2}\right).$$

4.3 Third inequality for multivariate beta function

For $m \ge 2$, let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$. If $0 < a \le c$ and $x > 0$, then

$$\left(\frac{c}{c+x}\right)^{m-1}\frac{\mathrm{B}(ca_1,ca_2,\dots,ca_m)}{\mathrm{B}((c+x)a_1,(c+x)a_2,\dots,(c+x)a_m)} \ge \left(\frac{a}{a+x}\right)^{m-1}\frac{\mathrm{B}(aa_1,aa_2,\dots,aa_m)}{\mathrm{B}((a+x)a_1,(a+x)a_2,\dots,(a+x)a_m)} \tag{4.19}$$

and the equality in (4.19) holds if and only if $a = c$. In particular, when $m = 2$,

$$\frac{c}{c+x}\frac{\mathrm{B}(a_1c,a_2c)}{\mathrm{B}(a_1(c+x),a_2(c+x))} \ge \frac{a}{a+x}\frac{\mathrm{B}(a_1a,a_2a)}{\mathrm{B}(a_1(a+x),a_2(a+x))} \tag{4.20}$$

and the equality in (4.20) holds if and only if $a = c$.

The inequality (4.19) implies that the function

$$\left(\frac{t}{t+x}\right)^{m-1}\frac{\mathrm{B}(a_1t,a_2t,\dots,a_mt)}{\mathrm{B}(a_1(t+x),a_2(t+x),\dots,a_m(t+x))}$$

for $m \ge 2$ and $x, a_i > 0$ is strictly increasing with respect to $t \in (0,\infty)$.

Remark 4.1. For more information on inequalities for the beta function $\mathrm{B}(x,y)$ and their applications, please refer to [**3, 6, 7, 8, 9, 26, 36**] and closely related references therein.

5. RECOVERING FOUR KNOWN RESULTS

From Theorems 3.3 to 3.5, we can recover inequalities and monotonicity for binomial coefficients in the papers [**2, 17**].

5.4 First recovery

Taking $a_1 = k \in \mathbb{N}$ and $a_2 = n-k \in \mathbb{N}$ in (3.12) results in

$$\binom{(\lambda_1x_1+\lambda_2x_2)n}{(\lambda_1x_1+\lambda_2x_2)k} \le \binom{x_1n}{x_1k}^{\lambda_1}\binom{x_2n}{x_2k}^{\lambda_2} \tag{5.21}$$

and the equality in (5.21) holds if and only if $x_1 = x_2 > 0$, where $x_1, x_2, \lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$. This recovers [**2**, Corollary 1]. When further letting $\lambda_1 = \lambda_2 = \frac{1}{2}$ and setting $x_1 = j-1$ and $x_2 = j+1$ in (5.21), we recover a combinatorial inequality in [**17**, p. 4, Section 4].

5.5 Second recovery

Setting $a_1 = k \in \mathbb{N}$ and $a_2 = n-k \in \mathbb{N}$ in (3.14) gives

$$\binom{nx_1}{kx_1}\binom{nx_2}{kx_2} < \binom{n(x_1+x_2)}{k(x_1+x_2)}, \quad x_1, x_2 > 0$$

which is a recovery of [**2**, Corollary 2].

5.6 Third recovery

Letting $a_1 = k \in \mathbb{N}$ and $a_2 = n - k \in \mathbb{N}$ in (3.16) deduces

$$\binom{(a+x)n}{(a+x)k}\binom{cn}{ck} \le \binom{an}{ak}\binom{(x+c)n}{(x+c)k} \tag{5.22}$$

for $0 < a \le c$ and $x > 0$ and the equality is valid if and only if $a = c > 0$. The inequality (5.22) is a recovery of [**2**, Corollary 3].

5.7 Fourth recovery

For $a_1, a_2, x > 0$ and $0 < a < c$, the inequality (3.16) means that the function

$$F_{a,c;a_1,a_2}(x) = \binom{(a+x)(a_1+a_2)}{(a+x)a_1} \Big/ \binom{(c+x)(a_1+a_2)}{(c+x)a_1}$$

is decreasing in $x > 0$ and

$$\lim_{x\to\infty} F_{a,c;a_1,a_2}(x) = \left(\frac{a_1}{a_1+a_2}\right)^{a_1}\left(\frac{a_2}{a_1+a_2}\right)^{a_2}.$$

This generalizes [**17**, Lemma 2.2] which reads that the sequence $T_{n,k}(j) = \frac{\binom{(j-1)n}{(j-1)k}}{\binom{n}{jk}}$ for integers $0 \le k \le n$ is decreasing in $j \ge 1$ and

$$\lim_{j\to\infty} T_{n,k}(j) = \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k}.$$

Remark 5.2. Because the restrictions $a_k < 1$ and $\sum_{k=1}^{n} a_k = 1$ appeared in [**20**, Theorem 2.1] are removed off in our Theorem 2.2, the conditions in our Theorem 2.2 are more relaxed than corresponding ones in [**20**, Theorem 2.1]. Because logarithmically complete monotonicity is stronger than complete monotonicity, just like that logarithmic convexity is stronger than convexity, our main conclusion in Theorem 2.2 is stronger than corresponding one in [**20**, Theorem 2.1].

Remark 5.3. This paper is a revised version of the preprint [**33**] whose first version was announced almost at the same time as the preprint [**21**] which has been formally published as [**20**]. This paper is a companion of the papers [**22, 23, 28, 31, 32**].

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