Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 560–578.
https://doi.org/10.2298/AADM190220038K

FRACTIONAL TRAPEZIUM-TYPE INEQUALITIES FOR STRONGLY EXPONENTIALLY GENERALIZED PREINVEX FUNCTIONS WITH APPLICATIONS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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The aim of this paper is to introduce a new extension of preinvexity called strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvexity. Some new integral inequalities of trapezium-type for strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex functions with modulus c via Riemann-Liouville fractional integral are established. Also, some new estimates with respect to trapezium-type integral inequalities for strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex functions with modulus c via general fractional integrals are obtained. We show that the class of strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex functions with modulus c includes several other classes of preinvex functions. At the end, some new error estimates for trapezoidal quadrature formula are provided as well. This results may stimulate further research in different areas of pure and applied sciences.

1. INTRODUCTION

The class of convex functions is well known in the literature and is usually defined in the following way:

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2010 Mathematics Subject Classification. Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15, 90C23.
Keywords and Phrases. Trapezium-type integral inequalities, preinvexit, exponential convex function, general fractional integrals.

Definition 1.1. *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be convex on I if the inequality*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1.0.1}$$

holds for all $a, b \in I$ and $t \in [0,1]$. Also, we say that f is concave, if the inequality in (1.0.1) holds in the reverse direction.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.0.2}$$

This inequality (1.0.2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.0.2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [**4**]-[**12**],[**15**]-[**21**],[**28, 33, 35, 36**].

Now, let us recall the following definitions.

Definition 1.3. *[**16**] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called strongly convex with modulus $c \in \mathbb{R}^+$, if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

holds for every $x, y \in I$ and $t \in [0,1]$.

Strongly convex functions have been introduced by Polyak, see [**16**] and references therein. Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.

Definition 1.4. *[**16, 17**] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called strongly m-convex with $m \in [0,1]$ and modulus $c \in \mathbb{R}^+$, if*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) - cmt(1-t)(y-x)^2$$

holds for every $x, y \in I$ and $t \in [0,1]$.

Definition 1.5. *[28] A function: $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, among $m \in (0,1]$, satisfies the following inequality*

$$f\big(tx + m(1-t)y\big) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.0.3}$$

Definition 1.6. *[3] A set $K \subseteq \mathbb{R}^n$ is said to be invex respecting the mapping $\tau : K \times K \to \mathbb{R}^n$, if $x + t\tau(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Definition 1.7. *[20] Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to τ, if*

$$f\big(x + t\tau(y,x)\big) \le h(1-t) f(x) + h(t) f(y) \tag{1.0.4}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Definition 1.8. *[11] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\tau : K \times K \to \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\tau(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1. *Taking $m = 1$ in definition 1.8, the mapping $\tau(y, mx)$ reduce to $\tau(y,x)$, and then we get definition 1.6.*

Definition 1.9. *[30] Let $K \subseteq \mathbb{R}$ be m-invex set respecting the mapping $\tau : K \times K \to \mathbb{R}$ and $h_1, h_2 : [0,1] \to [0, +\infty)$. A function $f : K \to \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if*

$$f\big(mx + t\tau(y, mx)\big) \le m h_1(t) f(x) + h_2(t) f(y) \tag{1.0.5}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

Definition 1.10. *[18] Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$.
Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Furthermore, let us define a function $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\phi(t)}{t}dt < \infty, \tag{1.0.6}$$

$$\frac{1}{A} \le \frac{\phi(s)}{\phi(r)} \le A \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2 \tag{1.0.7}$$

$$\frac{\phi(r)}{r^2} \le B\frac{\phi(s)}{s^2} \text{ for } s \le r \tag{1.0.8}$$

$$\left|\frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2}\right| \le C|r-s|\frac{\phi(r)}{r^2} \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2 \tag{1.0.9}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\phi(r)r^\alpha$ is increasing for some $\alpha \ge 0$ and $\frac{\phi(r)}{r^\beta}$ is decreasing for some $\beta \ge 0$, then ϕ satisfies (1.0.6)–(1.0.9), see reference [**34**]. Therefore, Sarikaya and Ertuğral [**33**] defined the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$${}_{a^+}I_\phi f(x) = \int_a^x \frac{\phi(x-t)}{x-t}f(t)dt, \quad x > a, \tag{1.0.10}$$

$${}_{b^-}I_\phi f(x) = \int_x^b \frac{\phi(t-x)}{t-x}f(t)dt, \quad x < b. \tag{1.0.11}$$

This fractional integral operators are a new generalization of fractional integrals such as the Riemann-Liouville fractional integral, the k-Riemann-Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc. To read more about fractional analysis, see references [**13, 14, 22, 32**].

An important class of convex functions, which is called exponential convex functions, was introduced and studied by Antczak [**2**], Dragomir et al [**9**] and Noor et al [**27**]. Alirezai and Mathar [**1**] have investigated their basic properties along with their potential applications in statistics and information theory. Awan et al. [**5**] and Pecarić and Jaksetić [**29**] defined another kind of exponential convex functions and have shown that the class of exponential convex functions unifies various unrelated concepts.

Definition 1.11. *[**2, 9, 26**] A function $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex function, if*

$$e^{f((1-t)a+tb)} \le (1-t)e^{f(a)} + te^{f(b)} \tag{1.0.12}$$

holds for all $a, b \in K$, $t \in [0, 1]$, where f is positive.

For the applications of exponentially convex functions and strongly exponentially convex functions in different field of statistics, information theory and mathematical sciences, see [**1, 2, 5**],[**23**]-[**27**] and the references therein.

Definition 1.12. *[**31**] A function $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially m-convex function, where $m \in (0,1]$, if*

$$e^{f((1-t)a+mtb)} \leq (1-t)e^{f(a)} + mte^{f(b)} \tag{1.0.13}$$

holds for all $a, b \in K$, $t \in [0,1]$, where f is positive.

Motivated by the above literatures, the main objective of this article is to establish in Section fractional integral inequalities using a new class of preinvex functions called strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex function with modulus c. Also, using a new identity pertaining differentiable functions defined on m-invex set as auxiliary result, some new Hermite-Hadamard inequalities for strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex functions with modulus c via Riemann-Liouville fractional integral will obtain. Also, some new estimates with respect to trapezium-type integral inequalities for strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex functions with modulus c via general fractional integrals will given. Various special cases will be discussed. In Section , some new error estimates for trapezoidal quadrature formula will be given. This results may stimulate further research in different areas of pure and applied sciences.

2. MAIN RESULTS

Now we introduce strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex functions with modulus c.

Definition 2.13. *Let $K \subseteq \mathbb{R}$ be m-invex set with respect to the mapping $\tau : K \times K \to \mathbb{R}$ for some fixed $m \in (0,1]$ and $h_1, h_2 : [0,1] \to [0,+\infty)$. A function $f : K \to (0,+\infty)$ is called strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex with modulus $c > 0$, if*

$$e^{f(mx+t\tau(y,mx))} \leq mh_1(t)e^{\omega_1 f(x)} + h_2(t)e^{\omega_2 f(y)} - cm(y-x)^2 h_1(t)h_2(t) \tag{2.0.14}$$

holds for all $x, y \in K$, $t \in [0,1]$ and ω_1, $\omega_2 \in \mathbb{R}$.

Remark 2. *In definition 2.13, taking $c \to 0^+$ and choose $\omega_1 = \omega_2 = 1$, $h_1(t) = 1-t$, $h_2(t) = t$ and $\tau(y, mx) = y - mx$, this definition reduce to the definition 1.12.*

Remark 3. *Under the conditions of remark 2, taking $m = 1$, we get definition 1.11.*

Remark 4. *Let us discuss some special cases in definition 2.13 as follows.*

(I) Taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get strongly exponentially generalized $(m, \omega_1, \omega_2, h)$-preinvex functions with modulus c.

(II) Taking $h_1(t) = h_2(t) = t(1-t)$, *then we get strongly exponentially generalized* $(m,\omega_1,\omega_2,tgs)$*-preinvex functions with modulus* c.
(III) Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *then we get strongly exponentially generalized* (m,ω_1,ω_2)*-MT-preinvex functions with modulus* c.

In this section, we obtain Hermite-Hadamard type inequalities for strongly exponentially generalized $(m,\omega_1,\omega_2,h_1,h_2)$-preinvex function with modulus c via Riemann-Liouville fractional integral.

Theorem 2.14. *Let* $K = [ma, ma+\tau(b,ma)] \subseteq \mathbb{R}$ *be* m*-invex set with respect to the mapping* $\tau : K\times K \to \mathbb{R}$ *for some fixed* $m \in (0,1]$, *where* $a<b$ *and* $\tau(b,ma) > 0$. *Suppose* $h_1,h_2 : [0,1] \to [0,+\infty)$ *be continuous functions. Let* $f,g : K \to (0,+\infty)$ *be strongly exponentially generalized* $(m,\omega_1,\omega_2,h_1,h_2)$*-preinvex functions with modulus* $c>0$. *If* $f,g \in L(K)$, *then for* ω_1, $\omega_2 \in \mathbb{R}$ *and* $\alpha > 0$, *the following inequality holds:*

$$\frac{\Gamma(\alpha)}{\tau^{\alpha}(b,ma)}\left\{J^{\alpha}_{(ma+\tau(b,ma))^-}e^{f(ma)} + J^{\alpha}_{(ma+\tau(b,ma))^-}e^{g(ma)}\right\}$$

$$\leq m\left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right)H_{h_1}(\alpha) + \left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right)H_{h_2}(\alpha) \tag{2.0.15}$$

$$-2cm(b-a)^2\Psi_{h_1,h_2}(\alpha),$$

where
(2.0.16)

$$\Psi_{h_1,h_2}(\alpha) := \int_0^1 t^{\alpha-1}h_1(t)h_2(t)dt, \quad H_{h_i}(\alpha) := \int_0^1 t^{\alpha-1}h_i(t)dt, \ \ \forall\, i=1,2.$$

Proof. From strongly exponentially generalized $(m,\omega_1,\omega_2,h_1,h_2)$-preinvexity with modulus c of f and g for all $t\in[0,1]$, we have

$$e^{f(ma+t\tau(b,ma))} \leq mh_1(t)e^{\omega_1 f(a)} + h_2(t)e^{\omega_2 f(b)} - cm(b-a)^2h_1(t)h_2(t)$$

and

$$e^{g(ma+t\tau(b,ma))} \leq mh_1(t)e^{\omega_1 g(a)} + h_2(t)e^{\omega_2 g(b)} - cm(b-a)^2h_1(t)h_2(t).$$

Adding both sides of the above inequalities, we get

$$e^{f(ma+t\tau(b,ma))} + e^{g(ma+t\tau(b,ma))} \leq m\left(e^{\omega_1 f(a)} + e^{\omega_1 g(a)}\right)h_1(t) \tag{2.0.17}$$

$$+\left(e^{\omega_2 f(b)} + e^{\omega_2 g(b)}\right)h_2(t) - 2cm(b-a)^2h_1(t)h_2(t).$$

Multiplying both sides of inequality (2.0.17) with $t^{\alpha-1}$ and integrating over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}\left[e^{f(ma+t\tau(b,ma))} + e^{g(ma+t\tau(b,ma))}\right]dt$$

$$\leq m\left(e^{\omega_1 f(a)}+e^{\omega_1 g(a)}\right)\int_0^1 t^{\alpha-1}h_1(t)dt+\left(e^{\omega_2 f(b)}+e^{\omega_2 g(b)}\right)\int_0^1 t^{\alpha-1}h_2(t)dt$$
$$-2cm(b-a)^2\int_0^1 t^{\alpha-1}h_1(t)h_2(t)dt.$$

Using definition 1.10, we get the required result. □

Corollary 2.15. *In Theorem 2.14, taking $c \to 0^+$, we get the following inequality:*

$$\frac{\Gamma(\alpha)}{\tau^\alpha(b,ma)}\left\{J^\alpha_{(ma+\tau(b,ma))^-}e^{f(ma)}+J^\alpha_{(ma+\tau(b,ma))^-}e^{g(ma)}\right\}$$

$$\leq m\left(e^{\omega_1 f(a)}+e^{\omega_1 g(a)}\right)H_{h_1}(\alpha)+\left(e^{\omega_2 f(b)}+e^{\omega_2 g(b)}\right)H_{h_2}(\alpha). \tag{2.0.18}$$

Corollary 2.16. *In Theorem 2.14, if we choose $m=1$ and $\tau(b,ma)=b-ma$, we get the following inequality:*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha}\left\{J^\alpha_{b^-}e^{f(a)}+J^\alpha_{b^-}e^{g(a)}\right\}$$

$$\leq\left(e^{\omega_1 f(a)}+e^{\omega_1 g(a)}\right)H_{h_1}(\alpha)+\left(e^{\omega_2 f(b)}+e^{\omega_2 g(b)}\right)H_{h_2}(\alpha) \tag{2.0.19}$$
$$-2c(b-a)^2\Psi_{h_1,h_2}(\alpha).$$

Corollary 2.17. *In Theorem 2.14, if we choose $\alpha=1$, we get the following inequality:*

$$\frac{1}{\tau(b,ma)}\int_{ma}^{ma+\tau(b,ma)}\left[e^{f(t)}+e^{g(t)}\right]dt$$

$$\leq m\left(e^{\omega_1 f(a)}+e^{\omega_1 g(a)}\right)H_{h_1}+\left(e^{\omega_2 f(b)}+e^{\omega_2 g(b)}\right)H_{h_2}-2cm(b-a)^2F_{h_1,h_2}, \tag{2.0.20}$$

where

$$F_{h_1,h_2}:=\int_0^1 h_1(t)h_2(t)dt,\quad H_{h_i}:=\int_0^1 h_i(t)dt,\quad \forall\, i=1,2. \tag{2.0.21}$$

Theorem 2.18. *Let $K=[ma,ma+\tau(b,ma)]\subseteq\mathbb{R}$ be m-invex set with respect to the mapping $\tau:K\times K\to\mathbb{R}$ for some fixed $m\in(0,1]$, where $a<b$ and $\tau(b,ma)>0$. Suppose $h_1,h_2:[0,1]\to[0,+\infty)$ be continuous functions. Let $f,g:K\to(0,+\infty)$ be strongly exponentially generalized $(m,\omega_1,\omega_2,h_1,h_2)$-preinvex functions with modulus $c>0$. If $f,g\in L(K)$, then for ω_1, $\omega_2\in\mathbb{R}$ and $\alpha>0$, the following inequality holds:*

$$\frac{\Gamma(\alpha)}{\tau^\alpha(b,ma)}\left\{J^\alpha_{(ma)^+}e^{f(ma+\tau(b,ma))}+J^\alpha_{(ma+\tau(b,ma))^-}e^{g(ma)}\right\}$$

$$\leq m\Big(e^{\omega_1 f(a)}C_{h_1}(\alpha)+e^{\omega_1 g(a)}H_{h_1}(\alpha)\Big)+e^{\omega_2 f(b)}C_{h_2}(\alpha)+e^{\omega_2 g(b)}H_{h_2}(\alpha) \tag{2.0.22}$$
$$-cm(b-a)^2D_{h_1,h_2}(\alpha),$$

where

$$D_{h_1,h_2}(\alpha):=\int_0^1\left[t^{\alpha}+(1-t)^{\alpha-1}\right]h_1(t)h_2(t)dt, \tag{2.0.23}$$

$$C_{h_i}(\alpha):=\int_0^1(1-t)^{\alpha-1}h_i(t)dt,\quad \forall\, i=1,2 \tag{2.0.24}$$

and $H_{h_1}(\alpha)$, $H_{h_2}(\alpha)$ *are defined as in Theorem 2.14.*

Proof. From strongly exponentially generalized $(m,\omega_1,\omega_2,h_1,h_2)$-preinvexity wth modulus c of f and g for all $t\in[0,1]$, we have

$$e^{f(ma+t\tau(b,ma))}\leq mh_1(t)e^{\omega_1 f(a)}+h_2(t)e^{\omega_2 f(b)}-cm(b-a)^2h_1(t)h_2(t)$$

and

$$e^{g(ma+t\tau(b,ma))}\leq mh_1(t)e^{\omega_1 g(a)}+h_2(t)e^{\omega_2 g(b)}-cm(b-a)^2h_1(t)h_2(t).$$

Multiplying first above inequality with $(1-t)^{\alpha-1}$, the second with $t^{\alpha-1}$ and adding both sides, we get

$$(1-t)^{\alpha-1}e^{f(ma+t\tau(b,ma))}+t^{\alpha-1}e^{g(ma+t\tau(b,ma))}$$

$$\leq(1-t)^{\alpha-1}\Big[mh_1(t)e^{\omega_1 f(a)}+h_2(t)e^{\omega_2 f(b)}\Big] \tag{2.0.25}$$

$$+t^{\alpha-1}\Big[mh_1(t)e^{\omega_1 g(a)}+h_2(t)e^{\omega_2 g(b)}\Big]-cm(b-a)^2\int_0^1\left[t^{\alpha}+(1-t)^{\alpha-1}\right]h_1(t)h_2(t)dt.$$

Integrating over $[0,1]$ both sides of inequality (2.0.25) and using definition 1.10, we get the required result. □

Corollary 2.19. *In Theorem 2.18, takng* $c\to 0^+$ *we get the following inequality:*

$$\frac{\Gamma(\alpha)}{\tau^{\alpha}(b,ma)}\Big\{J^{\alpha}_{(ma)^+}e^{f(ma+\tau(b,ma))}+J^{\alpha}_{(ma+\tau(b,ma))^-}e^{g(ma)}\Big\}$$

$$\leq m\Big(e^{\omega_1 f(a)}C_{h_1}(\alpha)+e^{\omega_1 g(a)}H_{h_1}(\alpha)\Big)+e^{\omega_2 f(b)}C_{h_2}(\alpha)+e^{\omega_2 g(b)}H_{h_2}(\alpha). \tag{2.0.26}$$

Corollary 2.20. *In Theorem 2.18, if we choose* $m=1$ *and* $\tau(b,ma)=b-ma$, *we get the following inequality:*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\Big\{J^{\alpha}_{a^+}e^{f(b)}+J^{\alpha}_{b^-}e^{g(a)}\Big\}\leq e^{\omega_1 f(a)}C_{h_1}(\alpha)+e^{\omega_1 g(a)}H_{h_1}(\alpha) \tag{2.0.27}$$
$$+e^{\omega_2 f(b)}C_{h_2}(\alpha)+e^{\omega_2 g(b)}H_{h_2}(\alpha)-c(b-a)^2D_{h_1,h_2}(\alpha).$$

Corollary 2.21. *In Theorem 2.18, if we choose $\alpha = 1$, we get Corollary 2.17.*

Remark 5. *Under the conditions of Theorems 2.14 and 2.18, using remark 4, we can get several new integral inequalities. The details are left to the interested reader.*

For establishing some new results regarding generalizations of Hermite-Hadamard type integral inequalities associated with strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvexity with modulus c via general fractional integrals, we need the following lemma.

Lemma 2.22. *Let $K = [ma, ma+\tau(b, ma)] \subseteq \mathbb{R}$ be m-invex set with respect to the mapping $\tau : K \times K \to \mathbb{R}$ for some fixed $m \in (0, 1]$, where $a < b$ and $\tau(b, ma) > 0$. If $f : K \to \mathbb{R}$ is a differentiable function on K° such that $f' \in L(K)$, then the following identity for generalized fractional integrals holds:*

$$\frac{e^{f(ma)} + e^{f(ma+\tau(b,ma))}}{2} - \frac{1}{2\Pi_m(1)\tau(b, ma)}$$

$$\times\Big\{ {}_{(ma)^+}I_\phi\, e^{f(ma+\tau(b,ma))} + {}_{(ma+\tau(b,ma))^-}I_\phi\, e^{f(ma)}\Big\}$$

$$= \frac{\tau(b, ma)}{2\Pi_m(1)} \int_0^1 \big[\Pi_m(t) - \Pi_m(1-t)\big] e^{f(ma+t\tau(b,ma))} f'(ma + t\tau(b, ma))dt, \tag{2.0.28}$$

where

$$\Pi_m(t) := \int_0^t \frac{\phi\,(\tau(b, ma)u)}{u} du < \infty. \tag{2.0.29}$$

We denote

$$\Xi_{f,\Pi_m}(a, b) := \frac{\tau(b, ma)}{2\Pi_m(1)} \tag{2.0.30}$$

$$\times \int_0^1 \big[\Pi_m(t) - \Pi_m(1-t)\big] e^{f(ma+t\tau(b,ma))} f'(ma + t\tau(b, ma))dt.$$

Proof. From (2.0.30), we have

$$\Xi_{f,\Pi_m}(a, b) = \frac{\tau(b, ma)}{2\Pi_m(1)} \times \Bigg[\int_0^1 \Pi_m(t) e^{f(ma+t\tau(b,ma))} f'(ma + t\tau(b, ma))dt$$

$$- \int_0^1 \Pi_m(1-t) e^{f(ma+t\tau(b,ma))} f'(ma + t\tau(b, ma))dt\Bigg]$$

$$= \frac{\tau(b, ma)}{2} \times \big[\Xi^{(1)}_{f,\Pi_m}(a, b) - \Xi^{(2)}_{f,\Pi_m}(a, b)\big], \tag{2.0.31}$$

where

$$\Xi^{(1)}_{f,\Pi_m}(a,b) := \int_0^1 \Pi_m(t) e^{f(ma+t\tau(b,ma))} f'(ma+t\tau(b,ma))dt \tag{2.0.32}$$

and

$$\Xi^{(2)}_{f,\Pi_m}(a,b) := \int_0^1 \Pi_m(1-t) e^{f(ma+t\tau(b,ma))} f'(ma+t\tau(b,ma))dt. \tag{2.0.33}$$

Now, integrating by parts (2.0.32), changing the variable $u = ma + t\tau(b,ma)$ and using definition 1.0.10, we get

$$\Xi^{(1)}_{f,\Pi_m}(a,b)$$
$$= \frac{\Pi_m(t)e^{f(ma+t\tau(b,ma))}}{\tau(b,ma)}\Big|_0^1 - \frac{1}{\tau(b,ma)}\int_0^1 \frac{\phi(\tau(b,ma)t)}{t} e^{f(ma+t\tau(b,ma))}dt$$

$$= \frac{\Pi_m(1)e^{f(ma+\tau(b,ma))}}{\tau(b,ma)} - \frac{1}{\tau^2(b,ma)} {}_{(ma+\tau(b,ma))^-}I_\phi\, e^{f(ma)}. \tag{2.0.34}$$

Similarly, using (2.0.33), we obtain

$$\Xi^{(2)}_{f,\Pi_m}(a,b) = -\frac{\Pi_m(1)e^{f(ma)}}{\tau(b,ma)} + \frac{1}{\tau^2(b,ma)} {}_{(ma)^+}I_\phi\, e^{f(ma+\tau(b,ma))}. \tag{2.0.35}$$

Substituting (2.0.34) and (2.0.35) in (2.0.31), we get (2.0.28). The proof is completed. □

Remark 6. *In Lemma 2.22, if we choose $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$, we get the following identity for fractional integrals:*

$$\frac{e^{f(ma)} + e^{f(ma+\tau(b,ma))}}{2} - \frac{\Gamma(\alpha+1)}{2\tau^\alpha(b,ma)}$$
$$\times\Big\{ J^\alpha_{(ma)^+} e^{f(ma+\tau(b,ma))} + J^\alpha_{(ma+\tau(b,ma))^-} e^{f(ma)} \Big\}$$

$$= \frac{\tau(b,ma)}{2}\int_0^1 \big[t^\alpha - (1-t)^\alpha\big] e^{f(ma+t\tau(b,ma))} f'(ma+t\tau(b,ma))dt. \tag{2.0.36}$$

Using Lemma 2.22, we now state the following theorem.

Theorem 2.23. *Let $K = [ma, ma+\tau(b,ma)] \subseteq \mathbb{R}$ be m-invex set with respect to the mapping $\tau : K \times K \to \mathbb{R}$ for some fixed $m \in (0,1]$, where $a < b$ and $\tau(b,ma) > 0$. Suppose $h_1, h_2 : [0,1] \to [0,+\infty)$ be continuous functions. Let $f : K \to (0,+\infty)$ be a differentiable strongly exponentially generalized $(m,\omega_1,\omega_2,h_1,h_2)$-preinvex function with modulus $c > 0$ on K° such that $f' \in L(K)$ and $\omega_1, \omega_2 \in \mathbb{R}$. If $|f'|^q$ is generalized*

(m,h_1,h_2)-preinvex function, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, the following inequality holds:

$$\left|\Xi_{f,\Pi_m}(a,b)\right| \leq \frac{\tau(b,ma)}{2\Pi_m(1)}\sqrt[p]{B_{\Pi_m}(p)} \tag{2.0.37}$$

$$\times\Big[m^2e^{q\omega_1 f(a)}\left|f'(a)\right|^q G_{h_1}+m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2}+e^{q\omega_2 f(b)}\left|f'(b)\right|^q G_{h_2}$$

$$-cm(b-a)^2\left(m\left|f'(a)\right|^q M_{h_1,h_2}+\left|f'(b)\right|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}},$$

where

$$B_{\Pi_m}(p):=\int_0^1\left|\Pi_m(t)-\Pi_m(1-t)\right|^p dt,\quad G_{h_i}:=\int_0^1[h_i(t)]^2dt,\ \ \forall\, i=1,2, \tag{2.0.38}$$

$$M_{h_1,h_2}:=\int_0^1[h_1(t)]^2h_2(t)dt,\quad N_{h_1,h_2}:=\int_0^1 h_1(t)[h_2(t)]^2dt, \tag{2.0.39}$$

$$\Delta_f(q;\omega_1,\omega_2,a,b):=e^{q\omega_1 f(a)}\left|f'(b)\right|^q+e^{q\omega_2 f(b)}\left|f'(a)\right|^q \tag{2.0.40}$$

and F_{h_1,h_2} is defined from (2.0.21).

Proof. From Lemma 2.22, strongly exponentially generalized $(m,\omega_1,\omega_2,h_1,h_2)$-preinvexity with modulus c of f, generalized (m,h_1,h_2)-preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|\Xi_{f,\Pi_m}(a,b)\right| \leq \frac{\tau(b,ma)}{2\Pi_m(1)}$$

$$\times\int_0^1\left|\Pi_m(t)-\Pi_m(1-t)\right|\left|e^{f(ma+t\tau(b,ma))}f'(ma+t\tau(b,ma))\right|dt$$

$$\leq\frac{\tau(b,ma)}{2\Pi_m(1)}\sqrt[p]{B_{\Pi_m}(p)}\left(\int_0^1 e^{qf(ma+t\tau(b,ma))}\left|f'(ma+t\tau(b,ma))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{\tau(b,ma)}{2\Pi_m(1)}\sqrt[p]{B_{\Pi_m}(p)}$$

$$\times\Bigg\{\int_0^1\Big[mh_1(t)e^{q\omega_1 f(a)}+h_2(t)e^{q\omega_2 f(b)}-cm(b-a)^2h_1(t)h_2(t)\Big]$$

$$\times\Big[mh_1(t)\left|f'(a)\right|^q+h_2(t)\left|f'(b)\right|^q\Big]dt\Bigg\}^{\frac{1}{q}}$$

$$=\frac{\tau(b,ma)}{2\Pi_m(1)}\sqrt[p]{B_{\Pi_m}(p)}$$

$$\times\Big[m^2 e^{q\omega_1 f(a)}\big|f'(a)\big|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\big|f'(b)\big|^q G_{h_2}$$

$$-cm(b-a)^2\left(m\big|f'(a)\big|^q M_{h_1,h_2} + \big|f'(b)\big|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}}.$$

So, the proof of this theorem is completed. □

We point out some special cases of Theorem 2.23.

Corollary 2.24. *In Theorem 2.23, taking $c \to 0^+$, we get the following inequality:*

$$\big|\Xi_{f,\Pi_m}(a,b)\big| \le \frac{\tau(b,ma)}{2\Pi_m(1)}\sqrt[p]{B_{\Pi_m}(p)} \tag{2.0.41}$$

$$\times\sqrt[q]{m^2 e^{q\omega_1 f(a)}\big|f'(a)\big|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\big|f'(b)\big|^q G_{h_2}}.$$

Corollary 2.25. *In Theorem 2.23, if we choose $\phi(t) = t$, we get the following inequality:*

$$\left|\frac{e^{f(ma)} + e^{f(ma+\tau(b,ma))}}{2} - \frac{1}{\tau(b,ma)}\int_{ma}^{ma+\tau(b,ma)} e^{f(t)}dt\right| \le \frac{\tau(b,ma)}{2\sqrt[p]{p+1}} \tag{2.0.42}$$

$$\times\Big[m^2 e^{q\omega_1 f(a)}\big|f'(a)\big|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\big|f'(b)\big|^q G_{h_2}$$

$$-cm(b-a)^2\left(m\big|f'(a)\big|^q M_{h_1,h_2} + \big|f'(b)\big|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}}.$$

Corollary 2.26. *In Theorem 2.23, if we choose $\phi(t) = \dfrac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$ and $\tau(b,ma) = b - ma$, we get the following inequality for fractional integrals:*

$$\left|\frac{e^{f(ma)} + e^{f(b)}}{2} - \frac{\Gamma(\alpha+1)}{2(b-ma)^\alpha}\times\left\{J^\alpha_{(ma)^+}e^{f(b)} + J^\alpha_{b^-}e^{f(ma)}\right\}\right|$$

$$\le \frac{(b-ma)}{2}\sqrt[p]{B(p,\alpha)} \tag{2.0.43}$$

$$\times\Big[m^2 e^{q\omega_1 f(a)}\big|f'(a)\big|^q G_{h_1} + m\Delta_f(q;\omega_1,\omega_2,a,b)F_{h_1,h_2} + e^{q\omega_2 f(b)}\big|f'(b)\big|^q G_{h_2}$$

$$-cm(b-a)^2\left(m\big|f'(a)\big|^q M_{h_1,h_2} + \big|f'(b)\big|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}},$$

where

$$B(p,\alpha) := \int_0^1 \big|t^\alpha - (1-t)^\alpha\big|^p dt.$$

Theorem 2.27. *Let $K = [ma, ma + \tau(b, ma)] \subseteq \mathbb{R}$ be m-invex set with respect to the mapping $\tau : K \times K \to \mathbb{R}$ for some fixed $m \in (0, 1]$, where $a < b$ and $\tau(b, ma) > 0$. Suppose $h_1, h_2 : [0, 1] \to [0, +\infty)$ be continuous functions. Let $f : K \to (0, +\infty)$ be a differentiable strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvex function with modulus $c > 0$ on K° such that $f' \in L(K)$ and $\omega_1, \omega_2 \in \mathbb{R}$. If $|f'|^q$ is generalized (m, h_1, h_2)-preinvex function, then for $q \geq 1$, the following inequality holds:*

$$\left|\Xi_{f,\Pi_m}(a, b)\right| \leq \frac{\tau(b, ma)}{2\Pi_m(1)} \left[B_{\Pi_m}(1)\right]^{1-\frac{1}{q}} \tag{2.0.44}$$

$$\times\Big[m^2 e^{q\omega_1 f(a)}\left|f'(a)\right|^q P_{\Pi_m,h_1} + m\Delta_f(q; \omega_1, \omega_2, a, b) S_{\Pi_m,h_1,h_2} + e^{q\omega_2 f(b)}\left|f'(b)\right|^q P_{\Pi_m,h_2}$$

$$-cm(b-a)^2 \left(m\left|f'(a)\right|^q U_{\Pi_m,h_1,h_2} + \left|f'(b)\right|^q T_{\Pi_m,h_1,h_2}\right)\Big]^{\frac{1}{q}},$$

where

$$S_{\Pi_m,h_1,h_2} := \int_0^1 \left|\Pi_m(t) - \Pi_m(1-t)\right| h_1(t)h_2(t)dt, \tag{2.0.45}$$

$$P_{\Pi_m,h_i} := \int_0^1 \left|\Pi_m(t) - \Pi_m(1-t)\right| [h_i(t)]^2 dt, \quad \forall\, i = 1, 2, \tag{2.0.46}$$

$$U_{\Pi_m,h_1,h_2} := \int_0^1 \left|\Pi_m(t) - \Pi_m(1-t)\right| [h_1(t)]^2 h_2(t)dt, \tag{2.0.47}$$

$$T_{\Pi_m,h_1,h_2} := \int_0^1 \left|\Pi_m(t) - \Pi_m(1-t)\right| h_1(t)[h_2(t)]^2 dt \tag{2.0.48}$$

and $\Delta_f(q; \omega_1, \omega_1, a, b)$, $B_{\Pi_m}(1)$ are defined as in Theorem 2.23.

Proof. From Lemma 2.22, strongly exponentially generalized $(m, \omega_1, \omega_2, h_1, h_2)$-preinvexity with modulus c of f, generalized (m, h_1, h_2)-preinvexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left|\Xi_{f,\Pi_m}(a, b)\right| \leq \frac{\tau(b, ma)}{2}$$

$$\times \int_0^1 \left|\Pi_m(t) - \Pi_m(1-t)\right| \left|e^{f(ma+t\tau(b,ma))} f'(ma + t\tau(b, ma))\right| dt$$

$$\leq \frac{\tau(b, ma)}{2\Pi_m(1)} \left[B_{\Pi_m}(1)\right]^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left|\Pi_m(t) - \Pi_m(1-t)\right| e^{qf(ma+t\tau(b,ma))} \left|f'(ma + t\tau(b, ma))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\tau(b,ma)}{2\Pi_m(1)}\left[B_{\Pi_m}(1)\right]^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1 \left|\Pi_m(t)-\Pi_m(1-t)\right|\left[mh_1(t)e^{q\omega_1 f(a)}+h_2(t)e^{q\omega_2 f(b)}-cm(b-a)^2h_1(t)h_2(t)\right]\right.$$

$$\left.\times\left[mh_1(t)\left|f'(a)\right|^q+h_2(t)\left|f'(b)\right|^q\right]dt\right\}^{\frac{1}{q}}$$

$$=\frac{\tau(b,ma)}{2\Pi_m(1)}\left[B_{\Pi_m}(1)\right]^{1-\frac{1}{q}}$$

$$\times\Big[m^2e^{q\omega_1 f(a)}\left|f'(a)\right|^qP_{\Pi_m,h_1}+m\Delta_f(q;\omega_1,\omega_2,a,b)S_{\Pi_m,h_1,h_2}+e^{q\omega_2 f(b)}\left|f'(b)\right|^qP_{\Pi_m,h_2}$$

$$-cm(b-a)^2\left(m\left|f'(a)\right|^qU_{\Pi_m,h_1,h_2}+\left|f'(b)\right|^qT_{\Pi_m,h_1,h_2}\right)\Big]^{\frac{1}{q}}.$$

So, the proof of this theorem is completed. □

We point out some special cases of Theorem 2.27.

Corollary 2.28. *In Theorem 2.27, taking $c\to 0^+$, we get the following inequality:*

$$\left|\Xi_{f,\Pi_m}(a,b)\right|\leq\frac{\tau(b,ma)}{2\Pi_m(1)}\left[B_{\Pi_m}(1)\right]^{1-\frac{1}{q}} \tag{2.0.49}$$

$$\times\sqrt[q]{m^2e^{q\omega_1 f(a)}\left|f'(a)\right|^qP_{\Pi_m,h_1}+m\Delta_f(q;\omega_1,\omega_2,a,b)S_{\Pi_m,h_1,h_2}+e^{q\omega_2 f(b)}\left|f'(b)\right|^qP_{\Pi_m,h_2}}.$$

Corollary 2.29. *In Theorem 2.27, if we choose $\phi(t)=t$, we get the following inequality:*

(2.0.50)

$$\left|\frac{e^{f(ma)}+e^{f(ma+\tau(b,ma))}}{2}-\frac{1}{\tau(b,ma)}\int_{ma}^{ma+\tau(b,ma)}e^{f(t)}dt\right|\leq 2^{\frac{1-2q}{q}}\tau(b,ma)$$

$$\times\Big\{m^2e^{q\omega_1 f(a)}\left|f'(a)\right|^q\Omega_{h_1}+m\Delta_f(q;\omega_1,\omega_2,a,b)\Theta_{h_1,h_2}+e^{q\omega_2 f(b)}\left|f'(b)\right|^q\Omega_{h_2}$$

$$-cm(b-a)^2\left(m\left|f'(a)\right|^qQ_{h_1,h_2}+\left|f'(b)\right|^qR_{h_1,h_2}\right)\Big\}^{\frac{1}{q}},$$

where

(2.0.51)

$$\Theta_{h_1,h_2}:=\int_0^1\left|2t-1\right|h_1(t)h_2(t)dt,\quad \Omega_{h_i}:=\int_0^1\left|2t-1\right|[h_i(t)]^2dt,\ \ \forall\, i=1,2,$$

$$Q_{h_1,h_2}:=\int_0^1\left|2t-1\right|[h_1(t)]^2h_2(t)dt,\quad R_{h_1,h_2}:=\int_0^1\left|2t-1\right|h_1(t)[h_2(t)]^2dt. \tag{2.0.52}$$

Corollary 2.30. *In Theorem 2.27, taking $c \to 0^+$, if we choose $\phi(t) = \dfrac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$ and $\tau(b, ma) = b - ma$, we get the following inequality:*

$$\left| \frac{e^{f(ma)} + e^{f(b)}}{2} - \frac{\Gamma(\alpha+1)}{2(b-ma)^\alpha} \times \left\{ J^\alpha_{(ma)^+} e^{f(b)} + J^\alpha_{b^-} e^{f(ma)} \right\} \right|$$

$$\leq \frac{(b-ma)}{2} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}} \tag{2.0.53}$$

$$\times \sqrt[q]{m^2 e^{q\omega_1 f(a)} \left|f'(a)\right|^q P_{\Pi_m, h_1} + m\Delta_f(q; \omega_1, \omega_2, a, b) S_{\Pi_m, h_1, h_2} + e^{q\omega_2 f(b)} \left|f'(b)\right|^q P_{\Pi_m, h_2}}.$$

Remark 7. *Under the conditions of Theorems 2.23 and 2.27, using remark 4, for the appropriate choices of function $\phi(t) = t$, $\dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\phi(t) = \dfrac{t}{\alpha} \exp\left[\left(-\dfrac{1-\alpha}{\alpha}\right)t\right]$ where $\alpha \in (0,1)$, etc., we can get several new integral inequalities. Also, under assumptions of Theorems 2.23 and 2.27, taking $K = \|f'\|_\infty := \sup_{x\in K} |f'(x)|$, we can obtain many other inequalities. The details are left to the interested reader.*

3. APPLICATIONS

In this section, we provide some new error estimates for trapezoidal quadrature formula. Let P be the partition of the points $a = x_0 < x_1 < \ldots < x_k = b$ of the interval $[a, b]$. Let consider the following quadrature formula:

$$\int_a^b e^{f(x)} dx = T(f, P) + E(f, P),$$

where

$$T(f, P) = \sum_{i=0}^{k-1} \left[\frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2} \right] (x_{i+1} - x_i)$$

is the trapezoidal version and $E(f, P)$ is denote the associated approximation error.

Proposition 3.31. *Let $f : [a, b] \to (0, +\infty)$ be a differentiable strongly exponentially generalized $(\omega_1, \omega_2, h_1, h_2)$-convex function with modulus $c > 0$ on (a, b), where $a < b$ and $\omega_1, \omega_2 \in \mathbb{R}$. Suppose $h_1, h_2 : [0, 1] \to [0, +\infty)$ be continuous functions. If $|f'|^q$ is generalized (h_1, h_2)-convex on $[a, b]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|E(f, P)\right| \leq \frac{1}{2\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \tag{3.0.54}$$

$$\times \Big[e^{q\omega_1 f(x_i)} \left|f'(x_i)\right|^q G_{h_1} + \Delta_f(q; \omega_1, \omega_2, x_i, x_{i+1}) F_{h_1, h_2} + e^{q\omega_2 f(x_{i+1})} \left|f'(x_{i+1})\right|^q G_{h_2}$$

$$-c(x_{i+1}-x_i)^2\left(\left|f'(x_i)\right|^q M_{h_1,h_2}+\left|f'(x_{i+1})\right|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}},$$

where

$$\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1}) := e^{q\omega_1 f(x_i)}\left|f'(x_{i+1})\right|^q + e^{q\omega_2 f(x_{i+1})}\left|f'(x_i)\right|^q \tag{3.0.55}$$

and F_{h_1,h_2}, G_{h_1}, G_{h_2}, M_{h_1,h_2}, N_{h_1,h_2} *are defined as in Theorem 2.23.*

Proof. Applying Theorem 2.23 for $m = 1$, $\tau(b, ma) = b - ma$ and $\phi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \dots, k-1)$ of the partition P, we have

$$\left|\frac{e^{f(x_i)}+e^{f(x_{i+1})}}{2}-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} e^{f(x)}dx\right| \le \frac{(x_{i+1}-x_i)}{2\sqrt[p]{p+1}} \tag{3.0.56}$$

$$\times\Big[e^{q\omega_1 f(x_i)}\left|f'(x_i)\right|^q G_{h_1}+\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})F_{h_1,h_2}+e^{q\omega_2 f(x_{i+1})}\left|f'(x_{i+1})\right|^q G_{h_2}$$

$$-c(x_{i+1}-x_i)^2\left(\left|f'(x_i)\right|^q M_{h_1,h_2}+\left|f'(x_{i+1})\right|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}}.$$

Hence from (3.0.56), we get

$$\left|E(f,P)\right| = \left|\int_a^b e^{f(x)}dx - T(f,P)\right|$$

$$\le \left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}} e^{f(x)}dx - \frac{e^{f(x_i)}+e^{f(x_{i+1})}}{2}(x_{i+1}-x_i)\right\}\right|$$

$$\le \sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}} e^{f(x)}dx - \frac{e^{f(x_i)}+e^{f(x_{i+1})}}{2}(x_{i+1}-x_i)\right\}\right|$$

$$\le \frac{1}{2\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2$$

$$\times\Big[e^{q\omega_1 f(x_i)}\left|f'(x_i)\right|^q G_{h_1}+\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})F_{h_1,h_2}+e^{q\omega_2 f(x_{i+1})}\left|f'(x_{i+1})\right|^q G_{h_2}$$

$$-c(x_{i+1}-x_i)^2\left(\left|f'(x_i)\right|^q M_{h_1,h_2}+\left|f'(x_{i+1})\right|^q N_{h_1,h_2}\right)\Big]^{\frac{1}{q}}.$$

The proof of this proposition is complete. □

Proposition 3.32. *Let $f : [a, b] \to (0, +\infty)$ be a differentiable strongly exponentially generalized $(\omega_1, \omega_2, h_1, h_2)$-convex function with modulus $c > 0$ on (a, b), where $a < b$ and $\omega_1, \omega_2 \in \mathbb{R}$. Suppose $h_1, h_2 : [0, 1] \to [0, +\infty)$ be continuous functions. If $|f'|^q$ is generalized (h_1, h_2)-convex on $[a, b]$ for $q \ge 1$, then the following inequality holds:*

$$\left|E(f,P)\right| \le 2^{\frac{1-2q}{q}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.0.57}$$

$$\times\Big[e^{q\omega_1 f(x_i)}\big|f'(x_i)\big|^q\Omega_{h_1}+\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})\Theta_{h_1,h_2}+e^{q\omega_2 f(x_{i+1})}\big|f'(x_{i+1})\big|^q\Omega_{h_2}$$

$$-c(x_{i+1}-x_i)^2\left(\big|f'(x_i)\big|^q Q_{h_1,h_2}+\big|f'(x_{i+1})\big|^q R_{h_1,h_2}\right)\Big]^{\frac{1}{q}},$$

where $\Delta_f(q;\omega_1,\omega_2,x_i,x_{i+1})$ *is defined from (3.0.55),* Θ_{h_1,h_2}, Ω_{h_1}, Ω_{h_2}; Q_{h_1,h_2}, R_{h_1,h_2} *are defined respectively from (2.0.51) and (2.0.52).*

Proof. The proof is analogous as to that of Proposition 3.31, taking $m = 1$, $\tau(b, ma) = b - ma$ and $\phi(t) = t$ in Theorem 2.27. □

Remark 8. *Under the conditions of Theorems 2.23 and 2.27, using remark 4, for the appropriate choices of function* $\phi(t) = t$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\phi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ *where* $\alpha \in (0,1)$, *etc., we can deduce several new bounds for the trapezoidal quadrature formula using above ideas and techniques. Also, under assumptions of Theorems 2.23 and 2.27, taking* $K = \|f'\|_\infty := \sup_{x\in K}|f'(x)|$, *we can obtain many other new bounds. The details are left to the interested reader.*

Acknowledgements

We thank anonymous referee for his/her valuable suggestion regarding the manuscript.

REFERENCES

1. G. Alirezaei and R. Mathar, On exponentially concave functions and their impact in information theory, Information Theory and Applications Workshop, San Diego, California, USA, (2018).
2. T. Antczak, (p,r)-invex sets and functions, J. Math. Anal. Appl., **263** (2001), pp. 355–379.
3. T. Antczak, Mean value in invexity analysis, Nonlinear Anal., **60** (2005), pp. 1473–1484.
4. S.M. Aslani, M.R. Delavar and S.M. Vaezpour, Inequalities of Fejér type related to generalized convex functions with applications, Int. J. Anal. Appl., **16**(1) (2018), pp. 38–49.
5. M.U. Awan, M.A. Noor and K.I. Noor, Hermite-Hadamard inequalities for exponentially convex functions, Appl. Math. Inform. Sci. **2**(12) (2018), pp. 405–409.
6. F.X. Chen and S.H. Wu, Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions, J. Nonlinear Sci. Appl., **9**(2) (2016), pp. 705–716.
7. Y.M. Chu, M.A. Khan, T.U. Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, J. Nonlinear Sci. Appl., **9**(5) (2016), pp. 4305–4316.

8. M.R. Delavar and M. De La Sen, Some generalizations of Hermite-Hadamard type inequalities, SpringerPlus, **5**(1661) (2016).

9. S.S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang J. Math., **33**(1) (2002), pp. 45–55.

10. T.S. Du, M.U. Awan, A. Kashuri and S. Zhao, Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m, h)-preinvexity, Appl. Anal., (2019).

11. T.S. Du, J.G. Liao and Y.J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions, J. Nonlinear Sci. Appl., **9** (2016), pp. 3112–3126.

12. M. Jleli and B. Samet, On Hermite-Hadamard type inequalities via fractional integral of a function with respect to another function, J. Nonlinear Sci. Appl., **9** (2016), pp. 1252–1260.

13. J. Hristov, Response functions in linear viscoelastic constitutive equations and related fractional operators, Math. Model. Nat. Phenom., **14** (2019), pp. 1–34.

14. U.N. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput., **218** (2011), pp. 860–865.

15. A. Kashuri and R. Liko, Some new Hermite-Hadamard type inequalities and their applications, Stud. Sci. Math. Hung., **56**(1) (2019), pp. 103–142.

16. T. Lara, N. Merentes, R. Quintero and E. Rosales, On strongly m-convex functions, Math. Aeterna, **5**(3) (2015), pp. 521–535.

17. T. Lara, N. Merentes and R. Quintero, On inequalities of Fejér and Hermite-Hadamard types for strongly m-convex functions, Math. Aeterna, **5**(5) (2015), pp. 777–793.

18. W. Liu, W. Wen and J. Park, Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, J. Nonlinear Sci. Appl., **9** (2016), pp. 766–777.

19. C. Luo, T.S. Du, M.A. Khan, A. Kashuri and Y. Shen, Some k-fractional integrals inequalities through generalized $\lambda_{\phi m}$-MT-preinvexity, J. Comput. Anal. Appl., **27**(4) (2019), pp. 690–705.

20. M. Matłoka, Inequalities for h-preinvex functions, Appl. Math. Comput., **234** (2014), pp. 52–57.

21. M.V. Mihai, Some Hermite-Hadamard type inequalities via Riemann-Liouville fractional calculus, Tamkang J. Math, **44**(4) (2013), pp. 411–416.

22. K.S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley: New York, NY, USA, (1993).

23. M.A. Noor and K.I. Noor, Strongly exponentially convex functions and their properties, Available online.

24. M.A. Noor and K.I. Noor, Strongly exponentially convex functions, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, Available online.

25. M.A. Noor and K.I. Noor, Exponential convex functions, Preprint.

26. M.A. Noor and K.I. Noor and S. Rashid, Fractal exponential convex functions and inequalities, Preprint.

27. M.A. Noor and K.I. Noor and S. Rashid, Exponential r-convex functions and inequalities, Preprint.

28. O. Omotoyinbo and A. Mogbodemu, Some new Hermite-Hadamard integral inequalities for convex functions, Int. J. Sci. Innovation Tech., **1**(1) (2014), pp. 1–12.

29. J. Pecarić and J. Jaksetić, Exponential convexity, Euler-Radau expansions and stolarsky means, Rad Hrvat. Matematicke Znanosti, **515** (2013), pp. 81–94.

30. C. Peng, C. Zhou and T.S. Du, Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2)-preinvexity, Ital. J. Pure Appl. Math., **38** (2017), pp. 345–367.

31. S. Rashid, M.A. Noor and K.I., Noor, Fractional exponentially m-convex functions and inequalities, Int. J. Anal. Appl., **17**(3) (2019), pp. 464–478.

32. M.A.F.D. Santos, Fractional Prabhakar derivative in diffusion equation with non-static stochastic resetting, Physics, **1** (2019), pp. 40–58.

33. M.Z. Sarikaya and F. Ertuğral, On the generalized Hermite-Hadamard inequalities, Available online.

34. M.Z. Sarikaya and H. Yildirim, On generalization of the Riesz potential, Indian J. Math. Math. Sci., **3** (2007), pp. 231–235.

35. H. Wang, T.S. Du and Y. Zhang, k-fractional integral trapezium-like inequalities through (h, m)-convex and (α, m)-convex mappings, J. Inequal. Appl., **2017**(311) (2017), pp. 20.

36. Y. Zhang, T.S. Du, H. Wang, Y.J. Shen and A. Kashuri, Extensions of different type parameterized inequalities for generalized (m, h)-preinvex mappings via k-fractional integrals, J. Inequal. Appl., **2018**(49) (2018), pp. 30.

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(Received 20.02.2019)
(Revised 17.07.2019)

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