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A CERTAIN CLASS OF STATISTICAL PROBABILITY CONVERGENCE AND ITS APPLICATIONS TO APPROXIMATION THEOREMS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In the present work, we introduce and study the notion of statistical probability convergence for sequences of random variables as well as the concept of statistical convergence for sequences of real numbers, which are defined over a Banach space via deferred weighted summability mean. We first establish a theorem presenting a connection between them. Based upon our proposed methods, we then prove a new Korovkin-type approximation theorem with periodic test functions for a sequence of random variables on a Banach space and demonstrate that our theorem effectively extends and improves most (if not all) of the previously existing results (in statistical versions). We also estimate the rate of deferred weighted statistical probability convergence and accordingly establish a new result. Finally, an illustrative example is presented here by means of the generalized Fejér convolution operators of a sequence of random variables in order to demonstrate that our established theorem is stronger than its traditional and statistical versions.

1. INTRODUCTION AND MOTIVATION

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In the study of sequence spaces, the classical convergence has got numerous applications where the convergence of a sequence requires that almost all elements are to satisfy the convergence condition. This means that, all of the elements of the sequence need to be in an arbitrarily small neighborhood of the limit. However, such restriction is relaxed in statistical convergence, where the validity of the convergence condition is achieved only for a majority of elements. Subsequently, the notion of statistical convergence was introduced by Fast [**8**] and Steinhaus [**29**]. Recently, statistical probability convergence has been a dynamic research area due to the fact that it is more general than the statistical convergence as well as the classical convergence. Moreover, such theory is discussed in the study of Fourier Analysis, Number Theory and Approximation Theory. For more details, see [**3**], [**4**], [**6**], [**9**], [**10**], [**20**] and [**22**].

Let $\mathbb{N}$ be the set of natural numbers and let $K \subseteq \mathbb{N}$. Also let

$$K_n = \{k : k \leqq n \quad \text{and} \quad k \in K\}$$

and suppose that $|K_n|$ is the cardinality of K_n. Then the natural density $d(K)$ of K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad k \in K\}|,$$

provided that the limit exists.

A given real sequence (x_n) is said to be statistically convergent to L if, for each $\epsilon > 0$, the set

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}$$

has zero natural density (see [**8**] and [**29**]). Thus, for each $\epsilon > 0$, we have

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad |x_k - L| \geqq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} x_n = L.$$

In the year 2002, Móricz [**15**] introduced the fundamental idea of statistical Cesàro summability. Later on, Mohiuddine *et al.* [**14**] established some Korovkin-type approximation theorems based upon statistical Cesàro summability. Subsequently, Karakaya and Chishti [**12**] introduced and studied the concept of weighted statistical convergence and their definition was later modified by Mursaleen *et al.* [**16**]. The fundamental concept of the deferred Cesàro statistical convergence as well as of the statistically-deferred Cesàro summability and associated approximation theorems was introduced by Jena *et al.* [**11**]. Recently, Srivastava *et al.* [**24**] introduced the notion of deferred weighted statistical convergence and proved analogous approximation theorems. For several other recent developments in this direction, see (for example) [**5**], [**17**], [**19**], [**24**], [**25**], [**26**] and [**27**].

Recalling the probability theory, let $X_n \quad (n \in \mathbb{N})$ be a random variable defined on an event space S with respect to a given class of events Δ. Let $P : \Delta \to \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers) be a probability density function. Then we denote the sequence $X_1, X_2, X_3, \cdots$ of random variables by $\{X_n\}_{n\in\mathbb{N}}$.

From the practical point of view, the discussion of a random variable X will be highly significant if it is known that there exists a real constant c for which

$$P(|X - c| < \epsilon) = 1,$$

where $\epsilon > 0$ is sufficiently small, that is, it is nearly certain that the values of X lie in a very small neighborhood of c.

For a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables, where each X_n may not have the above property, it may happen that the aforesaid property (with respect to a real constant c) becomes more and more distinguished as n gradually increases and then the question of existence of such a real constant c will be answered by a notion of probability convergence (that is, convergence in probability) of the sequence $\{X_n\}_{n\in\mathbb{N}}$.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, where each X_n is defined on the same event space S, with respect to a given class of subsets (of S) as the class Δ of events and a given probability function $P : \Delta \to \mathbb{R}$. The sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically probability convergent (or statistically convergent in probability) to a random variable X (where $X : S \to \mathbb{R}$) if, for any $\epsilon > 0$ and $\delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}|k : k \leqq n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta| = 0$$

or, equivalently,

$$\lim_{n\to\infty} \frac{1}{n}|k : k \leqq n \quad \text{and} \quad 1 - P(|X_n - X| \leqq \epsilon) \geqq \delta| = 0.$$

In this case, we write

$$\text{stat}_{\text{P}} \lim_{n\to\infty} P(|X_n - X| \geqq \epsilon) = 0 \qquad \text{or} \qquad \text{stat}_{\text{P}} \lim_{n\to\infty} P(|X_n - X| \leqq \epsilon) = 1.$$

We now show by means of the following example that every statistically convergent sequence is statistically probability convergent, but the converse is not necessarily true.

Example 1.1. Consider a probability density function of X_n of the following form:

$$f_n(x) = \begin{cases} \dfrac{1}{2} & (0 < x < 1;\ n = m^2\ \forall\ m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$f_n(x) = \begin{cases} \dfrac{nx^{n-1}}{3^n} & (0 < x < 3;\ n \neq m^2\ \forall\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P(|X_n - 3| \geqq \epsilon) = \begin{cases} 1 & (n = m^2\ \forall\ m \in \mathbb{N}) \\ 1 - P(|X_n - 3| < \epsilon) & \\ \quad = \left(1 - \dfrac{\epsilon}{3}\right)^n & (n \neq m^2\ \forall\ m \in \mathbb{N}). \end{cases}$$

This implies that

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad P(|X_n - 3| \geqq \epsilon) \geqq \delta\}|$$
$$\leqq \lim_{n\to\infty} \frac{1}{n} |\{1^2, 2^2, 3^2, \cdots, n^2\}| = 0.$$

Clearly, it is neither statistically convergent nor ordinarily convergent, while it is statistically probability convergent to 3.

Quite recently, Srivastava *et al.* [**23**] first introduced and studied the fundamental idea of deferred Cesàro statistical probability convergence of a sequence of random variables as follows.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, where each (X_n) is defined on the same event space S with respect to a given class of subsets (of S) as the class Δ of events and a given probability density function $P : \Delta \to \mathbb{R}$. A given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro statistical probability convergent to a random variable X (where $X : S \to \mathbb{R}$) if, for every $\delta > 0$ and $\epsilon > 0$, the set

$$\{k : a_n < k \leqq b_n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta\}$$

has natural density zero, that is, if

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leqq b_n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta\}| = 0,$$

In this case, we write

$$\text{stat}_{\text{DCP}} \lim_{n\to\infty} P(|X_n - X| \geqq \epsilon) = 0.$$

Several researchers have worked on extending or generalizing the Korovkin-type approximation theorems in many different ways and under several different settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces, and so on. This theory is very useful in Real Analysis, Functional Analysis, Harmonic Analysis, Measure Theory, Probability Theory and Summability Theory. In the year 2018, Jena *et al.* [**11**] introduced statistically-deferred Cesàro summability for single sequences in Korovkin-type approximation theorems. Recently, Paikray *et al.* [**18**] established a Korovkin-type theorem based upon the (p, q)-integers for statistically-deferred Cesàro summability mean. Subsequently, Dutta *et al.* [**7**] demonstrated the Korovkin theorem on $\mathcal{C}[0, \infty)$ by using the test functions 1, e^{-x} and e^{-2x} via the deferred Cesàro mean. In another recent work, Srivastava *et al.* [**24**] made use of the notion of the deferred weighted statistical convergence and accordingly proved a Korovkin-type approximation theorem. In this paper, we generalize the result of Srivastava *et al.* [**24**] via the notion of the deferred weighted statistical probability convergence for the same set of test functions 1, $\cos x$ and $\sin x$.

Motivated essentially by the above-mentioned investigations and results, we first introduce here the notion of the deferred weighted statistical convergence of a real sequence and the concept of the deferred weighted statistical probability convergence of sequences of random variables. We also establish an inclusion relation between them. Moreover, based upon our proposed methods, we prove a new Korovkin-type approximation theorem with periodic test functions for a sequence of random variables over a Banach space and demonstrate that our result is a non-trivial extension of some well-established ordinary and statistical versions of several known results. Finally, we estimate the rate of the deferred weighted statistical probability convergence and, as a consequence, establish a new result.

2. PRELIMINARIES AND DEFINITIONS

Let (a_n) and (b_n) be sequences of non-negative integers such that (i) $a_n < b_n$ and (ii) $\lim_{n\to\infty} b_n = \infty$. Suppose that (p_n) and (q_n) are sequences of non-negative real numbers such that

$$\mathcal{P}_n = \sum_{m=a_n+1}^{b_n} p_m \quad \text{and} \quad \mathcal{Q}_n = \sum_{m=a_n+1}^{b_n} q_m.$$

The convolution of the above sequences can be introduced as follows:

$$\mathcal{R}_n = \sum_{v=a_n+1}^{b_n} p_v q_v.$$

We now recall the deferred weighted mean $D_a^b(\overline{N}, p, q)$ as follows (see [**24**]):

$$t_n = \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_m q_m x_m,$$

and that a sequence (x_n) is summable to L by the deferred weighted mean method defined in terms of the sequences (p_n) and (q_n) (or, briefly, summable $D_a^b(\overline{N},p,q)$) if

$$\lim_{n\to\infty} t_n = L.$$

Moreover, a sequence (x_n) is statistically summable to L by the deferred weighted mean (or, briefly, summable $D_a^b(\overline{N},p,q)$) if

$$\text{stat}\lim_{n\to\infty} t_n = L.$$

It is well known that, $D_a^b(\overline{N},p,q)$ is regular under the above-mentioned conditions (i) and (ii) (see, for details, Agnew [**1**]).

We further recall the following definition.

Definition 2.2. (see [**24**]) Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) and (q_n) be the sequences of non-negative real numbers. A real sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be deferred weighted statistically convergent to L if, for each $\epsilon > 0$, the set given by

$$\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m |x_m - L| \geqq \epsilon\}$$

has its deferred weighted density equal to zero, that is, if

$$\lim_{n\to\infty} \frac{1}{\mathcal{R}_n} |\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m |x_m - L| \geqq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat}_{\text{D}\bar{\text{N}}} \lim x_n = L.$$

Let us now introduce the following definition which will be needed in connection with our proposed investigation here.

Definition 2.3. Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) and (q_n) be the sequences of non-negative real numbers. Suppose also that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables, where each (X_n) is defined on the same event space S with respect to a given class Δ of subsets of the event space S and a given probability density function $P : \Delta \to \mathbb{R}$. A given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred weighted statistically probability convergentto a random variable X (where $X : S \to \mathbb{R}$) if, for every $\delta > 0$ and $\epsilon > 0$, the set

$$\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_n - X| \geqq \epsilon) \geqq \delta\}$$

has its natural density equal to zero, that is, if

$$\lim_{n\to\infty} \frac{1}{\mathcal{R}_n} |\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_n - X| \geqq \epsilon) \geqq \delta\}| = 0$$

or, equivalently,

$$\lim_{n\to\infty}\frac{1}{\mathcal{R}_n}|\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad 1 - p_m q_m P(|X_n - X| \leqq \epsilon) \geqq \delta\}| = 0.$$

In this case, we write

$$\text{stat}_{\mathrm{D\bar{N}P}} \lim_{n\to\infty} p_m q_m P(|X_n - X| \geqq \epsilon) = 0$$

or

$$\text{stat}_{\mathrm{D\bar{N}P}} \lim_{n\to\infty} p_m q_m P(|X_n - X| \leqq \epsilon) = 1.$$

We next present a theorem in order to demonstrate that every deferred weighted statistically convergent sequence is deferred weighted statistically probability convergent. However, the converse is not true.

Theorem 2.4. *Let the sequence* $\{x_n\}$ *of constants be such that* $\text{stat}_{\mathrm{D\bar{N}}}\ x_n \to x$. *Then, assuming it to be a random variable having a one-point distribution at that point, the sequence* $\{X_n\}$ *of random variables is such that*

$$\text{stat}_{\mathrm{D\bar{N}P}}\ X_n \to X.$$

Proof. Let $\epsilon > 0$ be any arbitrarily small positive real number. Then, by Definition 2.2, we obtain

$$\lim_{n\to\infty}\frac{1}{\mathcal{R}_n}|\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m |x_m - L| \geqq \epsilon\}| = 0.$$

We now let $\delta > 0$, so that the set

$$\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_n - X| \geqq \epsilon) \geqq \delta\} \subseteq \mathcal{K},$$

where

$$\mathcal{K} = \{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m |x_m - L| \geqq \epsilon\}.$$

Thus, by Definition 2.3, we may write

$$\text{stat}_{\mathrm{D\bar{N}P}} X_n \to X.$$

□

We now present below an example to show that a sequence of random variables is deferred weighted statistically probability convergent, whenever it is not deferred weighted statistically convergent.

Example 2.5. Let $a_n = 2n$ and $b_n = 4n$. Also let $p_n = 2n$ and $q_n = 1$. Suppose that the probability density function of X_n is given by

$$f_n(x) = \begin{cases} 1 & (0 < x < 1;\ n = m^2\ \forall\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$f_n(x) = \begin{cases} \dfrac{(n+1)x^n}{3^{n+1}} & (0 < x < 3;\ n \neq m^2\ \forall\ m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P(|X_n - 3| \geqq \epsilon) = \begin{cases} 1 & \text{when} \quad n = m^2 \\ \\ 1 - P(|X_n - 3| < \epsilon) = \left(1 - \dfrac{\epsilon}{3}\right)^{n+1} & \text{when} \quad n \neq m^2. \end{cases}$$

Consequently, we have

$$\lim_{n\to\infty} \frac{1}{2n} |\{m : m \leqq 2n \quad \text{and} \quad 2nP(|X_n - 3| \geqq \epsilon) \geqq \delta\}| = 0,$$

Clearly, we observe that (X_n) is neither convergent nor deferred weighted statistically convergent; however, it is deferred weighted statistically probability convergent to 3.

3. A NEW KOROVKIN-TYPE THEOREM

In this section, we extend the result of Srivastava *et al.* [**24**] by using the notion of the deferred weighted statistical probability convergence of sequences of random variables over a Banach space.

Let $\mathcal{C}(\mathbb{R})$ be the space of all real-valued continuous probability density functions defined on $\mathbb{R}$ under the norm $\|.\|_\infty$. Also let $\mathcal{C}(\mathbb{R})$ be a Banach space. Then, for $f \in \mathcal{C}(\mathbb{R})$, the norm of f denoted by $\|f\|$ is given by

$$\|f\|_\infty = \sup_{x\in\mathbb{R}}\{|f(x)|\}.$$

We also let $C_{2\pi}(\mathbb{R})$ be the space of all 2π-periodic real-valued continuous probability functions f defined on $\mathbb{R}$ and suppose that $\mathfrak{L} : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be a linear operator. We say that the operator $\mathfrak{L}$ is a sequence of random variables of positive linear operator provided that

$$\mathfrak{L}(f; x) \geqq 0 \quad \text{whenever} \quad f \geqq 0.$$

It is also known that $C_{2\pi}(\mathbb{R})$ is a Banach space. For $f \in C_{2\pi}(\mathbb{R})$, the norm of the function f, denoted by $\|f\|$, is given by

$$\|f\|_{2\pi} = \sup_{x\in\mathbb{R}} |f(x)|.$$

We now state and prove the following theorem by using the notion of the deferred weighted statistical probability convergence.

Theorem 3.6. *Let*

$$\mathfrak{L}_m : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$$

be a sequence of random variables of positive linear operators. Then, for all $f \in C_{2\pi}(\mathbb{R})$,

$$\text{stat}_{\text{D}\bar{\text{N}}\text{P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} = 0 \tag{3.1}$$

if and only if

$$\text{stat}_{\text{D}\bar{\text{N}}\text{P}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_{2\pi} = 0, \tag{3.2}$$

$$\text{stat}_{\text{D}\bar{\text{N}}\text{P}} \lim_{m\to\infty} \|\mathfrak{L}_m(\cos x;x) - \cos x\|_{2\pi} = 0 \tag{3.3}$$

and

$$\text{stat}_{\text{D}\bar{\text{N}}\text{P}} \lim_{m\to\infty} \|\mathfrak{L}_m(\sin x;x) - \sin x\|_{2\pi} = 0. \tag{3.4}$$

Proof. Since each of the following functions:

$$f_0(x) = 1, \quad f_1(x) = \cos x \quad \text{and} \quad f_2(x) = \sin x$$

belongs to $C_{2\pi}(\mathbb{R})$ and is continuous, the assertion given by (3.1) implies that that the conditions (3.2) to (3.4) are obvious. In order to complete the proof of Theorem 3.6, we first assume that the conditions (3.2) to (3.4) hold true. If we let $f \in C_{2\pi}(\mathbb{R})$ and also let I be a closed sub-interval of length 2π of $\mathbb{R}$. Then there exists a constant $\mathcal{H} > 0$ such that

$$|f(x)| \leqq \mathcal{H} \qquad (\forall\ x \in \mathbb{R}).$$

We thus find that

$$|f(s) - f(x)| \leqq 2\mathcal{H} \quad (s, x \in I). \tag{3.5}$$

Clearly, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \epsilon \tag{3.6}$$

whenever

$$|e^{-s} - e^{-x}| < \delta \qquad (\forall\ s, x \in I).$$

Since f is bounded, it follows that

$$|f(s) - f(x)| < 2\|f\|_{2\pi} \quad (\forall\ s, x \in I). \tag{3.7}$$

From the equation (3.6) and (3.7), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}\varphi(s) \quad (s \in (x-\delta, 2\pi + x - \delta]), \tag{3.8}$$

where

$$\varphi(s) = \sin^2\left(\frac{s-x}{2}\right).$$

Since the function $f \in C_{2\pi}(\mathbb{R})$ is 2π-periodic, the inequality (3.8) holds true for $s \in \mathbb{R}$.

Now, since the operator $\mathfrak{L}_m(1;x)$ is monotone and linear, by applying this operator to the inequality in (3.8), we find that

$$\begin{aligned}
&|\mathfrak{L}_m(f;x) - f(x)| \\
&\quad \leqq (\epsilon + |f(x)|)|\mathfrak{L}_m(1;x) - 1| + \epsilon + \frac{\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}\{|\mathfrak{L}_m(1;x) - 1| \\
&\qquad + |\cos x||\mathfrak{L}_m(\cos s;x) - \cos x| + |\sin x||\mathfrak{L}_m(\sin s;x) - \sin x|\}. \\
&\quad \leqq \epsilon + \left(\epsilon + |f(x)| + \frac{\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}\right)\{|\mathfrak{L}_m(1;x) - 1| \\
&\qquad + |\mathfrak{L}_m(\cos s;x) - \cos x| + |\mathfrak{L}_m(\sin s;x) - \sin x|\}. \qquad (3.9)
\end{aligned}$$

Next, by taking $\sup_{x\in I}$ in both sides of (3.9), we get

$$\begin{aligned}
&\|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} \\
&\quad \leqq \epsilon + \mathcal{B}\{\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} + \|\mathfrak{L}_m(\cos t;x) - \cos x\|_{2\pi} \\
&\qquad + \|\mathfrak{L}_m(\sin t;x) - \sin x\|_{2\pi}\},
\end{aligned}$$

where

$$\mathcal{B} = \epsilon + |f(x)|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}.$$

This implies that

$$\begin{aligned}
p_m q_m \|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} \leqq \epsilon + \mathcal{B}\{&p_m q_m \|\mathfrak{L}_m(1;x) - 1\|_{2\pi} \\
&+ p_m q_m \|\mathfrak{L}_m(\cos s;x) - \cos x\|_{2\pi} \\
&+ p_m q_m \|\mathfrak{L}_m(\sin s;x) - \sin x\|_{2\pi}\}. \qquad (3.10)
\end{aligned}$$

Now, for a given $r > 0$, there exists $\delta, \epsilon > 0$ such that $0 < \epsilon < r$. Then, by setting

$$\Theta_m(x,r) = \{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P\left(|\mathfrak{L}_m(f,x) - f(x)| \geqq r\right)\} \geqq \delta,$$

we find for $i = 0, 1, 2$ that

$$\Theta_{i,m}(x;r) = \left\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P\left(|\mathfrak{L}_m(f_i;x) - f_i(x)| \geqq \frac{r-\epsilon}{3\mathcal{B}}\right)\right\} \geqq \delta,$$

so that

$$\Theta_m(x;r) \leqq \sum_{i=0}^{2} \Theta_{i,m}(x;r).$$

Thus, clearly, we have

$$\frac{\|\Theta_m(x;r)\|_{2\pi}}{\mathcal{R}_n} \leqq \sum_{i=0}^{2} \frac{\|\Theta_{i,m}(x;r)\|_{2\pi}}{\mathcal{R}_n}. \tag{3.11}$$

Next, by using the above assumption about the implications in (3.2) to (3.4) and by Definition 2.3, the right-hand side of (3.11) is seen to tend to zero as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\Theta_m(x;r)\|_{2\pi}}{\mathcal{R}_n} = 0 \ (\delta, r > 0).$$

Therefore, the implication (3.1) holds true. This completes the proof of Theorem 3.6. □

Now, by using the Definition 2.2, we present the following corollary as a consequence of Theorem 3.6.

Corollary 3.7. *Let $L_m : \mathcal{C}[1,\infty) \to \mathcal{C}[1,\infty)$ be a sequence of random variables of positive linear operators. Also let $f \in C_{2\pi}(\mathbb{R})$. Then*

$$\text{stat}_{\text{D}\bar{\text{N}}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} = 0$$

if and only if

$$\text{stat}_{\text{D}\bar{\text{N}}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_{2\pi} = 0,$$
$$\text{stat}_{\text{D}\bar{\text{N}}} \lim_{m\to\infty} \|\mathfrak{L}_m(\cos x;x) - \cos x\|_{2\pi} = 0$$

and

$$\text{stat}_{\text{D}\bar{\text{N}}} \lim_{m\to\infty} \|\mathfrak{L}_m(\sin x;x) - \sin x\|_{2\pi} = 0.$$

We present below an illustrative example (see Example 3.8 below) for the sequence of random variables of positive linear operators that does not satisfy the conditions of the Korovkin approximation theorems proved earlier by Srivastava *et al.* [**24**] and Paikray *et al.* [**17**], but which satisfies the conditions of our Theorem 3.6. Consequently, our Theorem 3.6 is stronger than the results established earlier by both Srivastava *et al.* [**24**] and Paikray *et al.* [**17**].

In order to first recall the *Fejér convolution operators*, we let $f \in C_{2\pi}(\mathbb{R})$ and also let the Fourier series of f at $t = x$ be given by

$$f_m(x) = \frac{a_0}{2} + \sum_{m=0}^{\infty} (a_m \cos mx + b_m \sin mx).$$

We denote the mth partial sum of the Fourier series of $f_m(x)$ by

$$S_m(f)(x) = \frac{a_0}{2} + \sum_{m=0}^{n} (a_m \cos mx + b_m \sin mx) \quad (\forall\ m \in \mathbb{N})$$

and we write the Cesàro mean of $f_m(x)$ as follows:

$$\mathfrak{F}_n(f;x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f;x).$$

Furthermore, by simple calculation, we obtain

$$\begin{aligned}
\mathfrak{F}_n(f;x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{1}{n+1}\right) \left(\sum_{m=0}^{n} \frac{\sin\left(\frac{(2m+1)(x-t)}{2}\right)}{\sin\left(\frac{x-t}{2}\right)} \right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{1}{n+1}\right) \left(\frac{\sin^2\left(\frac{(n+1)(x-t)}{2}\right)}{\sin^2\left(\frac{x-t}{2}\right)} \right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\phi_n(x-t)\, dt \qquad (n \in \mathbb{N}_0),
\end{aligned}$$

where the sequence $\{\phi_n(x)\}_{n\in\mathbb{N}_0}$ is given by

$$\phi_n(x) = \begin{cases} \left(\frac{1}{n+1}\right) \left(\frac{\sin^2\left(\frac{(n+1)x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)} \right) & (x \text{ is not a multiple of } 2\pi) \\[2ex] n+1 & (x \text{ is a multiple of } 2\pi). \end{cases}$$

The sequence $\{\phi_n(x)\}_{n\in\mathbb{N}_0}$ is a positive kernel which is known as the *Fejér kernel* and the corresponding operators $\mathfrak{F}_n(f;x)$ are called the *Fejér convolution operators.*

We now recall the operator

$$x(1+xD) \qquad \left(D = \frac{d}{dx}\right),$$

which was used by Al-Salam [**2**] and, more recently, by Viskov and Srivastava [**30**] (see also [**21**] and the monograph by Srivastava and Manocha [**28**] for various general families of operators of this kind). Here, in our Example 3.8 below, we use this operator in conjunction with the Fejér convolution operators.

Example 3.8. Let $\mathfrak{L}_m : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be defined by

$$\mathfrak{L}_m(f;x) = [1+X_m]x(1+xD)\mathfrak{F}_m(f) \quad \big(f \in C_{2\pi}(\mathbb{R})\big), \tag{3.12}$$

where (X_m) is a sequence of random variables defined as in Example 2.5. It is easily observed that

$$\mathfrak{L}_m(1;x) = [1+X_m]x(1+xD)1 = [1+X_m]x,$$

$$\begin{aligned}\mathfrak{L}_m(\cos s;x) &= [1+X_m]x(1+xD)\left(\frac{m-1}{m}\right)\cos x\\ &= [1+X_m]\left(\frac{m-1}{m}\right)(x\cos x - x^2\sin x)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}_m(\sin s;x) &= [1+X_m]x(1+xD)\left(\frac{m-1}{m}\right)\sin x\\ &= [1+X_m]\left(\frac{m-1}{m}\right)(x\sin x + x^2\cos x),\end{aligned}$$

so that we have

$$\begin{aligned}&\mathrm{stat}_{\mathrm{D\bar{N}P}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x)-1\|_{2\pi} = 0,\\ &\mathrm{stat}_{\mathrm{D\bar{N}P}} \lim_{m\to\infty} \|\mathfrak{L}_m(\cos x;x)-\cos x\|_{2\pi} = 0\end{aligned}$$

and

$$\mathrm{stat}_{\mathrm{D\bar{N}P}} \lim_{m\to\infty} \|\mathfrak{L}_m(\sin x;x)-\sin x\|_{2\pi} = 0,$$

that is, the sequence $\mathfrak{L}_m(f;x)$ satisfies the conditions (3.2) to (3.4). Therefore, by Theorem 3.6, we have

$$\mathrm{stat}_{\mathrm{D\bar{N}P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x)-f\|_{2\pi} = 0.$$

Hence, it is deferred weighted statistically probability convergent. However, since (X_m) is neither weighted statistically convergent nor deferred weighted statistically convergent, so we conclude that the earlier works in [24] and [17] are not valid for the operators defined by (3.12), whereas our Theorem 3.6 still works for the operators defined by (3.12).

4. RATE OF STATISTICAL PROBABILITY CONVERGENCE

In this section, we study the rates of the deferred weighted statistical probability convergence of a sequence of random variables of positive linear operators $\mathfrak{L}(f;x)$ defined on $C_{2\pi}(\mathbb{R})$ with the help of the modulus of continuity.

We begin by presenting the following definition.

Definition 4.9. Let (p_n) and (q_n) be sequences of non-negative real numbers and let (u_n) be a positive non-increasing sequence. A given sequence (X_m) of random variables is deferred weighted statistically probability convergent to a random variable X with the rate $o(u_n)$ if, for every $\epsilon > 0$ and $\delta > 0$,

$$\lim_{n\to\infty} \frac{1}{\mathcal{R}_n u_n} \left|\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_m - X| \geqq \epsilon) \geqq \delta\}\right| = 0.$$

In this case, we may write

$$X_m - L = \text{stat}_{\mathrm{D\bar{N}P}}\ o(u_n).$$

We now prove the following basic lemma.

Lemma 4.10. *Let (u_n) and (v_n) be two positive non-increasing sequences. Let (X_m) and (Y_m) be two sequences of random variables such that*

$$X_m - X_1 = \text{stat}_{\mathrm{D\bar{N}P}}\ o(u_n)$$

and

$$Y_m - X_2 = \text{stat}_{\mathrm{D\bar{N}P}}\ o(v_n),$$

respectively. Then the following conditions hold true:

(i) $(X_m + Y_m) - (X_1 + X_2) = \text{stat}_{\mathrm{D\bar{N}P}}\ o(w_n)$;

(ii) $(X_m - X_1)(Y_m - X_2) = \text{stat}_{\mathrm{D\bar{N}P}}\ o(u_n v_n)$;

(iii) $\lambda(X_m - X_1) = \text{stat}_{\mathrm{D\bar{N}P}}\ o(u_n)$ *(for any scalar λ)*;

(iv) $\sqrt{|X_m - X_1|} = \text{stat}_{\mathrm{D\bar{N}P}}\ o(u_n)$,

where

$$w_n = \max\{u_n, v_n\}.$$

Proof. In order to prove the implication (i) of Lemma 4.10, for $\epsilon > 0$ and $x \in \mathbb{R}$, we define the following sets:

$$\mathfrak{A}_n(x;\epsilon) = \left|\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_m + Y_m - X_1 + X_2| \geqq \epsilon) \geqq \delta\}\right|,$$

$$\mathfrak{A}_{0,n}(x;\epsilon) = \left|\left\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_m - X_1| \geqq \epsilon) \geqq \frac{\delta}{2}\right\}\right|$$

and

$$\mathfrak{A}_{1,n}(x;\epsilon) = \left|\left\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|Y_m - X_2| \geqq \epsilon) \geqq \frac{\delta}{2}\right\}\right|.$$

Clearly, we have

$$\mathfrak{A}_n(x;\epsilon) \subseteq \mathfrak{A}_{0,n}(x;\epsilon) \cup \mathfrak{A}_{1,n}(x;\epsilon).$$

Moreover, since

$$w_n = \max\{u_n, v_n\},$$

by the condition (3.1) of Theorem 3.6, we obtain

$$\frac{\|\mathfrak{A}_m(x;\epsilon)\|_{2\pi}}{w_n \mathcal{R}_n} \leqq \frac{\|\mathfrak{A}_{0,n}(x;\epsilon)\|_{2\pi}}{u_n \mathcal{R}_n} + \frac{\|\mathfrak{A}_{1,n}(x;\epsilon)\|_{2\pi}}{v_n \mathcal{R}_n}. \tag{4.13}$$

Now, by the conditions (3.2) to (3.4) of Theorem 3.6, we obtain

$$\frac{\|\mathfrak{A}_n(x;\epsilon)\|_{2\pi}}{w_n \mathcal{R}_n} = 0, \tag{4.14}$$

which establishes the implication (i) of Lemma 4.10. Since the proofs of the other implications (ii) to (iv) of Lemma 4.10 are similar, we choose to omit the analogous details involved. □

We now recall that the modulus of continuity of a function $f \in C_{2\pi}(\mathbb{R})$ is defined by

$$\omega(f,\delta) = \sup_{|y-x| \leqq \delta : x,y \in X} |f(y) - f(x)| \quad (\delta > 0),$$

which implies that

$$|f(y) - f(x)| \leqq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{4.15}$$

We state and prove a result in the form of the following theorem.

Theorem 4.11. Let $C_{2\pi}(\mathbb{R}) \subset \mathbb{R}$. Also let

$$\mathfrak{L}_m : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$$

be a sequence of random variables of positive linear operators. Assume that the following conditions hold true:

(i) $\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} = \text{stat}_{\text{D}\tilde{\text{N}}\text{P}}\ o(u_n)$

and

(ii) $\omega(f, \lambda_m) = \text{stat}_{\text{D}\tilde{\text{N}}\text{P}}\ o(v_n)$,

where

$$\lambda_m = \sqrt{\mathfrak{L}_m(\varphi^2;x)} \quad \textit{and} \quad \varphi(s,x) = \sin^2\left(\frac{s-x}{2}\right).$$

Then, for all $f \in C_{2\pi}(\mathbb{R})$, the following statement holds true:

$$\|\mathfrak{L}_m(f;x) - f\|_{2\pi} = \text{stat}_{\text{D}\tilde{\text{N}}\text{P}}\ o(w_n) \qquad \big(w_n := \max\{u_n, v_n\}\big). \tag{4.16}$$

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $x \in I$. Then, by using (4.15), we have

$$\begin{aligned}
|\mathfrak{L}_m(f;x) - f(x)| &\leqq \mathfrak{L}_m(|f(t) - f(x)|;x) + |f(x)||\mathfrak{L}_m(1;x) - 1| \\
&\leqq \mathfrak{L}_m\left(\frac{|x-t|}{\delta} + 1; x\right)\omega(f,\delta) + |f(x)||\mathfrak{L}_m(1;x) - 1| \\
&\leqq \mathfrak{L}_m\left(1 + \frac{\pi^2}{\delta^2}\sin^2\left(\frac{t-x}{2}\right); x\right)\omega(f,\delta) \\
&\quad + |f(x)||\mathfrak{L}_m(1;x) - 1| \\
&\leqq \left(\mathfrak{L}_m(1;x) + \frac{\pi^2}{\delta^2}\mathfrak{L}_m(\varphi(t);x)\right)\omega(f,\delta) \\
&\quad + |f(x)||\mathfrak{L}_m(1;x) - 1|.
\end{aligned}$$

Thus, by putting

$$\delta = \lambda_m = \sqrt{\mathfrak{L}_m(\varphi^2;x)},$$

we get

$$\begin{aligned}
\|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} &\leqq (1+\pi^2)\omega(f,\lambda_m) + \omega(f,\lambda_m)\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} \\
&\quad + \|f(x)\|_{2\pi}\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} \\
&\leqq \mu\{\omega(f,\lambda_m) + \omega(f,\lambda_m)\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} \\
&\quad + \|\mathfrak{L}_m(1;x) - 1\|_{2\pi}\},
\end{aligned}$$

where

$$\mu = \{\|f\|_{C_{2\pi}(\mathbb{R})}, 1 + \pi^2\}.$$

Therefore, we have

$$\begin{aligned}
p_m q_m\|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} \leqq \mu\{&\omega(f,\lambda_m)p_m q_m \\
&+ \omega(f,\lambda_m)p_m q_m\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} \\
&+ p_m q_m\|\mathfrak{L}_m(1;x) - 1\|_{2\pi}\}.
\end{aligned}$$

Now, by using the conditions (i) and (ii) of Theorem 4.11 in conjunction with Lemma 4.10, we arrive at the assertion (4.16) of Theorem 4.11. This completes the proof of Theorem 4.11. □

5. CONCLUDING REMARKS AND OBSERVATIONS

In this concluding section of our investigation, we present several further remarks and observations concerning the various results which we have proved here.

Remark 5.12. Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of random variables as given in Example $refexp2$. Then, since

$$\text{stat}_{\text{D}\bar{\text{N}}\text{P}} \lim_{m\to\infty} X_m = 3 \text{ on } [0, 2\pi],$$

we have

$$\operatorname{stat}_{\mathrm{D\tilde{N}P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f_i;x) - f_i(x)\|_{2\pi} = 0 \quad (i = 0,1,2). \tag{5.17}$$

Thus, by applying Theorem 3.6, we can write

$$\operatorname{stat}_{\mathrm{D\tilde{N}P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_{2\pi} = 0, \qquad (i = 0,1,2), \tag{5.18}$$

where

$$f_0(x) = 1, \quad f_1(x) = \cos x \quad \text{and} \quad f_2(x) = \sin x.$$

However, since (X_m) is neither statistically convergent nor convergent uniformly in the ordinary sense, the classical and the statistical Korovkin-type theorems do not work here for the operators defined by (3.12). Hence, clearly, this application indicates that our Theorem 3.6 is a non-trivial generalization of the classical as well as the statistical Korovkin-type theorems (see [13] and [8]).

Remark 5.13. Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of random variables as given in Example 2.5. Then, since

$$\operatorname{stat}_{\mathrm{D\tilde{N}P}} \lim_{m\to\infty} X_m = 3 \text{ on } [0,2\pi],$$

so (5.17) holds true. Now, by applying (5.17) and Theorem 3.6, the condition (5.18) holds true. However, since the sequence (X_m) of random variables is not deferred weighted statistically convergent, the result of Srivastava *et al.* (see [24]) does not work for our operator defined in (3.12). Thus, naturally, our Theorem 3.6 is also a non-trivial extension of the result of Srivastava *et al.* [24] (see also [17] and [19]). Based upon the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (3.12) and, therefore, our result is stronger than the classical and statistical versions of the Korovkin-type approximation theorems (see [17], [19] and [24]) which were established earlier.

Remark 5.14. Let us suppose that we replace the conditions (i) and (ii) in Theorem 4.11 by the following condition:

$$|\mathfrak{L}_m(f_i;x) - f_i(x)|_{2\pi} = stat_{\mathrm{D\tilde{N}P}}\ o(u_{n_i}) \quad (i = 0,1,2). \tag{5.19}$$

Then, since

$$\begin{aligned}\mathfrak{L}_m(\varphi^2;x) = |\mathfrak{L}_m(1;x) - 1| + |\cos x||\mathfrak{L}_m(\cos t;x) - \cos x| \\ + |\sin x||\mathfrak{L}_m(\sin x;x) - \sin x|,\end{aligned}$$

we can write

$$\mathfrak{L}_m(\varphi^2;x) \leqq \mathcal{M} \sum_{i=0}^{2} |\mathfrak{L}_m(f_i;x) - f_i(x)|_{2\pi}, \tag{5.20}$$

where

$$\mathcal{M} = 1 + \|f_1\|_{2\pi} + \|f_2\|_{2\pi}.$$

It now follows from (5.19), (5.20) and Lemma 4.10 that

$$\lambda_m = \sqrt{L_m(\varphi^2)} = \mathrm{stat}_{\mathrm{D\bar{N}P}}\ o(d_n), \tag{5.21}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

This implies that

$$\omega(f, \delta) = \mathrm{stat}_{\mathrm{D\bar{N}P}}\ o(d_n).$$

Now, by using (5.19) in Theorem 4.11, we immediately see for $f \in C_{2\pi}(\mathbb{R})$ that

$$\mathfrak{L}_m(f; x) - f(x) = \mathrm{stat}_{\mathrm{D\bar{N}P}}\ o(d_n).$$

Therefore, if we use the condition (5.19) in Theorem 4.11 instead of the conditions (i) and (ii), then we obtain the rates of the deferred weighted statistical probability convergence of the sequence of random variables of positive linear operators in Theorem 3.6.

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