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ERROR-FUNCTIONS IN DOUBLE-SIDED TAYLOR'S APPROXIMATIONS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In this paper we introduce the error-functions for one-sided and double-sided Taylor's approximations of real analytic functions. We illustrate the application of error-functions in the process of generalization of one trigonometric inequality.

1. INTRODUCTION

In papers [**8**], [**20**], [**24**], [**25**] [**26**], [**27**] and [**28**] double-sided Taylor's approximations are used to prove certain inequalities. In papers [**13**], [**15**], [**16**], [**17**], [**21**], [**22**], [**23**], [**30**], [**31**] i [**32**] similar techniques are used in the proofs of some mixed-trigonometric polynomial inequalities. Results of these papers are further organized and made more precise in [**33**], [**34**], where we established the order among the functions appearing within these inequalities.

Consider a real function $f:(a,b)\longrightarrow\mathbb{R}$ such that there exist finite limits $f^{(k)}(a+)=\lim\limits_{x\to a+}f^{(k)}(x)$, for $k=0,1,\ldots,n$. Let us denote by $T_n^{f,\,a+}(x)$ Taylor's polynomial of degree n, where $n\in\mathbb{N}_0$, for the function $f(x)$ in the right neighborhood of a:

$$T_n^{f,\,a+}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a+)}{k!}(x-a)^k.$$

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We will call $T_n^{f,\,a+}(x)$ the *first* TAYLOR*'s approximation in the right neighborhood of* a.

Similarly, the *first* TAYLOR*'s approximation in the left neighborhood of* b is defined by:

$$T_n^{f,\,b-}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k,$$

where $f^{(k)}(b-) = \lim\limits_{x \to b-} f^{(k)}(x)$, for $k = 0, 1, \ldots, n$.

Also, for $n \in \mathbb{N}_0$, the following functions:

$$R_n^{f,\,a+}(x) = f(x) - T_n^{f,\,a+}(x) \quad \text{and} \quad R_n^{f,\,b-}(x) = f(x) - T_n^{f,\,b-}(x)$$

are called the *remainder of the first* TAYLOR*'s approximation in the right neighborhood of* a, and the *remainder of the first* TAYLOR*'s approximation in the left neighborhood of* b, respectively [**33**].

Polynomials:

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) = \begin{cases} T_{n-1}^{f,\,a+}(x) + \dfrac{1}{(b-a)^n} R_{n-1}^{f,\,a+}(b-)(x-a)^n & : \; n \geq 1 \\ f(b-) & : \; n = 0, \end{cases}$$

and

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) = \begin{cases} T_{n-1}^{f,\,b-}(x) + \dfrac{1}{(a-b)^n} R_{n-1}^{f,\,b-}(a+)(x-b)^n & : \; n \geq 1 \\ f(a+) & : \; n = 0, \end{cases}$$

are called the *second* TAYLOR*'s approximation in the right neighborhood of* a, and the *second* TAYLOR*'s approximation in the left neighborhood of* b, respectively, $n \in \mathbb{N}_0$ [**33**].

Note that the second TAYLOR's approximation was first considered in the proof of the TAYLOR's formula with the LAGRANGE remainder back in 1851, in [**1**] by H. COX, see also [**19**].

For $n \in \mathbb{N}_0$, the following functions:

$$\mathbb{R}_n^{f;\,a+,b-}(x) = f(x) - \mathbb{T}_n^{f;\,a+,b-}(x) \quad \text{and} \quad \mathbb{R}_n^{f;\,b-,a+}(x) = f(x) - \mathbb{T}_n^{f;\,b-,a+}(x)$$

are called the *remainder of the second* TAYLOR*'s approximation in the right neighborhood of* a, and the *remainder of the second* TAYLOR*'s approximation in the left neighborhood of* b, recpectively [**33**].

Theorem 2 in [**7**] provides an important result regarding TAYLOR's approximations. We cite it below:

Theorem 1. Suppose that $f(x)$ is a real function on (a,b), and that n is a positive integer such that $f^{(k)}(a+), f^{(k)}(b-)$, for $k \in \{0,1,2,\ldots,n\}$, exist.

(i) Supposing that $(-1)^{(n)} f^{(n)}(x)$ is increasing on (a,b), then for all $x \in (a,b)$ the following inequality holds:

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) < f(x) < T_n^{f,\,b-}(x). \tag{1}$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on (a,b), then the reversed inequality of (1) holds.

(ii) Supposing that $f^{(n)}(x)$ is increasing on (a,b), then for all $x \in (a,b)$ the following inequality also holds:

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) > f(x) > T_n^{f,\,a+}(x). \tag{2}$$

Furthermore, if $f^{(n)}(x)$ is decreasing on (a,b), then the reversed inequality of (2) holds.

Let us name this theorem the *Theorem on double-sided* TAYLOR*'s approximations*. This theorem was proved in [**7**]. Some variants of this theorem were considered in papers [**4**], [**5**] and [**6**]. Note that in papers [**25**], [**29**], [**26**], [**27**] and [**28**] it was called Theorem WD.

The following proposition was proved in [**33**].

Proposition 1. *Consider a real function $f : (a,b) \longrightarrow \mathbb{R}$ such that there exist its first and second* TAYLOR*'s approximations on both sides, for some $n \in N_0$. Then,*

$$\operatorname{sgn}\Big(\mathbb{T}_n^{f,\,a+,\,b-}(x) \;-\; \mathbb{T}_{n+1}^{f,\,a+,\,b-}(x)\Big) = \operatorname{sgn}\Big(f(b-) \;-\; T_n^{f,\,a+}(b)\Big), \tag{3}$$

for all $x \in (a,b)$.

In [**33**], based on the above proposition, the following theorem holds.

Theorem 2. *Consider a real function $f : (a,b) \longrightarrow \mathbb{R}$ such that the derivatives $f^{(k)}(a+)$, $k \in \{0,1,2,\ldots,n+1\}$ exist, for some $n \in \mathbb{N}$.*

Suppose that $f^{(n)}(x)$ and $f^{(n+1)}(x)$ are increasing on (a,b), then for all $x \in (a,b)$ the following inequalities hold:

$$T_n^{f,\,a+}(x) < T_{n+1}^{f,\,a+}(x) < f(x) < \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) < \mathbb{T}_n^{f;\,a+,\,b-}(x), \tag{4}$$

for all $x \in (a,b)$. If $f^{(n)}(x)$ and $f^{(n+1)}(x)$ are decreasing on (a,b), then for all $x \in (a,b)$ the reversed inequalities hold.

Also, in [**33**] the case of real analytic functions was considered and the following theorem was proved.

Theorem 3. *Consider the real analytic functions $f : (a,b) \longrightarrow \mathbb{R}$ with the following power series expansion:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, \tag{5}$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for all $k \in \mathbb{N}_0$. Then,

$$\begin{array}{c} T_0^{f,\,a+}(x) \leq \ldots \leq T_n^{f,\,a+}(x) \leq T_{n+1}^{f,\,a+}(x) \leq \ldots \\ \ldots \leq f(x) \leq \ldots \\ \ldots \leq \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) \leq \mathbb{T}_n^{f;\,a+,\,b-}(x) \leq \ldots \leq \mathbb{T}_0^{f;\,a+,\,b-}(x), \end{array} \tag{6}$$

for all $x \in (a,b)$. If $c_k \in \mathbb{R}$ and $c_k \leq 0$ for all $k \in \mathbb{N}_0$, then the reversed inequalities hold.

2. MAIN RESULTS

In this section we introduce and consider some error-functions for double-sided TAYLOR's approximations of a real analytic functions $f : [a,b] \longrightarrow \mathbb{R}$. Note that for such functions $f(a) = f(a+)$ and $f(b) = f(b-)$.

Let us consider a real analytic function $f : [a,b] \longrightarrow \mathbb{R}$ with the following power series expansion:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for all $k \in \mathbb{N}_0$.

We will assume that for $\{c_k\}_{k \in N_0}$ there exists an infinite subsequence of positive coefficients.

The following theorem is the consequence of the fact that zeros of real analytic functions are isolated.

Theorem 4. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real analytic function with the following power series expansion:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

where $c_k \in \mathbb{R}$. Let the following conditions hold:

(i) $f(a) = f(b) = 0$,

(ii) $\big(\forall x \in (a,b)\big)\, f(x) > 0$.

Then, there exists a right neighborhood $\mathcal{U}_a = (a, a_1) \subseteq (a, b)$ *of* a *such that:*

$$f'(x) > 0 \ \text{for} \ x \in \mathcal{U}_a, \ \ f'(a_1) = 0 \ \text{and} \ f'(x) < 0 \ \text{for} \ x \in (a_1, a_2),$$

for some $a_2 \leq b$,
and there exists a left neighborhood $\mathcal{U}_b = (b_1, b) \subseteq (a, b)$ *of* b *such that:*

$$f'(x) < 0 \ \text{for} \ x \in \mathcal{U}_b, \ \ f'(b_1) = 0 \ \text{and} \ f'(x) > 0 \ \text{for} \ x \in (b_2, b_1),$$

for some $b_2 \geq a$.

Remark 5. *The condition* $f'(x) > 0$ *for* $x \in \mathcal{U}_a$ *provides that function* f *is increasing in the right neighborhood* $\mathcal{U}_a \subseteq (a, b)$ *of* a. *Based on [**15**] there exists* $k_1 \in N$ *such that:*

$$f(a) = \ldots = f^{(k_1-1)}(a) = 0 \quad \text{and} \quad f^{(k_1)}(a) > 0.$$

Similarly, the condition $f'(x) < 0$ *for* $x \in \mathcal{U}_b$ *provides that function* f *is decreasing in the left neighborhood* $\mathcal{U}_b \subseteq (a, b)$ *of* b. *Based on [**15**] there exists* $k_2 \in N$ *such that:*

$$f(b) = \ldots = f^{(k_2-1)}(b) = 0 \quad \text{and} \quad f^{(k_2)}(b) < 0.$$

Based on Theorem 2.1 from [**32**] the following theorem holds.

Theorem 6. *Let* $\phi : [a, b] \longrightarrow R$ *be an* m *times differentiable function (for some* $m \geq 2$, $m \in N$*) that satisfies the following conditions:*

(a) $\phi^{(m)}(x) < 0$ *for* $x \in (a, b)$*;*

(b) *There exists a right neighborhood* $\mathcal{U}_a \subseteq (a, b)$ *of* a*, such that:*

$$\phi(x) > 0, \, \phi'(x) > 0, \, \ldots, \phi^{(m-1)}(x) > 0;$$

(c) *There exists a left neighborhood* $\mathcal{U}_b \subseteq (a, b)$ *of* b*, such that:*

$$\phi(x) < 0, \, \phi'(x) < 0, \, \ldots, \phi^{(m-1)}(x) < 0.$$

Then, the function ϕ *has exactly one zero* $x_0 \in (a, b)$*, and* $\phi(x) > 0$ *for* $x \in (a, x_0)$*, and* $\phi(x) < 0$ *for* $x \in (x_0, b)$*.*

Also, the function ϕ *has exactly one local maximum in the interval* (a, b)*, i.e. there exists exactly one* $t_0 \in (a, x_0) \subset (a, b)$ *such that* $\phi(t_0) > 0$ *is the maximum value of* ϕ *on the interval* $(0, x_0)$ *i.e.* (a, b)*.*

Remark 7. *Note that functions described in Theorem 2.1. [**32**] i.e. in Theorem 6 are hook-shaped. Also, the above theorem represents an improvement of Theorem 3 in [**27**].*

Let us consider the remainder of the second TAYLOR's approximation in the right neighborhood of a:

$$\begin{aligned} \mathbb{R}_n^{f;\,a+,b-}(x) &= f(x) - \mathbb{T}_n^{f;\,a+,b-}(x) \\ &= \sum_{k=n}^{\infty} c_k(x-a)^k - \frac{1}{(b-a)^n} R_{n-1}^{f,\,a+}(b-)(x-a)^n, \end{aligned} \tag{7}$$

for $x \in (a,b)$ and $n \geq 1$. Note that according to **[33]** $\mathbb{T}_n^{f;\,a+,b-}(x) > f(x)$ for $x \in (a,b)$.

Let us introduce the function:

$$\varphi(x) = |\mathbb{R}_n^{f;\,a+,b-}(x)| = T_{n-1}^{f,a}(x) + \frac{1}{(b-a)^n} R_{n-1}^{f,a}(b)(x-a)^n - f(x)$$

as the *error-function of the second* TAYLOR*'s approximation of f with* $\mathbb{T}_n^{f;\,a+,b-}(x)$.

Theorem 8. *For the error-function $\varphi(x)$ the following holds*:

$$\varphi^{(n)}(x) = \frac{n!}{(b-a)^n} R_{n-1}^{f,a}(b) - f^{(n)}(x),$$

and

$$\varphi^{(n)}(a) > 0, \quad \varphi^{(n)}(b) < 0.$$

Proof. From the definition of $\varphi(x)$ it is easy to compute $\varphi^{(n)}(x)$.
Now we focus on the sign of $\varphi^{(n)}(a)$ and $\varphi^{(n)}(b)$.

$$\begin{aligned} \varphi^{(n)}(a) &= \frac{n!}{(b-a)^n} R_{n-1}^{f,a}(b) - f^{(n)}(a) = \frac{n!}{(b-a)^n}(f(b) - T_{n-1}^{f,a}(b)) - f^{(n)}(a) \\ &= \frac{n!}{(b-a)^n}(f(b) - T_n^{f,a}(b)) > 0 \end{aligned}$$

Based on Lagrange's form of remainder for TAYLOR's expansion, for some $\xi_b \in (a,b)$, the following holds true:

$$\begin{aligned} \varphi^{(n)}(b) &= \frac{n!}{(b-a)^n} R_{n-1}^{f,a}(b) - f^{(n)}(b) = \frac{n!}{(b-a)^n}(f(b) - T_{n-1}^{f,a}(b)) - f^{(n)}(b) \\ &= f^{(n)}(\xi_b) - f^{(n)}(b) < 0 \end{aligned}$$

This completes the proof. □

It is not hard to verify that $\varphi(x)$ satisfies the conditions (i) and (ii) of Theorem 4. Therefore, there exists the right neighborhood of a such that $\varphi'(x) > 0$, and there exists the left neighborhood of b such that $\varphi'(x) < 0$.

Now, let us consider the function:

$$\psi(x) = \varphi'(x).$$

The function ψ, as well as all its derivatives, are real analytic functions and, consequently, all zeros (if any) of such functions are isolated.

Now, for the function ψ we check the conditions (a), (b) and (c) of Theorem 6.

The condition (a) $\psi^{(n)}(x) < 0$ for $x \in (a,b)$ follows from the definition of function φ.

Let us now examine the condition (b) of Theorem 6. There exists $a_1 \in (a,b]$ such that $\psi(a_1) = 0$ and $\psi(x) > 0$ for $x \in (a, a_1)$. Let us notice that $\psi(a) = 0$ and $\psi(a_1) = 0$. Repeating the previous step on (a, a_1) there exists $a_2 \in (a, a_1)$ such that $\psi'(a_2) = 0$ and $\psi'(x) > 0$ for $x \in (a, a_2)$. Continuing this procedure we conclude that there exists a sequence $a < a_n < a_{n-1} < \ldots < a_i < \ldots < a_2 < a_1 \leq b$ such that:

$$\psi^{(i)}(a_{i+1}) = 0 \quad \text{and} \quad \psi^{(i)}(x) > 0 \ \text{ for } \ x \in (a, a_{i+1}), \quad i = 0,1,2,\ldots,n-1.$$

Thus, for every $x \in \mathcal{U}_a = (a, a_n)$ the following is true:

$$\psi(x) > 0, \psi'(x) > 0, \ldots, \psi^{(n-1)}(x) > 0.$$

Now, let us focus on the condition (c) of Theorem 6. There exists $b_1 \in (a,b)$ such that $\psi(b_1) = 0$ and $\psi(x) < 0$ for $x \in (b_1, b)$. As $\psi(a) = 0$ there exists $\tilde{b}_1 \in [a, b_1)$ such that $\psi(\tilde{b}_1) = 0$, $\psi(b_1) = 0$ and $\psi(x) > 0$ for $x \in (\tilde{b}_1, b_1)$. Applying Theorem 4, we conclude that there exists $b_2 \in (\tilde{b}_1, b_1)$ such that:

$$\psi'(x) > 0 \ \text{ for } \ x \in (\tilde{b}_1, b_2), \quad \psi'(b_2) = 0 \quad \text{and} \quad \psi'(x) < 0 \ \text{ for } \ x \in (b_2, b_1).$$

Continuing this reasoning we conclude that there exists a sequence $a < b_n < b_{n-1} < \ldots < b_i < \ldots < b_2 < b_1 < b$ such that $\psi^{(i)}(b_{i+1}) = 0$ for $i = 0,1,2,\ldots,n-1$.

Now, for the functions $\psi^{(i)}(x)$ we prove uniqueness of zeros $b_{i+1} \in (a,b)$, for $i = 0,1,2,\ldots,n-1$.

As $\psi^{(n-1)}(x) < 0$ for $x \in (a,b)$ (the condition (a)) and based on Theorem 8, we conclude that the function $\psi^{(n-1)}(x)$ is strictly decreasing with exactly one zero $b_n \in (a,b)$. Further, $\psi^{(n-2)}(a) = 0$ and $\psi^{(n-2)}(x)$ is increasing on (a, b_n) and decreasing on (b_n, b) and, therefore, it has a unique zero $b_{n-1} \in (b_n, b)$. Recursively, we conclude that the function $\psi(x)$ has a unique zero $b_1 \in (b_2, b)$.

Finally, we conclude that for all $x \in \mathcal{U}_b = (b_1, b)$ the following inequalities hold:

$$\psi(x) < 0, \psi'(x) < 0, \ldots, \psi^{(n-1)}(x) < 0.$$

Thus, we proved that $\psi(x)$ satisfies the conditions (a), (b) and (c) of Theorem 6. Hence, the assertion of the following theorem holds.

Theorem 9. *Let us consider the error-function of the second* TAYLOR*'s approximation of function f with $\mathbb{T}_m^{f;\,a+,b-}(x)$, $m \in N_0$:*

$$\varphi_m(x) = |\mathbb{R}_m^{f;\,a+,b-}(x)| = \mathbb{T}_m^{f;\,a+,b-}(x) - f(x) \quad \textit{for } x \in [a,b].$$

Then, for $m \in N$ the function $\varphi_m(x)$ reaches a local maximum at exactly one point $x_0 \in (a,b)$ and $\varphi_m(x)$ is increasing on (a,x_0) and decreasing on (x_0,b).

We define the *error of the second* TAYLOR*'s approximation*, for $m \in N_0$, by:

$$\Phi_m = \max_{x\in[a,b]} \Big(\mathbb{T}_m^{f;\,a+,b-}(x) - f(x) \Big). \tag{8}$$

From the previous theorem directly follows:

$$\Phi_m = \varphi_m(x_0) \tag{9}$$

for $m \in N$, and

$$\Phi_0 = f(b) - f(a). \tag{10}$$

It is not hard to check that the following assertion holds for real analytic functions with non-negative coefficients.

Theorem 10. *Let us consider the error-function of the first* TAYLOR*'s approximation of function f with $T_n^{f;\,a+}(x)$, $n \in N_0$:*

$$\gamma_n(x) = |R_n^{f;\,a+}(x)| = f(x) - T_n^{f,\,a+}(x), \quad \textit{for } x \in [a,b].$$

Then, for $n \in N$ the function $\gamma_n(x)$ is increasing on (a,b) and reaches its maximum at $x = b$.

We define the *error of the first* TAYLOR*'s approximation*, for $n \in N_0$, by:

$$\Upsilon_n = \max_{x\in[a,b]} \big(f(x) - T_n^{f;\,a+}(x) \big). \tag{11}$$

From the previous theorem directly follows:

$$\Upsilon_n = \gamma_n(b), \tag{12}$$

for $n \in N$, and

$$\Upsilon_0 = f(b) - f(a). \tag{13}$$

From (6), for $m, n \in N_0$ and $x \in (a,b)$, directly follows the *main inequality for double-sided* TAYLOR*'s approximations*:

$$\mathbb{T}_m^{f;\,a+,b-}(x) > f(x) > T_n^{f;\,a+}(x). \tag{14}$$

Now we define the *error-function of double-sided* TAYLOR*'s approximations*:

$$\omega_{m,n}(x) = \mathbb{T}_m^{f;\,a+,b-}(x) - T_n^{f;\,a+}(x), \tag{15}$$

for $m, n \in N_0$ and $x \in [a, b]$, and the *error of double-sided* TAYLOR*'s approximations*:

$$\Omega_{m,n} = \max_{x \in [a,b]} \omega_{m,n}(x). \tag{16}$$

In the following theorem we estimate the error of double-sided TAYLOR's approximations.

Theorem 11. *Za* $m, n \in N_0$ *važi*:

$$\Omega_{m,n} < \Phi_m + \Upsilon_n. \tag{17}$$

Remark 12. *Note that* $\omega_{m,n}(x)$ *is a polynomial and numerical values* $\Omega_{m,n}$ *can be computed by appropriate numerical methods.*

3. AN EXAMPLE

In this section we consider the error-functions in double-sided TAYLOR's approximations for the following result by J. SANDOR [**18**]:

Theorem 13.

$$\frac{3}{8} < \frac{1 - \dfrac{\cos x}{\cos \frac{x}{2}}}{x^2} < \frac{4}{\pi^2}, \tag{18}$$

for any $x \in (0, \pi/2)$.

In [**34**] the previous statement was proved using double-sided TAYLOR's approximations of the following function:

$$f(x) = \begin{cases} \dfrac{3}{8} & : \quad x = 0, \\ \dfrac{1 - \dfrac{\cos x}{\cos \frac{x}{2}}}{x^2} & : \quad x \in (0, \pi); \end{cases} \tag{19}$$

where f is a real analytic function over $[0, \pi)$ with power series

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{(|E_{2k}| - 2(-1)^k)}{2^{2k}(2k)!} x^{2(k-1)} \\ &= \frac{3}{8} + \frac{1}{128}\, x^2 + \frac{7}{5120}\, x^4 + \frac{461}{3440640}\, x^6 + \frac{16841}{1238630400}\, x^8 + \ldots \end{aligned} \tag{20}$$

($E_i -$are EULERs numbers). It is easy to verify that for the coefficients of the above power series the following is true:

$$c_{2(k-1)} = \frac{(|E_{2k}| - 2(-1)^k)}{2^{2k}(2k)!} > 0 \quad \text{and} \quad c_{2k-1} = 0, \tag{21}$$

for $k = 1, 2, \ldots$.

The following statement was proved in [**34**]:

Theorem 14. *For every $x \in (0,c)$, where $0 < c < \pi$ and $m,n \in N$, following inequalities are true*:

(22)
$$\frac{3}{8} = T_0^{f,\,0+}(x) < \ldots < T_{2n}^{f,\,0+}(x) < T_{2n+2}^{f,\,0+}(x) < \ldots$$
$$\ldots < f(x) < \ldots$$
$$\ldots < \mathbb{T}_{2m+2}^{f;\,0+,\,c-}(x) < \mathbb{T}_{2m}^{f;\,0+,\,c-}(x) < \ldots < \mathbb{T}_{0}^{f;\,0+,\,c-}(x) = \left(1 - \frac{\cos c}{\cos\frac{c}{2}}\right) \Big/ c^2.$$

Specially, for $c = \pi/2$ we obtain the result of J. Sandor stated by Theorem 13.

Now, in this example, we determine the error values of double-sided Taylor's approximations which appear in Theorem 14 up to the sixth degree, for $c = \pi/2$:

(23)
$$\begin{aligned}
T_0^{f,0}(x) &= \tfrac{3}{8}\\
T_2^{f,0}(x) &= \tfrac{3}{8} + \tfrac{x^2}{128}\\
T_4^{f,0}(x) &= \tfrac{3}{8} + \tfrac{x^2}{128} + \tfrac{7x^4}{5120}\\
T_6^{f,0}(x) &= \tfrac{3}{8} + \tfrac{x^2}{128} + \tfrac{7x^4}{5120} + \tfrac{461x^6}{3440640}
\end{aligned}$$

and

(24)
$$\begin{aligned}
\mathbb{T}_0^{f;\,0,\,\pi/2}(x) &= \tfrac{1}{(\pi/2)^2}\left(1 - \tfrac{\cos(\pi/2)}{\cos\frac{(\pi/2)}{2}}\right)\\
&= \tfrac{4}{\pi^2}\\
\mathbb{T}_2^{f;\,0,\,\pi/2}(x) &= \tfrac{3}{8} + \tfrac{x^2}{(\pi/2)^2}\left(\tfrac{1}{(\pi/2)^2}\left(1 - \tfrac{\cos(\pi/2)}{\cos\frac{(\pi/2)}{2}}\right) - \tfrac{3}{8}\right)\\
&= \tfrac{3}{8} + \tfrac{4\left(\frac{4}{\pi^2} - \frac{3}{8}\right)x^2}{\pi^2}\\
\mathbb{T}_4^{f;\,0,\,\pi/2}(x) &= \tfrac{3}{8} + \tfrac{x^2}{128} + \tfrac{x^4}{(\pi/2)^4}\left(\tfrac{1}{(\pi/2)^2}\left(1 - \tfrac{\cos(\pi/2)}{\cos\frac{(\pi/2)}{2}}\right) - \tfrac{3}{8} - \tfrac{(\pi/2)^2}{128}\right)\\
&= \tfrac{3}{8} + \tfrac{x^2}{128} + \tfrac{16\left(\frac{4}{\pi^2} - \frac{3}{8} - \frac{\pi^2}{512}\right)x^4}{\pi^4}\\
\mathbb{T}_6^{f;\,0,\,\pi/2}(x) &= \tfrac{3}{8} + \tfrac{x^2}{128} + \tfrac{7x^4}{5120} + \tfrac{x^6}{(\pi/2)^6}\left(\tfrac{1}{(\pi/2)^2}\left(1 - \tfrac{\cos(\pi/2)}{\cos\frac{(\pi/2)}{2}}\right)\right.\\
&\quad\left. - \tfrac{3}{8} - \tfrac{(\pi/2)^2}{128} - \tfrac{7(\pi/2)^4}{5120}\right)\\
&= \tfrac{3}{8} + \tfrac{x^2}{128} + \tfrac{7x^4}{5120} + \tfrac{64\left(\frac{4}{\pi^2} - \frac{3}{8} - \frac{\pi^2}{512} - \frac{7\pi^4}{81920}\right)x^6}{\pi^6},
\end{aligned}$$

for $x \in (0, \pi/2)$.

According to Theorem 14 it holds that

$$\begin{array}{c} \frac{3}{8} = T_0^{f,0}(x) < T_2^{f,0}(x) < T_4^{f,0}(x) < T_6^{f,0}(x) < \\ < f(x) < \\ < \mathbb{T}_6^{f;\,0,\,\pi/2}(x) < \mathbb{T}_4^{f;\,0,\,\pi/2}(x) < \mathbb{T}_2^{f;\,0,\,\pi/2}(x) < \mathbb{T}_0^{f;\,0,\,\pi/2}(x) = \frac{4}{\pi^2}, \end{array} \tag{25}$$

for any $x \in (0, \pi/2)$.

The error-functions of the first TAYLOR's approximations are:

$$\begin{array}{lclcl} \gamma_0(x) & = & R_0^{f;\,0}(x) & = & \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) - \frac{3}{8} \\ \gamma_2(x) & = & R_2^{f;\,0}(x) & = & \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) - \frac{3}{8} - \frac{x^2}{128} \\ \gamma_4(x) & = & R_4^{f;\,0}(x) & = & \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) - \frac{3}{8} - \frac{x^2}{128} - \frac{7x^4}{5120} \\ \gamma_6(x) & = & R_6^{f;\,0}(x) & = & \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) - \frac{3}{8} - \frac{x^2}{128} - \frac{7x^4}{5120} - \frac{461x^6}{3440640}, \end{array} \tag{26}$$

for $x \in (0, \pi/2)$. Numerical values of errors of the first TAYLOR's approximations are shown in the following table:

(27)

k	x_0	Υ_k
0	$1.57079\ldots$	$3.02847\ldots \cdot 10^{-2}$
2	$1.57079\ldots$	$1.10081\ldots \cdot 10^{-2}$
4	$1.57079\ldots$	$2.68463\ldots \cdot 10^{-3}$
6	$1.57079\ldots$	$6.71923\ldots \cdot 10^{-4}$

The error-functions of the second TAYLOR's approximations with even indices are:

$$\begin{array}{lclcl} \varphi_0(x) & = & \mathbb{R}_0^{f;\,0,\,\frac{\pi}{2}}(x) & = & \frac{4}{\pi^2} - \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) \\ \varphi_2(x) & = & \mathbb{R}_2^{f;\,0,\,\frac{\pi}{2}}(x) & = & \frac{3}{8} + \frac{4\left(\frac{4}{\pi^2} - \frac{3}{8}\right)x^2}{\pi^2} - \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) \\ \varphi_4(x) & = & \mathbb{R}_4^{f;\,0,\,\frac{\pi}{2}}(x) & = & \frac{3}{8} + \frac{x^2}{128} + \frac{16\left(\frac{4}{\pi^2} - \frac{3}{8} - \frac{\pi^2}{512}\right)x^4}{\pi^4} - \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right) \\ \varphi_6(x) & = & \mathbb{R}_6^{f;\,0,\,\frac{\pi}{2}}(x) & = & \frac{3}{8} + \frac{x^2}{128} + \frac{7x^4}{5120} + \frac{64\left(\frac{4}{\pi^2} - \frac{3}{8} - \frac{\pi^2}{512} - \frac{7\pi^4}{81920}\right)x^6}{\pi^6} \\ & & & & -\frac{1}{x^2}\left(1 - \frac{\cos x}{\cos\frac{x}{2}}\right), \end{array} \tag{28}$$

for $x \in (0, \pi/2)$. In accordance with Theorem 9 each of the functions $\varphi_k(x)$, for $k \in \{2, 4, 6\}$, reaches local maximum at exactly one point $x_0^{(k)} \in (0, \pi/2)$. Numerical values of errors of the second TAYLOR's approximations are determined with table

(29)

k	$x_0^{(k)}$	Φ_k
0	$0.00000\ldots$	$3.02847\ldots \cdot 10^{-2}$
2	$1.14909\ldots$	$3.15109\ldots \cdot 10^{-3}$
4	$1.30317\ldots$	$4.78929\ldots \cdot 10^{-4}$
6	$1.37292\ldots$	$8.74100\ldots \cdot 10^{-5}$

Finally, let us observe the polynomial functions:

$$\omega_{m.n}(x) = \mathbb{T}_m^{f;\,0,\,\frac{\pi}{2}}(x) - T_n^{f,0}(x) \tag{30}$$

for $m, n \in \{0, 2, 4, 6\}$ and $x \in (0, \pi/2)$. Using these functions we obtain the table of errors of double-sided TAYLOR's approximations with even indices:

(31)

$\Omega_{m,n}$	$n = 0$	$n = 2$	$n = 4$	$n = 6$
$m = 0$	$3.02847\ldots \cdot 10^{-2}$	$3.02847\ldots \cdot 10^{-2}$	$3.02847\ldots \cdot 10^{-2}$	$3.02847\ldots \cdot 10^{-2}$
$m = 2$	$3.02847\ldots \cdot 10^{-2}$	$1.10081\ldots \cdot 10^{-2}$	$3.63967\ldots \cdot 10^{-3}$	$3.20177\ldots \cdot 10^{-3}$
$m = 4$	$3.02847\ldots \cdot 10^{-2}$	$1.10081\ldots \cdot 10^{-2}$	$2.68463\ldots \cdot 10^{-3}$	$7.07601\ldots \cdot 10^{-4}$
$m = 6$	$3.02847\ldots \cdot 10^{-2}$	$1.10081\ldots \cdot 10^{-2}$	$2.68463\ldots \cdot 10^{-3}$	$6.71923\ldots \cdot 10^{-4}$

This table gives the error values of double-sided TAYLOR's approximations which appear in (25). Also, based on previous two tables and Theorem 11, we can obtain error estimates in considered double-sided TAYLOR's approximations.

4. CONCLUSION

TAYLOR's approximations represent a classic and well-known tool for various problems in mathematics and engineering. Many results on TAYLOR's approximations are presented in monographs [**2**] and [**11**].

In this paper we introduced the error-functions for one-sided and double-sided TAYLOR's approximations of real analytic functions. Also, we illustrated the application of error-functions in the process of generalization of SANDOR's trigonometric inequality from [**18**].

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