A GLOBALLY CONVERGENT MODIFIED VERSION OF THE METHOD OF MOVING ASYMPTOTES

*Dedicated to Academician Professor Gradimir Milovanović
on the occasion of his 70th birthday.*

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A new modified moving asymptotes method is presented. In each step of the iterative process, a strictly convex approximating subproblem is generated and explicitly solved. In doing so we propose a strategy to incorporate a modified second-order information for the moving asymptotes location. Under natural assumptions, we prove the geometrical convergence. In addition the experimental results reveal that the present method is significantly faster compared to the [**1**] method, Newton's method and the BFGS Method.

1. MOTIVATION AND THEORETICAL JUSTIFICATION

Consider the unconstrained optimization problem: Find $x^* \in \Omega$ such that

$$(1) \qquad f(x^*) = \min_{x \in \Omega} f(x),$$

where Ω is an open subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a given non-linear real-valued objective function, typically twice continuously differentiable, which could be non-convex. In order to evaluate the merit of using second order information an extension of the method of moving asymptotes, that accounts for the curvatures, was

proposed in [**1**]. Let us first briefly recall its main idea. Throughout, we assume that f' does not vanish at a given suitable initial point $x^{(0)} \in \Omega$, that is $f'(x^{(0)}) \neq 0$, since if this is not the case we have nothing to solve. Starting from the initial design point $x^{(0)}$ the iterates $x^{(k)}$ are computed successively by solving sub-problems of the form: Find $x^{(k+1)}$ such that

$$(2) \qquad f^{(k)}(x^{(k+1)}) = \min_{x \in \Omega} f^{(k)}(x),$$

where the approximating function $f^{(k)}$ of the objective function f at the $k-$th iteration has the following form:

$$(3) \quad \begin{aligned} f^{(k)}(x) &= a^{(k)} + b^{(k)}(x - x^{(k)}) \\ &+ c^{(k)} \left(\frac{1}{2} \frac{(x^{(k)} - d^{(k)})^3}{x - d^{(k)}} + \frac{1}{2}(x^{(k)} - d^{(k)})(x - 2x^{(k)} + d^{(k)}) \right), \end{aligned}$$

with

$$(4) \qquad d^{(k)} = \left\{ \begin{array}{ll} L^{(k)} & \text{if } f'(x^{(k)}) < 0, \text{and } L^{(k)} < x^{(k)} \\ U^{(k)} & \text{if } f'(x^{(k)}) > 0, \text{and } U^{(k)} > x^{(k)}, \end{array} \right.$$

here, the asymptotes $U^{(k)}$ and $L^{(k)}$ are adjusted heuristically during the iterations, or guided by a proposed given function where the first and second derivative are evaluated at the current iteration point $x^{(k)}$. In contrast with the classical Newton's method, here the approximation functions $f^{(k)}$ are of the form of a *linear* function plus a *rational* function. For each iteration, the approximate parameters $a^{(k)}, b^{(k)}$ and $c^{(k)}$ used in equation (3) are determined in such a way that the following set of interpolation conditions are satisfied:

$$(5) \qquad f^{(k)}(x^{(k)}) = f(x^{(k)}),$$
$$(6) \qquad (f^{(k)})'(x^{(k)}) = f'(x^{(k)}),$$
$$(7) \qquad (f^{(k)})''(x^{(k)}) = f''(x^{(k)}).$$

It follows from the above identities that $a^{(k)}, b^{(k)}$, and $c^{(k)}$ are given by

$$(8) \qquad \begin{aligned} a^{(k)} &= f(x^{(k)}), \\ b^{(k)} &= f'(x^{(k)}), \\ c^{(k)} &= f''(x^{(k)}). \end{aligned}$$

In order to apply this method, it is necessary that the objective function should fulfill for each iteration k the following condition:

$$(9) \qquad f''(x^{(k)}) > 0,$$

which is typically an important weakness to this approach. Hence, this method is very restrictive and also has the following disadvantages :

- It needs good initial solution $x^{(0)}$ close to the exact solution.

- It converges slowly, in many cases, to the optimum x^*.

- It does not always converge.

- Its performance degrades when it applied to nonconvex functions.

It is the intention of this contribution to extend the [1] method to a more general context, by removing the restrictive condition (9) on the objective functions making the present approach more efficient, more general, and therefore more competitive. It is shown, as will be proved below, that the new method converges geometrically. Comparative numerical studies also show the success of the proposed extensions for various kinds of different test functions.

2.   A MODIFIED MOVING ASYMPTOTES METHOD

Throughout this paper we assume that w is a function satisfying the following conditions:

(10) w is a real-valued function, defined and continuous on $\mathbb{R}$,

(11) $lim_{|x| \to +\infty} w(x) = 0.$

Our general modification of moving asymptotes method that we examine herein may be described as follows: Given the iteration point $\tilde{x}^{(k)}$ (at iteration k).

- The objective function f is iteratively approximated at the k-th iteration by the approximating function $\tilde{f}_w^{(k)}$ where:

(12) $\tilde{f}_w^{(k)}(x) \;\; = \;\; \tilde{a}^{(k)} + \tilde{b}^{(k)}(x - \tilde{x}^{(k)}) +$
$$\tilde{c}^{(k)}\left(\frac{1}{2} \frac{(\tilde{x}^{(k)} - \tilde{d}^{(k)})^3}{x - \tilde{d}^{(k)}} + \frac{1}{2}(\tilde{x}^{(k)} - \tilde{d}^{(k)})(x - 2\tilde{x}^{(k)} + \tilde{d}^{(k)}) \right).$$

- The approximating function $\tilde{f}_w^{(k)}$ is first order approximations of the original function f at the current iteration point $\tilde{x}^{(k)}$, i.e.,

(13) $\tilde{f}_w^{(k)}(\tilde{x}^{(k)}) \;\; = \;\; f(\tilde{x}^{(k)}),$
(14) $(\tilde{f}_w^{(k)})'(\tilde{x}^{(k)}) \;\; = \;\; f'(\tilde{x}^{(k)}).$

In addition to the above conditions (13) and (14), the approximating function should satisfy the more general condition (15) instead of (7):

(15) $(\tilde{f}_w^{(k)})''(\tilde{x}^{(k)}) \;\; = \;\; \left| f''(\tilde{x}^{(k)}) + w(\tilde{x}^{(k)})f'(\tilde{x}^{(k)}) \right|.$

Consequently, in the present situation, the approximate parameters $\tilde{a}^{(k)}, \tilde{b}^{(k)}$ and $\tilde{c}^{(k)}$ are here determined for each iteration such that:

$$
\begin{align}
(16) \qquad \tilde{a}^{(k)} &= f(\tilde{x}^{(k)}), \\
(17) \qquad \tilde{b}^{(k)} &= f'(\tilde{x}^{(k)}), \\
(18) \qquad \tilde{c}^{(k)} &= \left| f''(\tilde{x}^{(k)}) + w(\tilde{x}^{(k)})f'(\tilde{x}^{(k)}) \right|.
\end{align}
$$

Furthermore, in order to fully determine an explicit expression for the approximating function $\tilde{f}_w^{(k)}$, the parameter $\tilde{d}^{(k)}$ is chosen such that

$$
(19) \qquad \tilde{d}^{(k)} = \tilde{x}^{(k)} + 2\tilde{\alpha}^{(k)} \frac{f'(\tilde{x}^{(k)})}{\tilde{c}^{(k)}},
$$

where $\{\tilde{\alpha}^{(k)}\}_k$ is a sequence of real numbers with

$$
(20) \qquad \tilde{\alpha}^{(k)} > 1, (k \in \mathbb{N}).
$$

Different rules for how to choose these values (and possible weight functions in (15)) will be precisely chosen later. We note that our method does not use the interpolation condition (7), but instead we have incorporated a first- and second-order information, as given in (18). Moreover, in particular, if you take $w = 0$ and at each iteration condition (9) is fulfilled, then our iterative scheme obviously reduces to the one introduced in [1]. Hence, subsequent iterations of the [1] method are similar, except that in the proposed approximating function $\tilde{f}_w^{(k)}$ the parameters $c^{(k)}$ and $d^{(k)}$ are replaced by those computed in (18) and (19) respectively. It starts at an initial point $\tilde{x}^{(0)}$ and generates successive iterates by

$$
(21) \qquad f(\tilde{x}^{(k+1)}) \leftarrow \tilde{f}_w^{(k)}(\tilde{x}^{(k+1)}) = \min_{x \in \Omega} \tilde{f}_w^{(k)}(x).
$$

For simplicity, we have removed the index w in $\tilde{x}_w^{(k)}$.

We prefer to work with (18) instead of (7) for several reasons. First, as mentioned above, this allows us to apply our method to a large class of objective functions. There is also a significant difference from a numerical point of view: many experimental results reveal that the iterative scheme based on our modification (18) can yield significantly fewer iterations than the [1] method, Newton's method or the BFGS Method itself. In contrast to these three approaches, our method converges even if the starting point is very far from the true solution. In addition, as we will see, the key features of the present method are:

- It does not require us to build a good initial solution close to the exact solution.

- It converges geometrically for a large class of functions w that satisfy condition (11).

- It will succeed where the [**1**] method, Newton's method and the BFGS Method break down.

Newton's method and the BFGS Method have a well-studied convergence theory that guarantees the convergence to a solution under a standard set of assumptions. For these and other their variants, the interested reader should consult one of the many excellent books on this subject [**2**, pp. 48–75] and [**3**, pp. 75–89]. We refer the readers to [**1**] and the references therein for the method of moving asymptotes.

3. CONVERGENCE ANALYSIS

We start this section with a result concerning an explicit expression for the iterative sequence $\left\{\tilde{x}^{(k)}\right\}_k$ generated by the approximating function $\tilde{f}_w^{(k)}$. Here, we continue to denote by $\tilde{c}^{(k)}, \tilde{d}^{(k)}$ and $\tilde{\alpha}^{(k)}$ the coefficients given by (18), (19) and (20) respectively. Note that condition (20), imposed on the parameters $\tilde{\alpha}^{(k)}$, is crucial since it will guarantee strict convexity of the approximating function $\tilde{f}_w^{(k)}$. For brevity, in the following we use the notation:

$$(22) \qquad \mathcal{I}_k = \left]-\infty, \tilde{d}^{(k)}\right[\cup \left]\tilde{d}^{(k)}, +\infty\right[.$$

Now we are able to state the first main result.

Theorem 3.1. *With the above notation, let $\Omega \subset \mathbb{R}$ be an open subset of the real line, a given twice continuously differentiable function f in Ω, $\tilde{x}^{(0)} \in \Omega$ and $\tilde{x}^{(k)}$ being respectively the initial and a current point of the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$. Then, for each $k > 0$ the approximating function defined by (12) is a strictly convex function on $\mathcal{I}_k$. In addition, the function $\tilde{f}_w^{(k)}$ has an unique minimum at*

$$(23) \qquad \tilde{x}^{(k+1)} \leftarrow \tilde{x}_*^{(k)} = \tilde{d}^{(k)} + (\tilde{x}^{(k)} - \tilde{d}^{(k)})\sqrt{\tilde{s}^{(k)}}$$

where

$$(24) \qquad \tilde{s}^{(k)} = \frac{\tilde{\alpha}^{(k)}}{\tilde{\alpha}^{(k)} - 1}.$$

Proof. The proof of this theorem follows by arguments essentially identical to those given in [**1**]. The main ingredient here is a suitable application of the condition (20) imposed on the coefficient $\tilde{d}^{(k)}$. We first start by showing that the approximating function $\tilde{f}_w^{(k)}$ is well defined and strictly convex in $\mathcal{I}_k$. To this end we prove that $(\tilde{f}_w^{(k)})''$ is nonnegative in $\mathcal{I}_k$. Indeed, a simple calculation reveals that

$$(25) \qquad \left(\tilde{f}_w^{(k)}\right)''(x) = \tilde{c}^{(k)}\left(\frac{\tilde{x}^{(k)} - \tilde{d}^{(k)}}{x - \tilde{d}^{(k)}}\right)^3.$$

In view of (25), it remains to show that the term on the right-hand side of (25) is nonnegative for all $x \in \mathcal{I}_k$. Since $\tilde{c}^{(k)}$ is nonnegative (as can be seen in (18)), and the two terms $\tilde{x}_k - \tilde{d}^{(k)}$ and $x - \tilde{d}^{(k)}$ have the same sign in the interval $\mathcal{I}_k$, then $\tilde{f}_w^{(k)}$ is a convex function on $\mathcal{I}_k$. Furthermore, the function $\tilde{f}_w^{(k)}$ being continuous in $\mathcal{I}_k$, this implies the existence of a minimum, which by convexity is the unique critical point $\tilde{x}_*^{(k)}$. Now, looking for $(\tilde{f}_w^{(k)})'(x) = 0$ we conclude that the optimum $\tilde{x}_*^{(k)}$ is one solution of the equation

$$(26) \qquad f'\left(\tilde{x}^{(k)}\right) + \frac{1}{2}\tilde{c}^{(k)}\left(\tilde{x}^{(k)} - \tilde{d}^{(k)}\right)\left(1 - \frac{\left(\tilde{x}^{(k)} - \tilde{d}^{(k)}\right)^2}{\left(x - \tilde{d}^{(k)}\right)^2}\right) = 0,$$

which, after trivial calculations, implies

$$(27) \qquad \left(\frac{\tilde{x}^{(k)} - \tilde{d}^{(k)}}{x - \tilde{d}^{(k)}}\right)^2 = 1 + \frac{2f'\left(\tilde{x}^{(k)}\right)}{\tilde{c}^{(k)}\left(\tilde{x}^{(k)} - \tilde{d}^{(k)}\right)}.$$

Now using (19), we can write

$$(28) \qquad \frac{2f'\left(\tilde{x}^{(k)}\right)}{\tilde{c}^{(k)}\left(\tilde{x}^{(k)} - \tilde{d}^{(k)}\right)} = -\frac{1}{\tilde{\alpha}^{(k)}}.$$

Therefore, after some simplification, equation (27) becomes

$$(29) \qquad \left(\frac{\tilde{x}^{(k)} - \tilde{d}^{(k)}}{x - \tilde{d}^{(k)}}\right)^2 = 1 - \frac{1}{\tilde{\alpha}^{(k)}}.$$

By condition (20) imposed on the parameter $\alpha^{(k)}$, it can be deduced from (29) that the solvability of our subproblem can always be guaranteed. Indeed, under this condition, the required identity (23) immediately follows from (29) and the above mentioned fact that $\tilde{x}_k - \tilde{d}^{(k)}$ and $x - \tilde{d}^{(k)}$ have the same sign in the interval $\mathcal{I}_k$. This completes the proof of the theorem.   □

3.1   Convergence study

In this Section, we give the main result of this paper, that is sufficient conditions on the data (the point $\tilde{x}^{(0)}$, the function f' in a neighborhood of $\tilde{x}^{(0)}$, the family $f''\left(\tilde{x}^{(k)}\right), k \geq 0$), which guarantee that first derivative of f vanishes in a neighborhood of x^*, first, and secondly, the convergence of the method to this zero.
To establish our convergence results, we need the following assumptions. We assume that there exist positive constants r, M and $\xi < 1$ such that the following assumptions hold:

Assumption 1.
$$B_r := \left\{ x \in \mathbb{R} : \left| x - \tilde{x}^{(0)} \right| \leq r \right\} \subset \Omega.$$

Assumption 2.
$$0 < \frac{\tilde{\alpha}^{(k)}}{\tilde{\alpha}^{(k)} - 1} \leq \frac{M}{2} \tilde{c}^k, \quad (k > 0).$$

Assumption 3.
$$\sup_{k \geq 0} \sup_{x \in B} \left| f''(x) - \frac{f'\left(x^{(k-1)}\right)}{x^{(k-1)} - \tilde{x}^{(k)}} \right| \leq \frac{\xi}{M}.$$

Assumption 4.
$$0 < \left| f'\left(\tilde{x}^{(0)}\right) \right| \leq \frac{r}{M}(1 - \xi).$$

Assumption 2 enforces the quite natural conditions (20). Indeed, if condition 2 holds, then (20) is also satisfied. Furthermore, Assumption 3 tells us that the coefficient $f''(\tilde{x}^{(k)})$ does not change too much in a neighborhood of $\tilde{x}^{(0)}$, and finally for any k the functions $f''(\tilde{x}^{(k)})$ and $\frac{f'\left(\tilde{x}^{(k-1)}\right)}{\tilde{x}^{(k-1)} - \tilde{x}^{(k)}}$ have not to change too much in a neighborhood of $\tilde{x}^{(k)}$. Finally, Assumption 4 only says that $\left| f'\left(\tilde{x}^{(0)}\right) \right|$ is small enough and that $f'\left(\tilde{x}^{(0)}\right)$ is non zero.

Throughout this subsection, we assume that Assumptions 1-4 hold. The constants r, M and $\xi < 1$ that appear in the subsequent analysis are always the constants from Assumptions 1-4. Our aim is to show that the sequence $\left\{ \tilde{x}^{(k)} \right\}_{k \geq 0}$ defined by (30) converges geometrically to a point x^* in the sense that

$$\left| \tilde{x}^{(k)} - x^* \right| \leq \frac{\xi^k}{1 - \xi} \left| x^{(1)} - x^{(0)} \right|.$$

Theorem 3.2. *Assume Assumptions 1-4 hold. Let the assumptions of theorem 3.1 be valid and let $\tilde{s}^{(k)}$ be defined by (24). Then the sequence $\left\{ \tilde{x}^{(k)} \right\}_{k \geq 0}$ given by*

$$(30) \qquad \tilde{x}^{(k+1)} = \tilde{d}^{(k)} + (\tilde{x}^{(k)} - \tilde{d}^{(k)})\sqrt{\tilde{s}^{(k)}}$$

is completely contained in the interval B_r, and converges to the unique zero of f' in B_r.

Before we embark on the proof of Theorem 3.2, we first prove some technical lemmas.

Lemma 3.3. *Let Assumption 2 be satisfied and let the sequence $\left\{ \tilde{x}^{(k)} \right\}_{k \geq 0}$ be as defined in Theorem 3.2. Then, for any positive integer k the following inequality holds.*

$$(31) \qquad \left| \tilde{x}^{(k)} - \tilde{x}^{(k-1)} \right| \leq M \left| f'(\tilde{x}^{(k-1)}) \right|.$$

Proof. Let us fix a positive integer k. Using (30) we may write

$$
\begin{aligned}
\tilde{x}^{(k)} - \tilde{x}^{(k-1)} &= \tilde{d}^{(k-1)} + (\tilde{x}^{(k-1)} - \tilde{d}^{(k-1)})\sqrt{\tilde{s}^{(k-1)}} - \tilde{x}^{(k-1)} \\
&= (\tilde{x}^{(k-1)} - \tilde{d}^{(k-1)})\left(\sqrt{\tilde{s}^{(k-1)}} - 1\right)
\end{aligned}
$$

Now, from (20) we have

$$
(32) \qquad\qquad \tilde{s}^{(k-1)} > 1, \quad (k \geq 1),
$$

we then immediately deduce

$$
\sqrt{\tilde{s}^{(k-1)}} < \tilde{s}^{(k-1)}.
$$

Therefore, by (20) and (32), we arrive at

$$
(33) \qquad \left|\tilde{x}^{(k)} - \tilde{x}^{(k-1)}\right| \leq \left|\tilde{x}^{(k-1)} - \tilde{d}^{(k-1)}\right|\left(\tilde{s}^{(k-1)} - 1\right),
$$

$$
(34) \qquad\qquad\qquad \leq \frac{2\tilde{\alpha}^{(k-1)}}{(\tilde{\alpha}^{(k-1)} - 1)\tilde{c}^{(k-1)}}\left|f'(\tilde{x}^{(k-1)})\right|.
$$

Thus, we have

$$
\left|\tilde{x}^{(k)} - \tilde{x}^{(k-1)}\right| \leq \frac{2\tilde{\alpha}^{(k-1)}}{(\tilde{\alpha}^{(k-1)} - 1)\tilde{c}^{(k-1)}}\left|f'(\tilde{x}^{(k-1)})\right|.
$$

Finally, combing Assumption 2 and this last inequaly, we get the required inequality in (31). $\qquad\square$

In order to prove that the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ converges geometrically, we need some further preparatory results.

Lemma 3.4. *Let Assumption 3 be satisfied and let the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ be defined as in Theorem 3.2. Then, for any positive k the following inequality holds.*

$$
(35) \qquad\qquad \left|f'(\tilde{x}^{(k)})\right| \leq \frac{\xi}{M}\left|\tilde{x}^{(k)} - \tilde{x}^{(k-1)}\right|.
$$

Proof. Fix a positive integer k. Let us define $\tilde{t}^{(k-1)}$ by

$$
(36) \qquad \tilde{t}^{(k-1)} := \frac{\tilde{c}^{(k-1)}}{2}(\tilde{x}^{(k-1)} - \tilde{d}^{(k-1)}) - f'\left(\tilde{x}^{(k-1)}\right),
$$

and the auxiliary function $\varphi : B_r \to \mathbb{R}$ as follows:

$$
\varphi(x) := f'(x) - \frac{f'(\tilde{x}^{(k-1)})}{\frac{1}{2}\tilde{c}^{(k-1)}(\tilde{x}^{(k)} - \tilde{x}^{(k-1)})}h(x),
$$

where

$$
h(x) := -f'\left(\tilde{x}^{(k-1)}\right) - \frac{\tilde{c}^{(k-1)}}{2}\left(x - \tilde{x}^{(k)} + \tilde{x}^{(k-1)} - \tilde{d}^{(k-1)}\right) - \tilde{t}^{(k-1)}.
$$

Using (36), it is easily checked that φ satisfies

$$
\begin{aligned}
\varphi(\tilde{x}^{(k-1)}) &= 0, \\
\varphi(\tilde{x}^{(k)}) &= f'(\tilde{x}^{(k)}).
\end{aligned}
$$

Then, from the mean-value theorem and Assumption 3 we get

(37)
$$
\begin{aligned}
\left| f'\left(\tilde{x}^{(k)}\right) \right| &\leq \sup_{x \in B_r} \left| f''(x) - \frac{f'\left(\tilde{x}^{(k-1)}\right)}{\tilde{x}^{(k-1)} - \tilde{x}^{(k)}} \right| \left| \tilde{x}^{(k)} - \tilde{x}^{(k-1)} \right| \\
&\leq \frac{\xi}{M} \left| \tilde{x}^{(k)} - \tilde{x}^{(k-1)} \right|.
\end{aligned}
$$

This shows that the required inequality (35) holds true for any positive integer k. $\qquad \square$

   The next result shows that for all k the iterate $\tilde{x}^{(k)}$ remains in the interval B_r.

Lemma 3.5. *Let Assumption 2-3 be satisfied and let the sequence $\left\{\tilde{x}^{(k)}\right\}_{k \geq 0}$ be as defined in Theorem 3.2. Assume that the starting point $\tilde{x}^{(0)}$ belongs to the interval B_r, where r is defined in Assumption 1. Then, all terms of the sequence $\left\{\tilde{x}^{(k)}\right\}_{k \geq 0}$ lie inside the interval B_r.*

Proof. Indeed, combining inequalities (31) and (35) of Lemmas 3.3 and 3.4 respectively, we immediately obtain

(38)
$$
\left| \tilde{x}^{(k)} - \tilde{x}^{(k-1)} \right| \leq \xi \left| \tilde{x}^{(k-1)} - \tilde{x}^{(k-2)} \right| \leq \ldots \leq \xi^{k-1} \left| \tilde{x}^{(1)} - \tilde{x}^{(0)} \right|,
$$

and therefore we have

(39)
$$
\begin{aligned}
\left| \tilde{x}^{(k)} - \tilde{x}^{(0)} \right| &\leq \sum_{l=1}^{k} \left| \tilde{x}^{(l)} - \tilde{x}^{(l-1)} \right| \\
&\leq \left(\sum_{l=1}^{k} \xi^{l-1} \right) \left| \tilde{x}^{(1)} - \tilde{x}^{(0)} \right| \\
&\leq \frac{\left| \tilde{x}^{(1)} - \tilde{x}^{(0)} \right|}{1 - \xi}.
\end{aligned}
$$

Finally, applying inequality (31) for $k = 1$ and using Assumption 4 we easily get

(40)
$$
\begin{aligned}
\left| \tilde{x}^{(k)} - \tilde{x}^{(0)} \right| &\leq \frac{M}{1-\xi} \left| f'\left(\tilde{x}^{(0)}\right) \right| \\
&\leq r,
\end{aligned}
$$

which shows that each $\tilde{x}^{(k)}$ belongs to B_r. $\qquad \square$

   As a consequence of the previous three lemmas, we are now in a position to prove Theorem 3.2.
Proof of Theorem 3.2. Since the entire sequence $\left\{\tilde{x}^{(k)}\right\}_{k \geq 0}$ remains in the (closed) interval B_r by Lemma 3.5, every limit point of this sequence belongs to this set,

too. Hence, it remains to show that the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ converges. To this end, we first note that, for $k \geq 0$ and $l \geq 0$, we have

$$
(41) \quad
\begin{aligned}
\left|\tilde{x}^{(k+l)} - \tilde{x}^{(k)}\right| &\leq \sum_{v=0}^{l-1}\left|\tilde{x}^{(k+v+1)} - \tilde{x}^{(k+v)}\right| \\
&\leq \xi^k \sum_{v=0}^{l-1}\xi^v \left|\tilde{x}^{(1)} - \tilde{x}^{(0)}\right| \\
&\leq \frac{\xi^k}{1-\xi}\left|\tilde{x}^{(1)} - \tilde{x}^{(0)}\right|,
\end{aligned}
$$

then the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ is a Cauchy sequence. Being Cauchy in B_r (closed interval in $\mathbb{R}$), it has a limit, x^*, in B_r. Now, thanks to the continuity of f' on B_r, (35) the continuity of f' on B_r and the convergence of the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ imply

$$
(42) \qquad \left|f'\left(\tilde{x}^*\right)\right| = \lim_{k\to+\infty}\left|f'\left(\tilde{x}^{(k)}\right)\right| \leq \frac{\xi}{M}\lim_{k\to+\infty}\left|\tilde{x}^{(k)} - \tilde{x}^{(k-1)}\right| = 0,
$$

and then $f'\left(\tilde{x}^*\right) = 0$.

Passing to the limit for l tending to ∞ in (41), we deduce that

$$
(43) \qquad \left|\tilde{x}^{(k)} - \tilde{x}^*\right| \leq \frac{\xi^k}{1 - \xi}\left|\tilde{x}^{(1)} - \tilde{x}^{(0)}\right|,
$$

which shows the geometric convergence of the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ to $\tilde{x}^*$.

çà We are now in a position to prove that f' has an unique zero in B_r. To this end, we proceed by contradiction, assuming that f' has another zero $\tilde{y}^* \in B$. Let us introduce the auxiliary function

$$
\lambda(x) = \frac{\tilde{x}^{(1)} - \tilde{x}^{(0)}}{f'(\tilde{x}^{(0)})}\left(f'(x) - \frac{f'(\tilde{x}^{(0)})}{\tilde{x}^{(0)} - \tilde{x}^{(1)}}(x - \tilde{x}^*)\right),
$$

which satisfies $\lambda(\tilde{x}^*) = 0$ and $\lambda(\tilde{y}^*) = \tilde{y}^* - x^*$. Therefore, applying (31) for $k = 1$, it follows from the mean value theorem and Lemma 3.3, inequality (31) for $k = 1$,

$$
(44) \quad
\begin{aligned}
\left|\tilde{x}^* - \tilde{y}^*\right| &\leq \left|\frac{\tilde{x}^{(1)} - \tilde{x}^{(0)}}{f'(\tilde{x}^0)}\right|\sup_{x\in B}\left|f''(x) - \frac{f'(\tilde{x}_0)}{\tilde{x}^0 - \tilde{x}^{(1)}}\right|\left|\tilde{x}^* - \tilde{y}^*\right| \\
&\leq M\frac{\xi}{M}\left|\tilde{x}^* - \tilde{y}^*\right| \\
&\leq \xi\left|\tilde{x}^* - \tilde{y}^*\right|.
\end{aligned}
$$

This yields $x^* = \tilde{y}^*$ since $\xi < 1$. Thus, the theorem is proved.

3.2   Description of algorithm

The results of the previous section may be used to construct the following algorithm.

Algorithm 1 Modified Method of Moving Asymptotes

1: Input: $\tilde{x}^{(0)}$, w, $M_1, M_2 \geq 1$, (and an optional error tolerance $\varepsilon > 0$).
2: $k = 0$
3: REPEAT
4: $\tilde{c}^{(k)} = \left| f''(\tilde{x}^{(k)}) + w(\tilde{x}^{(k)}) f'(\tilde{x}^{(k)}) \right|$,
5: $\tilde{\alpha}^{(k)} = M_1 \left(1 + \frac{2}{M_2 \tilde{c}^{(k)}} \right)$,
6: $\tilde{d}^{(k)} = \tilde{x}^{(k)} + 2\tilde{\alpha}^{(k)} \frac{f'(\tilde{x}^{(k)})}{\tilde{c}^{(k)}}$,
7: $\tilde{s}^{(k)} = \frac{\tilde{\alpha}^{(k)}}{\tilde{\alpha}^{(k)} - 1}$,
8: $\tilde{x}^{(k+1)} = \tilde{d}^{(k)} + (\tilde{x}^{(k)} - \tilde{d}^{(k)}) \sqrt{\tilde{s}^{(k)}}$,
9: while $\left| f'(\tilde{x}^{(k)}) \right| > \epsilon$.

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4. NUMERICAL EXAMPLES

We employ the present method (designated as present) to solve some non-linear, non-convex optimization problems and compare it with the [1] method, Newton's method and the BFGS method. For our numerical tests we use two kinds of weight functions and the following two test functions:

$$
\begin{aligned}
f_1(x) &= \frac{1}{3}(\sin^3(x) - x^3) + x, \\
f_2(x) &= \frac{1}{2}\exp(x^2) + \frac{1}{2}(x - \frac{1}{2}\sin(2x)) + 3\sin(x) + 5x.
\end{aligned}
$$

It is assumed that all methods use the finite difference method to compute first and second derivatives. Numerical results are summarized in Tables 1 and 2, where for each weight function w, we present the objective functions, the starting points, the methods used, the number of iterations N to obtain the objective value of the obtained optimal solution $x^{(N)}$ and $f(x)$ at $x^{(N)}$. We use the following stopping criteria $\left| f'(\tilde{x}^{(N)}) \right| \leq \epsilon$, (the absolute value of the derivative of the function is less than or equal to the tolerance). For numerical illustrations we used different values of ϵ. Therefore, when the stopping criterion is satisfied, $\tilde{x}_* = \tilde{x}^{(N)}$ is taken as the optimal solution. In Tables 1 and 2, div. means that the stopping criteria is not satisfied. The test results in Tables 1 and 2 show that for all of the functions we tested, the present method is better than the [1] method, Newton's method and the BFGS Method. It always converges even if the starting point is very far from the true solution, and it never requires more iterations than the other three methods. These characteristics give a strong advantage over these other methods.

Function	Method	N	x_N	$f(x_N) \simeq minf(x)$	ϵ
$f_1(x)$	.	.	.	.	
$\tilde{x}^{(0)} = 10^{-12}$	.	.	.	.	10^{-14}
.	Newton	div.	.	.	.
.	BFGS	div.	.	.	.
.	[1]	div.	.	.	.
.	Present	6	-1.156436	$-896.585243e - 003$	.
$\tilde{x}^{(0)} = -\frac{1}{4}$	.	.	.	.	
.	Newton	13	.	.	.
.	BFGS	13	.	.	.
.	[1]	5	-1.156436	$-896.585243e - 003$	.
.	Present	5	.	.	.
$f_2(x)$	.	.	.	.	
$\tilde{x}^{(0)} = 0.25$	.	.	.	.	10^{-15}
.	Newton	div.	.	.	.
.	BFGS	div.	.	.	.
.	[1]	div.	.	.	.
.	Present	8	-1.231394	-7.166822	.
$\tilde{x}^{(0)} = -10$	.	.	.	.	
.	Newton	110	.	.	.
.	BFGS	110	.	.	.
.	[1]	114	-1.231394	-7.166822	.
.	Present	107	.	.	.

Table 1: Numerical comparisons of the [1] method, Newton's method, the BFGS Method and the present method. Here $w(x) = (1 + |x|)^{1/2} exp(-2 |x|)$ and $\tilde{\alpha}^{(k)} = 2 \left(1 + \frac{1}{4\tilde{c}^{(k)}}\right)$.

Function	$f(x_0)$	Method	N	x_N	$f(x_N) \simeq minf(x)$	ϵ
$f_1(x)$	.	.	.	.	.	
$\tilde{x}^{(0)} = -62 * 10^{100}$	79.44e+303	.	.	.	.	10^{-7}
	.	Newton	div.	.	.	
	.	BFGS	div.	.	.	
	.	[1]	div.	.	.	
	.	Present	241	-1.156436	$-896.585243e - 003$	
$\tilde{x}^{(0)} = 40 * 10^{60}$	-21.33e+183	.	.	.	.	10^{-12}
	.	Newton	div.	.	.	
	.	BFGS	div.	.	.	
	.	[1]	div.	.	.	
	.	Present	152	-1.156436	$-896.585243e - 003$	
$\tilde{x}^{(0)} = -30 * 10^{10}$	9.00e+033	.	.	.	.	10^{-15}
.	.	Newton	44	.	.	
.	.	BFGS	44	.	.	
.	.	[1]	div.	-1.156436	$-896.585243e - 003$	
.	.	Present	40	.	.	
$f_2(x)$	.	.	.	.	.	
$\tilde{x}^{(0)} = 26$	19.14e+292	.	.	.	.	10^{-12}
.	.	Newton	div.	.	.	
.	.	BFGS	div.	.	.	
.	.	[1]	div.	.	.	

.	.	Present	556	-1.231394	-7.166822	
$\tilde{x}^{(0)} = 10$	13.44e+042	.	.	.	.	10^{-16}
.	.	Newton	div.	.	.	
.	.	BFGS	div.	.	.	
.	.	[1]	div.	-1.231394	-7.166822	
.	.	Present	136	.	.	

Table 2: Numerical comparisons of the [1] method, Newton's method, the BFGS Method and the present paper. Here $w(x) = (1 + |x|)^{-4} exp(-10 |x|^{0.5}) \log(e + |x|)^{10}$ and $\tilde{\alpha}^{(k)} = 3 \left(1 + \frac{1}{10\tilde{c}^{(k)}}\right)$.

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