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SOME RESULT FOR BINOMIAL CONVOLUTION SUMS OF RESTRICTED DIVISOR FUNCTIONS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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Besge presented the result about the convolution sum of divisor functions. Since then Liouville obtained the generalized version of Besge's formula, which is the binomial convolution sum of divisor functions. In 2004, Hahn obtained the results about the convolution sums of $\sum_{d|n}(-1)^{d-1}d$ and $\sum_{d|n}(-1)^{n/d-1}d$. In this paper, we present the results for the binomial convoltion sums, generalized convolution sums of Hahn, of these divisor functions.

1. INTRODUCTION

For a positive integer n and a nonnegative integer k, let $\sigma_k(n)$ be the usual divisor function defined by $\sigma_k(n) = \sum_{d|n} d^k$. The well-known identity

$$\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = \frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma_1(n))$$

first appeared in a letter from Besge to Liouville in 1862(see [**1**]). The generalized version of Besge's identity, that is said to the binomial convolution sums of divisor functions was obtained by Liouville(see [**6**]).

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$$\sum_{s=0}^{k-1}\binom{2k}{2k+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)=\frac{2k+3}{4k+2}\sigma_{2k+1}(n)$$
$$+\left(\frac{k}{6}-n\right)\sigma_{2k-1}(n)+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(n),$$

where for a nonnegative integer n, B_n is the n-th Bernoulli number. We rewrite another version of the above identity in terms of the divisor function and Bernoulli polynomial $B_n(x)$ as follows:

$$\sum_{s=0}^{k-1}\binom{2k}{2k+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)$$
$$=\frac{1}{2}\sigma_{2k+1}(n)+\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)+\frac{1}{2k+1}\sum_{d|n}B_{2k+1}(d). \tag{1.1}$$

We introduce another divisor functions. For a positive integer n and a non-negative integer k,

$$\sigma_k^*(n)=\sum_{\substack{d|n\\ n/d\equiv 1(2)}}d^k \qquad \textit{and} \qquad \sigma_{k,i}(n;2)=\sum_{\substack{d|n\\ d\equiv i(2)}}d^k \quad (i\in\{0,1\}).$$

Recently, Kim and Bayad(see [**4**]) obtained the convolution sum of $\sigma_k^*(n)$

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m)$$
$$=\frac{1}{2}\sigma^*_{2k+1}(n)-\frac{n}{2}\sigma^*_{2k-1}(n) \tag{1.2}$$

and Kim, Bayad and Park(see [**5**]) computed the convolution sum of $\sigma_{k,1}(n;2)$

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2)$$
$$=2^{2k-1}\sigma_{2k+1}(n/2)+\frac{2^{2k}}{2k+1}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}\left(\frac{d+1}{2}\right). \tag{1.3}$$

Hahn has defined for $k,r\in\mathbb{N}$

$$\tilde{\sigma}_k(n)=\sum_{d|n}(-1)^{d-1}d^k \qquad \text{and} \qquad \hat{\sigma}_k(n)=\sum_{d|n}(-1)^{n/d-1}d^k.$$

Hahn obtained the convolution sums of the these divisor functions(see [**2**]).

$$\sum_{m=1}^{n-1}\tilde{\sigma}_1(m)\tilde{\sigma}_1(n-m) = -\frac{1}{4}\tilde{\sigma}_3(n) + \left(\frac{1}{2}n - \frac{1}{4}\right)\tilde{\sigma}_1(n),$$
$$\sum_{m=1}^{n-1}\hat{\sigma}_1(m)\tilde{\sigma}_1(n-m) = -\frac{1}{12}\tilde{\sigma}_3(n) + \left(\frac{1}{4}n - \frac{1}{8}\right)\hat{\sigma}_1(n) - \frac{1}{24}\tilde{\sigma}_1(n).$$

Later, Williams(see [**7**]) computed

$$\sum_{m=1}^{n-1}\hat{\sigma}_1(m)\hat{\sigma}_1(n-m) = \frac{5}{42}\hat{\sigma}_3(n) - \frac{1}{28}\tilde{\sigma}_3(n) - \frac{1}{12}\hat{\sigma}_1(n).$$

In this article we present the binomial convolution sums of two version of (restricted) divisor functions. The first binomial convolution sums(Theorem 1) have a combination of $\sigma_k(n)$, $\sigma_k^*(n)$ and $\sigma_{k,1}(n;2)$. The second binomial convolution sums(Theorem 2) are the generalized versions of the results of Hahn and Williams.

Theorem 1. *For each $k, n \in \mathbb{N}$, we have*

(i)
$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2) = \frac{1}{4}\sigma^*_{2k+1}(n) - 2^{2k-2}\sigma^*_{2k+1}(n/2)$$
$$-\frac{n}{4}\Big(\sigma_{2k-1,1}(n;2) - 2^{2k-1}\sigma^*_{2k-1}(n/2)\Big) + \frac{1}{4}\sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}}E_{2k}(2d) + \frac{2}{2k+1}\sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}}B_{2k+1}(d)$$
$$+\frac{2^{2k-1}}{2k+1}\Big(\sum_{\substack{d|n\\ \frac{n}{d}\equiv 1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big) - \sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}}B_{2k+1}\Big(\frac{2d+1}{2}\Big)\Big),$$

(ii)
$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1}(n-m) = \frac{1}{2}\sigma_{2k+1}(n) + \frac{1}{2}\sigma_{2k+1}(n/2)$$
$$+\frac{3}{4}\sigma_{2k}(n) + \frac{1}{4}\sigma_{2k}(n/2) - \frac{5n}{4}\sigma_{2k-1}(n) - \frac{n}{2}\sigma_{2k-1}(n/2)$$
$$+\frac{1}{2(2k+1)}\Big(3\sum_{d|n}B_{2k+1}(d) + \sum_{d|n/2}B_{2k+1}(d)\Big),$$

(iii)
$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2) = \frac{1}{4}(\sigma_{2k+1}(n) - 2^{2k}\sigma(n/2)) + \frac{1}{4}\sigma_{2k,1}(n;2)$$
$$-\frac{n}{2}\sigma_{2k-1,1}(n;2) + 2^{2k-2}\sigma_{2k-1}(n/2) - \frac{1}{2(2k+1)}\Big(\sum_{d|n}B_{2k+1}(d) + 2^{2k}\sum_{d|\frac{n}{2}}B_{2k+1}(d)$$
$$-2^{2k}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big)\Big),$$

where for a nonnegative integer n, $E_n(x)$ is the Euler polynomial.

Theorem 2. *For each $n, k \in \mathbb{N}$, we have*

$$(i)\quad \sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m)=\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\hat{\sigma}_{2k}(n)$$
$$-\frac{1}{2k+1}\sum_{d|n}B_{2k+1}(d)+\frac{2}{2k+1}\sum_{d|\frac{n}{2}}B_{2k+1}(d),$$

$$(ii)\quad \sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\tilde{\sigma}_{2k-2s-1}(m)\tilde{\sigma}_{2s+1}(n-m)=-\frac{1}{2}\tilde{\sigma}_{2k+1}(n)-2^{2k}\sigma_{2k+1}(n/2)$$
$$-\frac{1}{2}\tilde{\sigma}_{2k}(n)+n\tilde{\sigma}_{2k-1}(n)+2^{2k}\sigma_{2k-1}(n/2)-\frac{1}{2k+1}\Big(\sum_{d|n}B_{2k+1}(d)$$
$$-2^{2k+1}\sum_{d|\frac{n}{2}}B_{2k+1}(d)-2^{2k+1}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big)\Big),$$

$$(iii)\quad \sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\tilde{\sigma}_{2s+1}(n-m)=-2^{2k-1}\hat{\sigma}_{2k+1}(n/2)-\frac{1}{4}(\hat{\sigma}_{2k}(n)+\tilde{\sigma}_{2k}(n))$$
$$+\frac{n}{2}\hat{\sigma}_{2k-1}(n)+2^{2k-1}n\sigma^{*}_{2k-1}(n/2)-2^{2k-1}\sigma_{2k-1}(n/2)+\sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}}E_{2k}(2d)$$
$$+\frac{8}{2k+1}\sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}}B_{2k+1}(d)+\frac{2^{2k+1}}{2k+1}\left(\sum_{\substack{d|n\\ \frac{n}{d}\equiv 1(2)}}B_{2k+1}\left(\frac{d+1}{2}\right)-\sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}}B_{2k+1}\left(\frac{2d+1}{2}\right)\right)$$
$$-\frac{1}{2k+1}\sum_{d|n}B_{2k+1}(d)+\frac{1+2^{2k}}{2k+1}\sum_{d|\frac{n}{2}}B_{2k+1}(d)-\frac{2^{2k}}{2k+1}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}\left(\frac{d+1}{2}\right).$$

2. PROOF OF THEOREM 1

We introduce the identity of Huard, Ou, Spearman and Williams to prove (i) of Theorem 1.

Theorem 3. *([3]) Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$f(a,b,x,y)-f(x,y,a,b)=f(-a,-b,x,y)-f(x,y,a,b)$$

for all $a, b, x, y \in \mathbb{Z}$. Let $n \in \mathbb{N}$. Then

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}(f(a,b,x,-y)-f(a,-b,x,y)+f(a,a-b,x+y,y)$$
$$-f(a,a+b,y-x,y)+f(b-a,b,x,x+y)-f(a+b,b,x,x-y))$$
$$=\sum_{d|n}\sum_{\substack{x\in\mathbb{N}\\ x<d}}(f(0,n/d,x,d)+f(n/d,0,d,x)+f(n/d,n/d,d-x,-x)$$
$$-f(x,x-d,n/d,n/d)-f(x,d,0,n/d)-f(d,x,n/d,0)).$$

Proof of (i) *in Theorem 1.* For $N \in \mathbb{N}$ and $a, i \in \mathbb{Z}$, we define $F_{i,N}(a)$ is 1 if $a \equiv i \pmod{N}$ and is 0 if $a \not\equiv i \pmod{N}$. Also, $F_N(a) := F_{0,N}(a)$. By definition we see $F_{i,N}(-a) = F_{-i,N}(a)$.

We take $f(a,b,x,y) = F_{1,2}(a)F_{1,2}(y)(x-y)^{2k}$ for a positive integer k in Theorem 3. Then the left hand side is equal to

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (F_{1,2}(a)F_{1,2}(y)(x+y)^{2k} - F_{1,2}(a)F_{1,2}(y)(x-y)^{2k} \\
= &\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} F_{1,2}(a)F_{1,2}(y)\sum_{s=0}^{2k}\binom{2k}{s}(x^{2k-s}y^s - x^{2k-s}(-y)^s) \\
= &\ 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} F_{1,2}(a)F_{1,2}(y)x^{2k-2s-1}y^{2s+1} \\
= &\ 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sum_{\substack{x|m \\ m/x\equiv 1(2)}} x^{2k-2s-1}\sum_{\substack{y|n-m \\ y\equiv 1(2)}} y^{2s+1} \\
= &\ 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2).
\end{aligned}$$

The right hand side is $U_1 + U_2 - U_3$, where

$$\begin{aligned}
U_1 &= \sum_{d|n}\sum_{x=1}^{d-1} F_{1,2}(n/d)F_{1,2}(x)(d-x)^{2k}, \qquad U_2 = \sum_{d|n}\sum_{x=1}^{d-1} F_{1,2}(n/d)F_{1,2}(x)d^{2k}, \\
U_3 &= \sum_{d|n}\sum_{x=1}^{d-1} F_{1,2}(x)F_{1,2}(n/d)(n/d)^{2k}.
\end{aligned}$$

From $\sum_{x=1}^{d} x^k = \frac{1}{k+1}(B_{k+1}(d+1) - B_{k+1}(0))$ and $E_k(x) = \frac{2}{k+1}(B_{k+1}(x) -$

$2^{k+1}B_{k+1}(x/2))$ for $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
U_1 &= \sum_{d|n} F_{1,2}(n/d) \sum_{x=1}^{d-1} F_{1,2}(x)(d-x)^{2k} \\
&= \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} \sum_{\substack{x=1 \\ x\equiv 1(2)}}^{d-1} x^{2k} + \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2\nmid d}} \sum_{\substack{x=1 \\ x\equiv 0(2)}}^{d-1} x^{2k} \\
&= \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} \left(\frac{1}{2k+1}(B_{2k+1}(d) - B_{2k+1}(0)) - \frac{2^{2k}}{2k+1}(B_{2k+1}(d/2) - B_{2k+1}(0) \right) \\
&\quad + \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2\nmid d}} \left(\frac{2^{2k}}{2k+1}(B_{2k+1}((d+1)/2) - B_{2k+1}(0)) \right) \\
&= \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} \left(\frac{1}{2k+1}(B_{2k+1}(d) - 2^{2k}B_{2k+1}(0)) - \frac{2^{2k}}{2k+1}(B_{2k+1}(d/2) - B_{2k+1}(0) \right) \\
&\quad + \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2\nmid d}} \left(\frac{2^{2k}}{2k+1}(B_{2k+1}((d+1)/2) - B_{2k+1}(0)) \right) \\
&= \frac{1}{2} \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} E_{2k}(d) + \frac{4}{2k+1} \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} B_{2k+1}(d/2) + \frac{2^{2k}}{2k+1} \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2\nmid d}} B_{2k+1}((d+1)/2).
\end{aligned}$$

Replacing x by $d-x$, U_2 equals to

$$\begin{aligned}
& \sum_{d|n} F_{1,2}(n/d) d^{2k} \sum_{x=1}^{d-1} (d-x) \\
&= \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} d^{2k}(d/2) + \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2\nmid d}} d^{2k}((d-1)/2) \\
&= \frac{1}{2} \sum_{\substack{d|n/2 \\ n/d\not\equiv 1(2)}} (2d)^{2k+1} + \frac{1}{2} \left(\sum_{\substack{d|n \\ n/d\equiv 1(2)}} d^{2k}(d-1) - \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} d^{2k}(d-1) \right) \\
&= \frac{1}{2}\sigma^*_{2k+1}(n) - \frac{1}{2}\sigma^*_{2k}(n) + 2^{2k-1}\sigma^*_{2k}(n/2).
\end{aligned}$$

Now we compute U_3.

$$\begin{aligned}
U_3 &= \sum_{d|n} F_{1,2}(n/d)(n/d)^{2k} \sum_{x=1}^{d-1} F_{1,2}(x) \\
&= \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} (n/d)^{2k} \sum_{\substack{x=1 \\ x\equiv 1(2)}}^{d-1} 1 + \sum_{\substack{d|n \\ d\equiv 1(2) \\ d\nmid n/d}} d^{2k}((n/d-1)/2) \\
&= \frac{n}{2} \sum_{\substack{d|n/2 \\ d\equiv 1(2)}} d^{2k-1} + \frac{1}{2}\left(\sum_{\substack{d|n \\ 2\nmid n/d}} d^{2k}(n/d-1) - \sum_{\substack{d|n \\ 2\nmid n/d \\ 2|d}} d^{2k}(n/d-1) \right) \\
&= \frac{n}{2}\sigma_{2k-1,1}(n/2;2) + \frac{n}{2}(\sigma_{2k-1}^*(n) - 2^{2k}\sigma_{2k-1}^*(n/2) - \frac{1}{2}(\sigma_{2k}^*(n) - 2^{2k+1}\sigma_{2k}^*(n/2)).
\end{aligned}$$

Hence the right hand side is

$$\frac{1}{2} \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} E_{2k}(d) + \frac{4}{2k+1} \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2|d}} B_{2k+1}(d/2) + \frac{2^{2k}}{2k+1} \sum_{\substack{d|n \\ n/d\equiv 1(2) \\ 2\nmid d}} B_{2k+1}((d+1)/2)$$
$$+\frac{1}{2}\sigma_{2k+1}^*(n) - \frac{1}{2}\sigma_{2k}^*(n) + 2^{2k-1}\sigma_{2k}^*(n/2) - \Big(\frac{n}{2}\sigma_{2k-1,1}(n/2;2) + \frac{n}{2}(\sigma_{2k-1}^*(n)$$
$$-2^{2k}\sigma_{2k-1}^*(n/2) - \frac{1}{2}(\sigma_{2k}^*(n) - 2^{2k+1}\sigma_{2k}^*(n/2))\Big).$$

□

Lemma 4. *For $k, n \in \mathbb{N}$, we have*

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m) = \frac{1}{2}\sigma_{2k+1}(n/2) + \frac{1}{4}\sigma_{2k}(n) + \frac{1}{4}\sigma_{2k}(n/2)$$
$$-\frac{n}{4}\sigma_{2k-1}(n) - \frac{n}{2}\sigma_{2k-1}(n/2) + \frac{1}{2(2k+1)}\left(\sum_{d|n} B_{2k+1}(d) + \sum_{d|n/2} B_{2k+1}(d)\right).$$

Proof. Let $k, n \in \mathbb{N}$. Then

$$\begin{aligned}
&\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}^*(m)\sigma_{2s+1}^*(n-m) \\
= &\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \Big(\sigma_{2k-2s-1}(m) - \sigma_{2k-2s-1}(m/2))(\sigma_{2s+1}(n-m) - \sigma_{2k+1}((n-m)/2)\Big) \\
= &\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m) + \sigma_{2s-2s-1}(m/2)\sigma_{2s+1}((n-m)/2) \\
&\qquad -2\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\Big).
\end{aligned}$$

From (1.1) and (1.2), we have

$$
\begin{aligned}
&2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\\
= &\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)+\sigma_{2s-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\
&\qquad -\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m)\Big)\\
= &\ \sigma_{2k+1}(n/2)+\frac{1}{2}\sigma_{2k}(n)+\frac{1}{2}\sigma_{2k}(n/2)-\frac{n}{2}\sigma_{2k-1}(n)-n\sigma_{2k-1}(n/2)\\
&+\frac{1}{2k+1}\left(\sum_{d|n}B_{2k+1}(d)+\sum_{d|n/2}B_{2k+1}(d)\right).
\end{aligned}
$$

□

Proof of (ii) in Theorem 1. By (1.1) and Lemma 4, we have

$$
\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
= &\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-\sigma_{2k-2s-1}(m/2)\Big)\sigma_{2s+1}(n-m)\\
= &\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)-\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\Big)\\
= &\ \frac{1}{2}\sigma_{2k+1}(n)+\frac{1}{2}\sigma_{2k+1}(n/2)+\frac{3}{4}\sigma_{2k}(n)+\frac{1}{4}\sigma_{2k}(n/2)-\frac{5n}{4}\sigma_{2k-1}(n)-\frac{n}{2}\sigma_{2k-1}(n/2)\\
&+\frac{1}{2(2k+1)}\Big(3\sum_{d|n}B_{2k+1}(d)+\sum_{d|n/2}B_{2k+1}(d)\Big).
\end{aligned}
$$

□

Lemma 5. *For $k, n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}2^{2k-2s-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)=\frac{1}{4}\sigma_{2k+1}(n)+2^{2k-2}\sigma_{2k+1}(n/2)\\
&+\frac{1}{4}\sigma_{2k}(n)+2^{2k-2}\sigma_{2k}(n/2)-\frac{n}{2}\sigma_{2k-1}(n)-(n+1)2^{2k-2}\sigma_{2k-1}(n/2)\\
&+\frac{1}{2(2k+1)}\Big(\sum_{d|n}B_{2k+1}(d)+2^{2k}\sum_{d|n/2}B_{2k+1}(d)-2^{2k}\sum_{\substack{d|n\\ d\equiv 1(2)}}((d+1)/2)\Big).
\end{aligned}
$$

Proof. Let $k, n \in \mathbb{N}$. Then

$$
\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-2^{2k-2s-1}\sigma_{2k-2s-1}(m/2)\Big)\\
&\qquad\times\Big(\sigma_{2s+1}(n-m)-2^{2s+1}\sigma_{2s+1}((n-m)/2)\Big)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)-2^{2k}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\
&\qquad -2\cdot 2^{2k-2s-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\Big).
\end{aligned}
$$

Hence we obtain the result for $\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}2^{2k-2s-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)$ from (1.1), (1.3) and (iii) in Theorem 1. □

Proof of (iii) in Theorem 1. From (1.1) and Lemma 5, we obtain

$$
\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\Big(\sigma_{2s+1}(n-m)-2^{2s+1}\sigma_{2s+1}((n-m)/2)\Big)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)-2^{2k-2s-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\Big)\\
=&\frac{1}{4}(\sigma_{2k+1}(n)-2^{2k}\sigma(n/2))+\frac{1}{4}\sigma_{2k,1}(n;2)-\frac{n}{2}\sigma_{2k-1,1}(n;2)+2^{2k-2}\sigma_{2k-1}(n/2)\\
&-\frac{1}{2(2k+1)}\Big(\sum_{d|n}B_{2k+1}(d)+2^{2k}\sum_{d|\frac{n}{2}}B_{2k+1}(d)-2^{2k}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big)\Big).
\end{aligned}
$$

□

3. PROOF OF THEOREM 2

Proof of (i) in Theorem 2. From (1.1) and Lemma 4, we obtain

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-2\sigma_{2s-2s-1}(m/2)\Big)\Big(\sigma_{2s+1}(n-m)-2\sigma_{2s+1}((n-m)/2)\Big)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)+4\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\
&\qquad -4\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\Big)\\
=&\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\hat{\sigma}_{2k}(n)-\frac{1}{2k+1}\sum_{d|n}B_{2k+1}(d)+\frac{2}{2k+1}\sum_{d|\frac{n}{2}}B_{2k+1}(d).
\end{aligned}$$

□

Proof of (ii) in Theorem 2. From (1.1) and Lemma 5, we have

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\tilde{\sigma}_{2k-2s-1}(m)\tilde{\sigma}_{2s+1}(n-m)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-2^{2k-2s}\sigma_{2k-2s-1}(m/2)\Big)\\
&\qquad\times\Big(\sigma_{2s+1}(n-m)-2^{2s+2}\sigma_{2s+1}((n-m)/2)\Big)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)+2^{2s+2}\sigma_{2s-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\
&-4\cdot 2^{2k-2s-1}\sigma_{2s-2s-1}(m)\sigma_{2s+1}(n-2m)\Big)\\
=&-\frac{1}{2}\tilde{\sigma}_{2k+1}(n)-2^{2k}\sigma_{2k+1}(n/2)-\frac{1}{2}\tilde{\sigma}_{2k}(n)+n\tilde{\sigma}_{2k-1}(n)+2^{2k}\sigma_{2k-1}(n/2)\\
&-\frac{1}{2k+1}\Big(\sum_{d|n}B_{2k+1}(d)-2^{2k+1}\sum_{d|\frac{n}{2}}B_{2k+1}(d)-2^{2k+1}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big)\Big).
\end{aligned}$$

□

Lemma 6. *For $k,n\in\mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}2^{2s+1}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)=\frac{1}{4}\sigma_{2k+1}(n/2)+2^{2k-2}\sigma_{2k+1}(n/4)\\
&+\frac{1}{4}\sigma_{2k}(n/2)+2^{2k-2}\sigma_{2k}(n/2)-\frac{n}{2}\sigma_{2k-1}(n/2)-2^{2k-2}\sigma_{2k-1}(n/2)-2^{2k-2}n\sigma_{2k-1}(n/4)\\
&+\frac{2}{2k+1}\sum_{\substack{d|n/2\\ \frac{n/2}{d}\equiv 1(2)}}B_{2k+1}(d)+\frac{2^{2k-1}}{2k+1}\Big(B_{2k+1}((d+1)/2)-\sum_{\substack{d|n/2\\ \frac{n/2}{d}\equiv 1(2)}}B_{2k+1}((2d+1)/2\Big)\\
&+\frac{1}{2(2k+1)}\Big((2^{2k}+1)\sum_{d|n/2}B_{2k+1}(d)-2^{2k}\sum_{\substack{d|n\\ d\equiv 1(2)}}B_{2k+1}((d+1)/2)\Big)+\frac{1}{4}\sum_{\substack{d|n/2\\ \frac{n/2}{d}\equiv 1(2)}}E_{2k}(2d).
\end{aligned}$$

Proof. From (i) in Theorem 1, we have

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-\sigma_{2k-2s-1}(m/2)\Big)\\
&\times\Big(\sigma_{2s+1}(n-m)-2^{2s+1}\sigma_{2s+1}((n-m)/2)\Big).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}2^{2s+1}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma^*_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2)-\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
&+\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)+2^{2k-2s-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)\Big).
\end{aligned}$$

From (1.1), Lemma 4 and Lemma 5, we obtain the result for convolution sum of lemma. □

Proof of (iii) in Theorem 2. Let $k,n\in\mathbb{N}$. Then we have

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat\sigma_{2k-2s-1}(m)\tilde\sigma_{2s+1}(n-m)\\
=&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)-2\sigma_{2s-2s-1}(m)\sigma_{2s+1}(n-2m)\\
&-2^{2k-2s}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-2m)+2^{2s+3}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\Big)\\
=&-2^{2k-1}\hat\sigma_{2k+1}(n/2)-\frac{1}{4}(\hat\sigma_{2k}(n)+\tilde\sigma_{2k}(n))+\frac{n}{2}\hat\sigma_{2k-1}(n)+2^{2k-1}n\sigma^*_{2k-1}(n/2).
\end{aligned}$$

From (1.1), Lemma 4, Lemma 5 and Lemma 6, we obtain the result of (iii) in Theorem 2. □

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REFERENCES

1. M. Besge, *Extrait d'une lettre de M. Besge á M. Liouville*, J. Math. Pures Appl. **7**(1862), 256.
2. H. Hahn, *Eisenstein series, analogues of the Roger-Ramanujan functions, and partition identities*, Ph. D. thesis, University of Illinois at Urbana-Champaign, 2004.
3. J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams, *Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions, Number theory for the millennium, II*, (2002), 229–274.
4. D. Kim and A. Bayad, *Convolution identities for twisted Eisenstein series and twisted divisor functions*, Fixed Point Theory and Applications, **2013**:81, (2013).
5. D. Kim, A. Bayad and J. Park, *Euler Polynomials and Combinatoric Convolution Sums of Divisor Functions with Even Indices*, Abstract and Applied Analysis, Art. ID 289187, 6pp., (2014).
6. J. Liouville, Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres, N. Math. Pures Appl. **3**(1858), 143–152.
7. K. Williams, "Number Theory in the Spirit of Liouville", Cambridge University Press, 2011.

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