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APPROXIMATION PROPERTIES OF JAIN-APPELL OPERATORS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In this paper, we introduce the Jain-Appell operators by applying Gamma transform to the Jakimovski-Leviatan operators. In their special cases they include not only the Jain-Pethe operators, but also new families of operators, where we call them Appell-Baskakov and Appell-Lupaş operators, since their special cases contain Baskakov and Lupaş operators, respectively. We investigate their weighted approximation properties and compute the error of approximation by using certain Lipschitz class functions. Furthermore, we obtain their $A-$statistical approximation property.

1. INTRODUCTION

The well-known discrete type linear positive operators for the approximation to a function on $[0,\infty)$ are the Szász-Mirakjan operators [**42**], defined by

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

where f is sufficiently nice function which guarantees the convergence of the above sum and belonging to a subspace of $C[0,\infty)$,the space of all continuous functions on $[0,\infty)$. These operators have been intensively investigated especially in the recent years (see [**4**],[**6**],[**7**],[**16**],[**17**],[**35**],[**36**] and [**37**]).

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Jain and Pethe [**26**], introduced the generalization of these operators by

$$\mathbf{S}_n^{(\alpha)}(f;x)=\frac{1}{(1+n\alpha)^{(x/\alpha)}}\sum_{k=0}^{\infty}\mathbf{f}\left(\frac{k}{n}\right)\left(\frac{n}{1+n\alpha}\right)^k\frac{\mathbf{x}^{(k,-\alpha)}}{k!}, \tag{1}$$

where

$$\begin{aligned}\mathbf{x}^{(k,-\alpha)} &= x(x+\alpha)(x+2\alpha)\cdots(x+(k-i-1)\alpha), \quad (k\in\mathbb{N}:=\{1,2,\cdots\})\\ \mathbf{x}^{(0,-\alpha)} &= 1.\end{aligned}$$

Here the function f can be taken as above. These operators and their different variants have been considered by several authors in [**2**],[**11**],[**12**],[**20**],[**21**],[**24**],[**31**] and [**39**]. As it is mentioned by many authors (see for instance [**2**], [**33**]), the operators in (1) are the gamma transform of the Szasz-Mirakyan operators. More precisely

$$\mathbf{S}_n^{(\alpha)}(f;x)=\frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^{\infty}e^{-t}t^{(x/\alpha)-1}S_n(f;\alpha t)\,dt, \quad \alpha>0.$$

On the other hand, the Szasz-Mirakyan operators are one of the main representative of the Jakimovski Leviatan operators [**27**] which are defined by

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \; x\geq 0, \tag{3}$$

where

$$g(u)=\sum_{n=0}^{\infty}a_nu^n$$

is an analytic function in the disc $|u|<r$ $(r>1)$ with $g(1)\neq 0$ and

$$p_k(x)=\sum_{i=0}^{k}a_i\frac{x^{k-i}}{(k-i)!}$$

are the corresponding Appell polynomials generated by

$$g(u)e^{ux}=\sum_{k=0}^{\infty}p_k(x)u^k.$$

The operators in (3) are linear. On the other hand, Wood [**43**] showed that they are positive on $[0,\infty)$ if and only if $\frac{a_n}{g(1)}\geq 0$ for all $n\in\mathbb{N}$. These operators and their variants have attracted the attention of many mathematicians (see [**1**],[**3**],[**8**],[**10**],[**28**] and [**41**]).

In the present paper, we consider the Gamma transform of the Jakimovski-Leviatan operators, which we will call the transformed operator as Jain-Appell operators. In section 2, we construct our operators and introduce some special cases, which we will call Baskakov-Appell and Lupaş-Appell operators. We also compute the first few moments of the Jain-Appell operators and investigate their transformation properties. In section 3, we compute the rate of convergence of these operators by using the modulus of continuity and state the Korovkin type approximation theorem. We further introduce the Lipschitz type function space, which is the modification of the previous versions introduced in [**42**] and [**37**], and obtain a global estimate in this space. Finally, A-statistical approximation theorem is proved for the Jain-Appell operators.

2. CONSTRUCTION AND TRANSFORMATION PROPERTIES OF THE OPERATORS

In this section, we construct the Jain-Appell operators by taking the gamma transform of the Jakimovski-Leviatan operators and introduce new families of linear positive operators as special cases. We compute the first few moments and investigate the transformation properties of these operators.

For fixed $\alpha > 0$, $\forall x \in (0, \infty)$ and sufficiently nice function such that the series in (3) is uniformly convergent, direct calculations yield

$$\begin{aligned} P_n^{(\alpha)}(f;x) &= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)} \int_0^\infty e^{-t} t^{(x/\alpha)-1} P_n(f;\alpha t)\,dt \\ &= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)} \int_0^\infty e^{-t} t^{(x/\alpha)-1} \frac{e^{-n\alpha t}}{g(1)} \sum_{k=0}^\infty p_k(n\alpha t) f\left(\frac{k}{n}\right) dt \\ &= \frac{1}{g(1)\Gamma\left(\frac{x}{\alpha}\right)} \sum_{k=0}^\infty f\left(\frac{k}{n}\right) \sum_{i=0}^k \frac{a_i (n\alpha)^{k-i}}{(k-i)!} \int_0^\infty e^{-(1+n\alpha)t} t^{(x/\alpha)+k-i-1} dt \\ &= \frac{1}{g(1)\Gamma\left(\frac{x}{\alpha}\right)} \sum_{k=0}^\infty f\left(\frac{k}{n}\right) \sum_{i=0}^k \frac{a_i (n\alpha)^{k-i}}{(k-i)!} \frac{\Gamma\left(\frac{x}{\alpha}+k-i\right)}{(1+n\alpha)^{(x/\alpha)+k-i}}. \end{aligned}$$

On the other hand since

$$\begin{aligned} \frac{\Gamma\left(\frac{x}{\alpha}+k-i\right)}{\Gamma\left(\frac{x}{\alpha}\right)} &= \left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}+1\right)\cdots\left(\frac{x}{\alpha}+k-i-1\right) \\ &= \frac{x(x+\alpha)(x+2\alpha)\cdots(x+(k-i-1)\,\alpha)}{\alpha^{k-i}} = \frac{x^{(k-i,-\alpha)}}{\alpha^{k-i}} \end{aligned}$$

with $x^{(0,-\alpha)} = 1$, we get the following family of linear positive operators:

$$\mathbf{P}_n^{(\alpha)}(f;x) = \frac{1}{(1+n\alpha)^{(x/\alpha)}\, g(1)} \sum_{k=0}^\infty \mathbf{f}\left(\frac{k}{n}\right) \widetilde{p_k}^{(\alpha)}(x;n) \tag{4}$$

where

$$\widetilde{p_k}^{(\alpha)}(x;n)=\sum_{i=0}^{k}\frac{a_i}{(k-i)!}\left(\frac{n}{1+n\alpha}\right)^{k-i}\mathbf{x}^{(k-i,-\alpha)}.$$

We call the operators in (4) as the Jain-Appell operators.

Remark 1. *Taking $g(x)=1$ in (4), we get the Jain-Pethe operators defined in (1). Letting $\alpha\to 0^{+}$ in (4), we recover the Jakimovski Leviatan operators given in (3). Furthermore, choosing $g(x)=1$, we get the well known Szasz-Mirakyan operators (1).*

Remark 2. *Choosing $\alpha=\alpha_n(x)=\frac{x}{n}$, where $x\geq 0$ and $n\in\mathbb{N}$ in (4), we have*

(5)

$$\mathbb{B}_n(f;x):=\mathbf{P}_n^{(\frac{x}{n})}(f;x)=\frac{1}{(1+x)^n g(1)}\sum_{k=0}^{\infty}\mathbf{f}\left(\frac{k}{n}\right)\sum_{i=0}^{k}a_i\binom{n+k-i-1}{n-1}\left(\frac{x}{1+x}\right)^{k-i}$$

which we call them as Appell-Baskakov operators. Further choosing $g(x)=1$ in (5), we recover the usual Baskakov operators

$$B_n(f;x)=\frac{1}{(1+x)^n}\sum_{k=0}^{\infty}\mathbf{f}\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^{k}.$$

.

Remark 3. *Choosing $\alpha=\alpha_n(x)=\frac{1}{n}$ $(n\in\mathbb{N})$ in (1), we have*

$$\mathbb{L}_n(f;x):=\mathbf{P}_n^{(\frac{1}{n})}(f;x)=\frac{1}{2^{nx}g(1)}\sum_{k=0}^{\infty}\mathbf{f}\left(\frac{k}{n}\right)\sum_{i=0}^{k}a_i\frac{(nx)_{k-i}}{2^{k-i}(k-i)!}\tag{6}$$

where $(a)_k$ is known as the Pochhammer's symbol defined as

$$(a)_0=1\ (a\neq 0)$$
$$(a)_k=a(a+1)\cdots(a+k-1)\quad k\in\mathbb{N}\ .$$

*We call these operators as Appell-Lupaş operators. Further choosing $g(x)=1$ in (6), we recover the usual Lupaş operators [**30**]*

$$L_n(f;x)=\frac{1}{2^{nx}}\sum_{k=0}^{\infty}\mathbf{f}\left(\frac{k}{n}\right)\frac{(nx)_k}{2^k k!}.$$

*These operators and their variants have recently been studied in [**5**],[**18**] and [**25**].*

We start by obtaining the first few moments of the operators defined in (4).

Lemma 4. *For the first three moments and central moments, we have*

$$
\begin{aligned}
P_n^{(\alpha)}(1;x) &= 1,\\
P_n^{(\alpha)}(y;x) &= x+\frac{g'(1)}{ng(1)},\\
P_n^{(\alpha)}\left(y^2;x\right) &= x^2+\alpha x+(2\frac{g'(1)}{g(1)}+1)\frac{x}{n}+\frac{g''(1)+g'(1)}{g(1)n^2},\\
P_n^{(\alpha)}\left((y-x)^2;x\right) &= \alpha x+\frac{x}{n}+\frac{g''(1)+g'(1)}{g(1)n^2}.
\end{aligned}
$$

Proof. Using the fact that

$$
\begin{aligned}
P_n(1;x) &= \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)=1,\\
P_n(y;x) &= \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}\frac{k}{n}p_k(nx)=x+\frac{g'(1)}{g(1)},\\
P_n(y^2;x) &= \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^2p_k(nx)=x^2+\frac{x}{n}(\frac{2g'(1)}{g(1)}+1)+\frac{g''(1)+g'(1)}{n^2g(1)},
\end{aligned}
$$

we get

$$
P_n^{(\alpha)}(1;x)=\frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^{\infty}e^{-t}t^{(x/\alpha)-1}P_n(1;\alpha t)\,dt=1
$$

and

$$
\begin{aligned}
P_n^{(\alpha)}(y;x) &= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^{\infty}e^{-t}t^{(x/\alpha)-1}P_n(y;\alpha t)\,dt\\
&= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^{\infty}e^{-t}t^{(x/\alpha)-1}\left[\alpha t+\frac{g'(1)}{ng(1)}\right]dt\\
&= \alpha\frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^{\infty}e^{-t}t^{x/\alpha+1-1}dt+\frac{g'(1)}{ng(1)}\\
&= \alpha\frac{\Gamma\left(\frac{x}{\alpha}+1\right)}{\Gamma\left(\frac{x}{\alpha}\right)}+\frac{g'(1)}{ng(1)}=x+\frac{g'(1)}{ng(1)}.
\end{aligned}
$$

In a similar manner

$$
\begin{aligned}
P_n^{(\alpha)}\left(y^2;x\right) &= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^\infty e^{-t}t^{(x/\alpha)-1}P_n\left(y^2;\alpha t\right)dt \\
&= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^\infty e^{-t}t^{(x/\alpha)-1}\left[\alpha^2t^2+\frac{\alpha t}{n}(2g'(1)+g(1))+\frac{g''(1)+g'(1)}{n^2}\right]dt \\
&= \frac{\alpha^2}{\Gamma\left(\frac{x}{\alpha}\right)}\int_0^\infty e^{-t}t^{(x/\alpha)+2-1}dt+\frac{x}{n}(2g'(1)+g(1))+\frac{g''(1)+g'(1)}{n^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha^2\Gamma\left(\frac{x}{\alpha}+2\right)}{\Gamma\left(\frac{x}{\alpha}\right)}+\frac{x}{n}(2g'(1)+g(1))+\frac{g''(1)+g'(1)}{n^2} \\
&= \alpha^2\left(\left(\frac{x}{\alpha}\right)^2+\frac{x}{\alpha}\right)+\frac{x}{n}(2g'(1)+g(1))+\frac{g''(1)+g'(1)}{n^2} \\
&= x^2+\alpha x+\frac{x}{n}(2g'(1)+g(1))+\frac{g''(1)+g'(1)}{n^2}.
\end{aligned}
$$

Finally, using linearity of the operators, we get

$$
\begin{aligned}
&P_n^{(\alpha)}\left((y-x)^2;x\right) \\
&= P_n^{(\alpha)}(y^2,x)-2xP_n^{(\alpha)}(y,x)+x^2P_n^{(\alpha)}(1,x) \\
&= x^2+\alpha x+\frac{x}{n}(2\frac{g'(1)}{g(1)}+1)+\frac{g''(1)+g'(1)}{g(1)n^2}-2x\left(x+\frac{g'(1)}{ng(1)}\right)+x^2 \\
&= \alpha x+\frac{x}{n}+\frac{g''(1)+g'(1)}{g(1)n^2}.
\end{aligned}
$$

Whence the result. □

Remark 5. *It should be remarked that, if the determining function g is chosen such that $g'(1)=0$, then the corresponding operators preserve the linear functions. On the other hand, for the sub family of operators such that their determining function satisfy $g'(1)=g''(1)=0$, the second central moment $P_n^{(\alpha)}\left((y-x)^2;x\right)$, which determines the error is the same.*

In the present paper, we work in the following function spaces:
Let $C[0,\infty)$ denote the space of continuous functions defined on $[0,\infty)$, $B_{1+x^2}[0,\infty)$ denote the set of functions f satisfying $|f(x)|\le A_f(1+x^2)$, where $x\in[0,\infty)$ and A_f is a positive constant depending only on f. Also let

$$C_{1+x^2}[0,\infty):=B_{1+x^2}[0,\infty)\cap C[0,\infty)$$

and

$$E_{1+x^2} := \left\{ f \in C_{1+x^2}[0,\infty) : \exists \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty \right\},$$

endowed with the norm

$$\|f\|_{1+x^2} := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

In considering the special cases of the operators (4), we observe that α is chosen to be certain function sequences. Since we will work in the space E_{1+x^2}, we impose the following condition on α.
(1°) $\alpha := (\alpha_n(x))$ such that $0 \leq \frac{x\alpha_n(x)}{1+x^2} \leq c_n \leq M$ for all $x \in [0,\infty)$ and $n \in \mathbb{N}$, where (c_n) is positive numerical sequence and M is a positive constant.

Now we continue with showing that the operators $\mathbf{P}_n^{(\alpha)}(f;x)$ defined in (4) is bounded in E_{1+x^2}.

Lemma 6. *There exists a constant C such that, for $\omega(x) = \left(1+x^2\right)^{-1}$, the inequality*

$$\omega(x)\mathbf{P}_n^{(\alpha_n(x))}(\frac{1}{\omega};x) \leq C$$

holds for all $x \in [0,\infty)$ and $n \in \mathbb{N}$. Furthermore, for all E_{1+x^2}, we have

$$\left\|\mathbf{P}_n^{(\alpha_n(x))}(f;x)\right\|_{1+x^2} \leq C \|f\|_{1+x^2}.$$

Proof. We can write by using Lemma 4 and (1°) that

$$\omega(x)\mathbf{P}_n^{(\alpha_n(x))}(\frac{1}{\omega};x) = \frac{1}{1+x^2}\left[\mathbf{P}_n^{(\alpha_n(x))}(1;x) + \mathbf{P}_n^{(\alpha_n(x))}(t^2;x)\right]$$
$$\leq \frac{1}{1+x^2}\left[1+x^2+x\alpha_n(x)+\frac{x}{n}(2g'(1)+g(1))+\frac{g''(1)+g'(1)}{n^2}\right] \leq C,$$

Which is the first inequality. Since

$$\omega(x)\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)\right| = \omega(x)\left|\mathbf{P}_n^{(\alpha_n(x))}(\omega\frac{f}{\omega};x)\right|$$
$$\leq \|f\|_{1+x^2}\,\omega(x)\mathbf{P}_n^{(\alpha_n(x))}(\frac{1}{\omega};x) \leq C\|f\|_{1+x^2}.$$

The second inequality follows by taking supremum over $x \in [0,\infty)$ in the above expression. □

3. RATE OF CONVERGENCE

Now we compute the rate of convergence of the operators in terms of the usual modulus of continuity, which is defined on the closed interval $[0, B]$ by

$$\omega_\theta(f,\delta) = \sup_{\substack{|t-x|\le\delta \\ x,t\in[0,\theta]}} |f(t) - f(x)|.$$

Note that, for a function $f \in E_{1+x^2}$, we have $\lim_{\delta\to\infty} \omega_\theta(f,\delta) = 0$.

Theorem 7. *Let $f \in E_{1+x^2}$ and $\omega_{\theta+1}(f,\delta)$ $(\theta > 0)$ be its modulus of continuity on the finite interval $[0, \theta+1] \subset [0,\infty)$. Then*

$$\left\| \mathbf{P}_n^{(\alpha_n(x))}(f;x) - f(x) \right\|_{C[0,\theta]} \le 6M_f \left(1+\theta^2\right)^2 \delta_n + 2\omega_{\theta+1}(f,(\delta_n)^{1/2}).$$

where

$$\delta = \delta_n = \left[c_n + \frac{\theta}{n} + \left| \frac{g''(1) + g'(1)}{g(1)n^2} \right| \right]$$

and M_f is an absolute constant depending on f and (c_n) is the same as in condition (1°).

Proof. Let $x \in [0,\theta]$ and $y \le \theta + 1$. From the well known property of the modulus of continuity, we can write for any $\delta > 0$ that

$$|f(t) - f(x)| \le \omega_{B+1}(f, |y-x|) \le \left(1 + \frac{|y-x|}{\delta}\right) \omega_{\theta+1}(f,\delta). \tag{7}$$

Now, let $x \in [0,\theta]$ and $y > \theta + 1$. In this case, since $y - x > 1$, we have

$$|f(y) - f(x)| \le M_f(1+x^2+y^2) \le M_f(2+3x^2+2(y-x)^2) \le 6M_f \left(1+\theta^2\right)(y-x)^2 \tag{8}$$

where M_f is a positive constant depending on f. Combining (7) and (8), we get for all $x \in [0,\theta]$ and $y \ge 0$ that

$$|f(y) - f(x)| \le 6M_f \left(1+\theta^2\right)(y-x)^2 + \left(1 + \frac{|y-x|}{\delta}\right) \omega_{\theta+1}(f,\delta).$$

Hence,

$$\begin{aligned} &\left| \mathbf{P}_n^{(\alpha_n(x))}(f;x) - f(x) \right| \\ &\le 6M_f \left(1+\theta^2\right) \mathbf{P}_n^{(\alpha)}((y-x)^2;x) + \left(1 + \frac{\mathbf{P}_n^{(\alpha)}(|y-x|\,;x)}{\delta}\right) \omega_{\theta+1}(f,\delta). \end{aligned}$$

Using the Cauchy-Schwarz inequality and Lemma 4, we get by using (1°) that

$$
\begin{aligned}
&\left|\mathbf{P}_n^{(\alpha)}(f;x) - f(x)\right| \\
&\leq 6M_f\left(1+\theta^2\right)P_n^{(\alpha)}\left((y-x)^2;x\right) + \left(1 + \frac{\left[P_n^{(\alpha)}\left((y-x)^2;x\right)\right]^{1/2}}{\delta}\right)\omega_{\theta+1}(f,\delta) \\
&\leq 6M_f\left(1+\theta^2\right)^2\left[c_n + \frac{\theta}{n} + \left|\frac{g''(1)+g'(1)}{g(1)n^2}\right|\right] \\
&+ \left(1 + \frac{\left[c_n + \frac{\theta}{n} + \left|\frac{g''(1)+g'(1)}{g(1)n^2}\right|\right]^{1/2}}{\delta}\right)\omega_{\theta+1}(f,\delta) \\
&\leq 6M_f\left(1+\theta^2\right)^2\delta_n + 2\omega_{\theta+1}(f,(\delta_n)^{1/2}),
\end{aligned}
$$

where

$$
\delta = \delta_n = \left[c_n + \frac{\theta}{n} + \left|\frac{g''(1)+g'(1)}{g(1)n^2}\right|\right].
$$

Whence the result. □

Corollary 8. *If the sequence* (c_n) *converges to* 0*, then for all* $f \in E_{1+x^2}$*, the sequence* $\left(\mathbf{P}_n^{(\alpha)}(f;x)\right)$ *converges uniformly to* f *on* $[0,\theta]$ *(*$\theta > 0$*).*

In 1950, in order to state global estimate for the Szasz-Mirakyan operators, Otto Szasz [**42**] considered the following Lipschitz-type space

$$
Lip_M^{(*)}(\lambda) := \left\{ f \in C_B[0,\infty) : |f(y)-f(x)| \leq M\frac{|y-x|^\lambda}{(y+x)^{\lambda/2}};\ x,y \in (0,\infty)\right\},
$$

where M is any positive constant, $0 < \lambda \leq 1$ and $C_B[0,\infty)$ denote the space of bounded continuous functions on $[0,\infty)$. A two parameter Lipschitz-type space, which provides a global approximation result for a family of linear positive operators including Szasz-Mirakyan, Baskakov, Post-Widder and Stancu operators, was introduced in [**38**] as follows:

$$
Lip_M^{(a,b)}(\lambda) := \left\{ f \in C_B[0,\infty) : |f(y)-f(x)| \leq M\frac{|y-x|^\lambda}{(y+ax^2+bx)^{\lambda/2}};\ x,y \in (0,\infty)\right\},
$$

where a and b are positive fixed parameters.

In the present investigation, we consider the following Lipschitz type space

$$
\widetilde{Lip}_M(\lambda) := \left\{ f \in C_B[0,\infty) : |f(y)-f(x)| \leq M\frac{|y-x|^\lambda}{(y+x^2+1)^{\lambda/2}};\ x,y \in (0,\infty)\right\},
$$

where M is any positive constant and $0<\lambda\le 1$. We have the following approximation result.

Theorem 9. *For any $f\in \widetilde{Lip}_M(\lambda)$, $\lambda\in(0,1]$,, we have*

$$\begin{aligned}&\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)-f(x)\right|\\&\le M\left\{c_n+\frac{1}{n}+\frac{g''(1)+g'(1)}{g(1)n^2}\right\}^{\lambda/2}\end{aligned}$$

uniformly for $x\in(0,\infty)$ as $n\to\infty$.

Proof. Let $\lambda=1$. Then, for $f\in\widetilde{Lip}_M(1)$ and $x\in(0,\infty)$, we have

$$\begin{aligned}\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)-f(x)\right|&\le\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\left|f(\frac{k}{n})-f(x)\right|\widetilde{p_k}^{(\alpha_n(x))}(x;n)\\&\le\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\frac{\left|\frac{k}{n}-x\right|}{\left(\frac{k}{n}+x^2+1\right)^{1/2}}\widetilde{p_k}^{(\alpha_n(x))}(x;n)\\&\le\frac{M}{(x^2+1)^{1/2}(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|\widetilde{p_k}^{(\alpha_n(x))}(x;n).\end{aligned}$$

Applying Cauchy-Schwarz inequality, and using (1°) we obtain

$$\begin{aligned}&\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)-f(x)\right|\\&\le\frac{M}{(x^2+1)^{1/2}}\sqrt{\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2\widetilde{p_k}^{(\alpha_n(x))}(x;n)}\\&=\frac{M}{(x^2+1)^{1/2}}\left\{x\alpha_n(x)+\frac{x}{n}+\frac{g''(1)+g'(1)}{g(1)n^2}\right\}^{1/2}\\&\le M\left\{c_n+\frac{1}{n}+\frac{g''(1)+g'(1)}{g(1)n^2}\right\}^{1/2}.\end{aligned}$$

Now assume that $\lambda\in(0,1)$. Then, for $f\in\widetilde{Lip}_M(\lambda)$ and $x\in(0,\infty)$, we have

$$\begin{aligned}\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)-f(x)\right|&\le\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\left|f(\frac{k}{n})-f(x)\right|\widetilde{p_k}^{(\alpha_n(x))}(x;n)\\&\le\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\frac{\left|\frac{k}{n}-x\right|^{\lambda}}{\left(\frac{k}{n}+x^2+1\right)^{\lambda/2}}\widetilde{p_k}^{(\alpha_n(x))}(x;n)\\&\le\frac{M}{(x^2+1)^{\lambda/2}(1+n\alpha_n(x))^{(x/\alpha_n(x))}g(1)}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{\lambda}\widetilde{p_k}^{(\alpha_n(x))}(x;n).\end{aligned}$$

Taking $p=\frac{1}{\lambda}$ and $q=\frac{1}{1-\lambda}$, and applying the Hölder inequality, we have

$$\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)-f(x)\right| \le \frac{M}{(x^2+1)^{\lambda/2}}\left(\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}\,g(1)}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|\widetilde{p_k}^{(\alpha_n(x))}(x;n)\right)^{\lambda}.$$

Finally, by Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathbf{P}_n^{(\alpha_n(x))}(f;x)-f(x)\right| &\le \frac{M}{(x^2+1)^{\lambda/2}}\left(\frac{1}{(1+n\alpha_n(x))^{(x/\alpha_n(x))}\,g(1)}\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2\widetilde{p_k}^{(\alpha_n(x))}(x;n)\right)^{\lambda/2}\\ &\le \frac{M}{(x^2+1)^{\lambda/2}}\left\{x\alpha_n(x)+\frac{x}{n}+\frac{g''(1)+g'(1)}{g(1)n^2}\right\}^{\lambda/2}.\end{aligned}$$

Whence the result. □

4. A-STATISTICAL CONVERGENCE

The main aim of this section is to give the A-statistical approximation properties of the operators $\mathbf{P}_n^{(\alpha)}(f;x)$. The origin of the method depends on the non-negative regular summability matrix $A=(a_{jk})$. It was Freedman and Sember [**22**], who defined the $A-$density of $K\subset\mathbb{N}$ by

(1) $$\delta_A(K)=\lim_{j\to\infty}\sum_{k\in K}a_{j,k},$$

provided that the limit exists. The sequence $x=(x_n)$ is said to be $A-$statistically convergent to l and denoted by

$$st_A-\lim_n x_n=l,$$

if for every $\varepsilon>0$, $\delta_A\{n\in\mathbb{N}:|x_n-l|\ge\varepsilon\}=0$ (see[**19**], [**40**]). In the particular case when $A=C_1$, the Cesaro matrix of order one, $A-$statistical convergence coincides with the statistical convergence [**23**],[**32**]. Choosing $A=I$, the identity matrix then $A-$statistical convergence reduces to the ordinary convergence. It was Kolk [**29**] who proved that in the case of $\lim_j\max_n|a_{j,n}|=0$, A-statistical convergence is stronger than ordinary convergence.

In this section, in addition to (1°) we impose the following condition:

(2°) $st_A-\lim_n c_n=0$.

Such a sequence $(\alpha_n(x))_{n\in\mathbb{N}}$ satisfying the above conditions, can be given as follows: Take $A = C_1$, and define

$$\alpha_n(x) := \begin{cases} 5, & \text{if } n = m^2 \ (m \in \mathbb{N}) \\ \dfrac{x}{n}, & \text{otherwise.} \end{cases} \tag{9}$$

Then obviously

$$c_n = \begin{cases} 5, & \text{if } n = m^2 \ (m \in \mathbb{N}) \\ \dfrac{1}{n}, & \text{otherwise.} \end{cases}$$

and $st_{C_1} - \lim c_n = 0$.

Theorem 10. *Let $A = (a_{jk})$ be a non-negative regular summability matrix. If the condition (1°) and (2°) holds, then*

$$st_A - \lim_n \left\| \mathbf{P}_n^{(\alpha)}(f;x) - f(x) \right\|_{C[0,\theta]} = 0$$

holds for every $f \in E_{1+x^2}$.

Proof. Given $\varepsilon > 0$ choose $\varepsilon^* > 0$ such that $\varepsilon^* < \varepsilon$ and consider the following sets:

$$\Omega := \{n : \delta_n \geq \varepsilon\},$$
$$\Omega_1 := \left\{ n : c_n \geq \frac{\varepsilon - \varepsilon^*}{3M_{g,\theta}} \right\},$$
$$\Omega_2 := \left\{ n : \frac{1}{n} \geq \frac{\varepsilon - \varepsilon^*}{3M_{g,\theta}} \right\}$$
$$\Omega_3 := \left\{ n : \frac{1}{n^2} \geq \frac{\varepsilon - \varepsilon^*}{3M_{g,\theta}} \right\}$$

where $M_{g,\theta} = \max\left\{1, \theta, \left|\dfrac{g''(1)+g'(1)}{g(1)}\right|\right\}$ and $\delta_n = c_n + \frac{\theta}{n} + \left|\frac{g''(1)+g'(1)}{g(1)n^2}\right|$. It is obvious that $\Omega \subseteq \Omega_1 \cup \Omega_2 \cup \Omega_3$, which gives

$$\sum_{k\in\Omega} a_{jk} \leq \sum_{k\in\Omega_1} a_{jk} + \sum_{k\in\Omega_2} a_{jk} + \sum_{k\in\Omega_3} a_{jk}. \tag{10}$$

Letting $j \to \infty$ in (10) and considering (2°), we have $\lim_j \sum_{k\in\Omega} a_{jk} = 0$. This guarantees that $st_A - \lim_n \delta_n = 0$, which further implies $st_A - \lim_n \omega_{\theta+1}(f, (\delta_n)^{1/2}) = 0$. Using Theorem 7 the proof is completed. □

We should remark that if the sequence $(\alpha_n(x))_{n\in\mathbb{N}}$ is chosen for example as in (9), then the statistical approximation results (Theorem 10) works, however its classical case (Corollary 8) does not work.

5. ILLUSTRATIVE EXAMPLE

Figures 1 and 2 illustrate the approximation of $f(x) = x^2e^{-x}$ by the Jakimoski-Leviatan, Appell-Baskakov, Lupaş-Baskakov operators with $g(x) = e^x$, for $n = 10$ and $n = 20$ respectively. In both figures, the blue curve represents the function $f(x)$, the red curve represents $P_n(f;x)$ given in (3), the yellow curve represents $\mathbb{B}_n(f;x)$ given in (4) and the green curve represents $\mathbb{L}_n(f;x)$ given in (5).

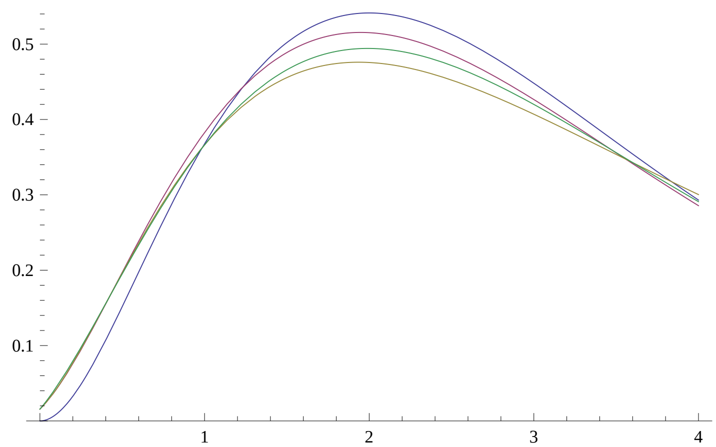

Figure 1: Approximation of $f(x) = x^2e^{-x}$ on $[0, 4]$,when $n = 10$.

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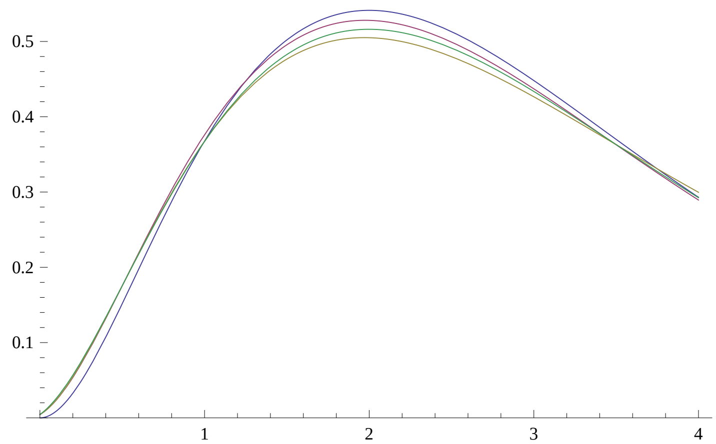

Figure 2: Approximation of $f(x) = x^2e^{-x}$ on $[0, 4]$,when $n = 20$.

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