Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 670–684.
https://doi.org/10.2298/AADM190221045V

ON LEVINSON'S INEQUALITY INVOLVING AVERAGES OF 3-CONVEX FUNCTIONS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

Ana Vukelić

By using the Levinson inequality we give the extension for 3-convex functions of Wulbert's result from *Favard's Inequality on Average Values of Convex Functions*, Math. Comput. Model. **37** (2003), 1383–1391. Also, we obtain inequalities with divided differences, and as a consequence, the convexity of higher order for function defined by divided difference is proved. Further, we construct a new family of exponentially convex functions and Cauchy-type means by looking at linear functionals associated with these new inequalities.

1. Introduction

Let f be a continuous function on an interval I with a nonempty interior. Then, define

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I,\ x \neq y, \\ f(x), & x = y \in I. \end{cases} \tag{1}$$

Wulbert in [**16**], proved that the integral arithmetic mean F defined in (1) is convex on I^2 if f is convex on I. Zhang and Chu, in [**17**], rediscovered (without referring to Wulbert's result) that the necessary and sufficient condition for the convexity of the integral arithmetic mean F is for f to be convex on I.

2010 Mathematics Subject Classification. Primary 26D15. Secondary 26D07, 26A51
Keywords and Phrases. Levinson's inequality, divided difference, n-convex function, (n,m)-convex function, mean

Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0, ..., x_n \in [a,b]$, is defined recursively (see [**1**], [**10**]) by

$$f[x_i] = f(x_i), \quad (i = 0, \dots, n)$$

and

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \dots, x_n]$ is independent of the order of the points $x_0, \dots, x_n$. The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x, \dots, x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{2}$$

For divided differences the following holds

$$f[x_0, \dots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}, \text{ where } \omega(x) = \prod_{j=0}^{n}(x - x_j),$$

so we have that

$$f[x_0, \dots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0, j\neq i}^{n}(x_i - x_j)}.$$

If the function f has continuous n-th derivative on $[a,b]$, the divided difference $f[x_0, \dots, x_n]$ can be represented in integral form (see [**10**, p. 15]) by

$$f[x_0, \dots, x_n] = \int_{\Delta_n} f^{(n)}\left(\sum_{i=0}^{n} u_i x_i\right) du_0 \dots du_{n-1},$$

where

$$\Delta_n = \left\{(u_0, \dots, u_{n-1}) : u_i \geq 0, \sum_{i=0}^{n-1} u_i \leq 1\right\}$$

and $u_n = 1 - \sum_{i=0}^{n-1} u_i$.

The notion of n-convexity goes back to Popoviciu ([**12**]). We follow the definition given by Karlin ([**6**]).

Definition 1. *A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, the n-th order divided difference of f satisfies*

$$f[x_0, ..., x_n] \geq 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[a, b]$ is the uniform limit of a sequence of n-convex polynomials. Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [**7**].

The following Jensen inequality for divided differences is proved in [**4**].

Theorem 1. *Let f be an $(n+2)$-convex function on (a, b) and $\mathbf{x} \in (a, b)^{n+1}$. Then*

$$G(\mathbf{x}) = f[x_0, \ldots, x_n]$$

is a convex function of the vector $\mathbf{x} = (x_0, \ldots, x_n)$. Consequently,

$$(3) \qquad f\left[\sum_{i=0}^{m} a_i x_0^i, \ldots, \sum_{i=0}^{m} a_i x_n^i\right] \le \sum_{i=0}^{m} a_i f[x_0^i, \ldots, x_n^i] \quad (i \text{ is an upper index})$$

holds for all $a_i \ge 0$ such that $\sum_{i=0}^{m} a_i = 1$.

The Schur polynomial in $n+1$ variables $x_0, \ldots, x_n$ of degree $d = d_0 + \ldots + d_n$ (the d_j's form a nonincreasing sequence of nonnegative integers, i.e. $d_0 \ge \ldots \ge d_n$) is defined as

$$S_{(d_0,\ldots,d_n)}(x_0, \ldots, x_n) = \frac{\det\left[x_i^{d_{n-j}+j}\right]_{i,j=0}^{n}}{\det\left[\;x_i^j\;\right]_{i,j=0}^{n}} \quad (d_{n-j} + j \text{ and } j \text{ are powers}).$$

The numerator consists of alternating polynomials (they change the sign under any transposition of the variables) and so they are all divisible by the denominator which is a Vandermonde determinant. The Schur polynomial is also symmetric because the numerator and denominator are both alternating.

Using the Schur polynomial and the the Vandermonde determinant (extended with logarithmic function)

$$V(\mathbf{x}; p, q) = \det \begin{bmatrix} 1 & x_0 & {x_0}^2 & \ldots & {x_0}^{n-1} & {x_0}^p \ln^q x_0 \\ 1 & x_1 & {x_1}^2 & \ldots & {x_1}^{n-1} & {x_1}^p \ln^q x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & {x_n}^2 & \ldots & {x_n}^{n-1} & {x_n}^p \ln^q x_n \end{bmatrix}$$

we obtain (see [**9**])

Proposition 1. *For the monomial function $h(x) = x^{n+k}$, where $k \ge 1$ is an integer, it holds*

$$h[x_0, \ldots, x_n] = S_{(k,\underbrace{0,\ldots,0}_{n-times})}(x_0, \ldots, x_n) = \frac{V(\mathbf{x}; n+k, 0)}{V(\mathbf{x}; n, 0)}.$$

For the potential function $f(x) = x^p = x^{n+p-n}$, where p is a real number, it holds

$$f[x_0, \ldots, x_n] = \frac{V(\mathbf{x}; p, 0)}{V(\mathbf{x}; n, 0)}.$$

Let $f(x,y)$ be a real-valued function defined on $I\times J$ ($I=[a,b], J=[c,d]$). Then the (n,m) divided difference of the function f at distinct points $x_0,\dots,x_n\in I, y_0,\dots,y_m\in J$, is defined by (see [**10**, p. 18])

$$f\begin{bmatrix} x_0,\dots,x_n \\ y_0,\dots,y_m \end{bmatrix} = f[y_0,\dots,y_m][x_0,\dots,x_n] = f[x_0,\dots,x_n][y_0,\dots,y_m] = \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{f(x_i,y_j)}{\omega'(x_i)w'(y_j)}, \tag{4}$$

where $\omega(x)=\prod_{i=0}^{n}(x-x_i),\ w(y)=\prod_{j=0}^{m}(y-y_j)$.

Definition 2. *([**10**, p. 18]) A function $f: I\times J\to\mathbb{R}$ is said to be (n,m)-convex or convex of order (n,m) if for all distinct points $x_0,\dots,x_n\in I, y_0,\dots,y_m\in J$,*

$$f\begin{bmatrix} x_0,\dots,x_n \\ y_0,\dots,y_m \end{bmatrix}\ge 0. \tag{5}$$

If this inequality is strict, then f is said to be strictly (n,m)-convex.

Popoviciu, in [**13**], proved the following theorem.

Theorem 2. *If the partial derivative $f^{(n+m)}_{x^n y^m}$ $(\partial^{(n+m)}f/\partial x^n\partial y^m)$ of f exists, then f is (n,m)-convex iff*

$$f^{(n+m)}_{x^n y^m}\ge 0. \tag{6}$$

If the inequality in (6) is strict, then f is strictly (n,m)-convex.

The well known Levinson inequality is given in the following theorem (see [**8**] and [**14**]).

Theorem 3. *Let f be a real valued 3-convex function on $[0,2a]$. Then for $0\le x_k\le a,\ p_k>0$ $(1\le k\le n)$ and $P_k=\sum_{i=1}^{k}p_i$ $(2\le k\le n)$ we have*

$$\frac{1}{P_n}\sum_{k=1}^{n}p_kf(x_k)-f\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kx_k\right) \le \frac{1}{P_n}\sum_{k=1}^{n}p_kf(2a-x_k)-f\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k(2a-x_k)\right). \tag{7}$$

If $f'''>0$, then the equality holds iff $x_1=\dots=x_n$.

Bullen [**2**] proved the following generalization of Theorem 3.

Theorem 4. *a) Let f be a real-valued 3-convex function on $[a,b]$ and $x_k,y_k (1\le k\le n)$ be $2n$ points on $[a,b]$ such that*

$$\max\{x_1,\dots,x_n\}\le\min\{y_1,\dots,y_n\},\quad x_1+y_1=\dots=x_n+y_n. \tag{8}$$

If $p_k > 0$ $(1 \le k \le n)$, then
(9)
$$\frac{1}{P_n}\sum_{k=1}^{n} p_k f(x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right) \le \frac{1}{P_n}\sum_{k=1}^{n} p_k f(y_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k y_k\right).$$

If f is strictly 3-convex there is equality in (9) if and only if $x_1 = \cdots = x_n$.

b) *If (9) holds for a continuous function f, (8) is satisfied by $2n$-distinct points and $p_k > 0$ for $k \in [1, n]$, then f is 3-convex.*

The goal of this paper is to give the extension for 3-convex function of Wulbert's result from [**16**] and also to obtain inequalities with divided differences using the Levinson inequality. As a consequence, we will prove the convexity of higher order for function defined by divided difference. In the last section, a new family of exponentially convex functions and Cauchy-type means will be constructed by looking at the linear functionals associated with the obtained inequalities.

2. Inequalities involving averages

The following result is an extension of Wulbert's result for 3-convex functions.

Theorem 5. *Let f be a real valued 3-convex function on $[0, 2a]$ and let F be defined in (1). Then for $0 \le x_k, y_k \le a$, $p_k > 0$ $(1 \le k \le n)$ and $P_k = \sum_{i=1}^{k} p_i$ $(2 \le k \le n)$ we have*

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k F(x_k, y_k) - F(\bar{x}, \bar{y}) \le \frac{1}{P_n}\sum_{k=1}^{n} p_k F(2a - x_k, 2a - y_k) - F(2a - \bar{x}, 2a - \bar{y}), \tag{10}$$

where $\bar{x} = \frac{1}{P_n}\sum_{k=1}^{n} p_k x_k$ and $\bar{y} = \frac{1}{P_n}\sum_{k=1}^{n} p_k y_k$.

If f is strictly 3-convex there is equality in (9) if and only if $x_1 = \cdots = x_n = y_1 = \cdots = y_n$.

Consequently, for $l + m = 3$ the integral arithmetic mean (1) is (l, m)-convex on $[0, 2a]^2$.

Proof. By using the Levinson inequality (7) we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{k=1}^{n} p_k F(x_k, y_k) - F(\bar{x}, \bar{y}) \\
= \; & \frac{1}{P_n}\sum_{k=1}^{n} p_k \int_0^1 f(sy_k + (1-s)x_k)ds \\
& - \int_0^1 f\left(s\frac{1}{P_n}\sum_{k=1}^{n} p_k y_k + (1-s)\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right) ds \\
= \; & \int_0^1 \left[\frac{1}{P_n}\sum_{k=1}^{n} p_k f(sy_k + (1-s)x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \left(sy_k + (1-s)x_k\right)\right)\right] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \left[\frac{1}{P_n}\sum_{k=1}^n p_k f(2a - sy_k - (1-s)x_k) \right. \\
&\quad \left. -f\left(\frac{1}{P_n}\sum_{k=1}^n p_k(2a - sy_k - (1-s)x_k)\right)\right] ds \\
&= \frac{1}{P_n}\sum_{k=1}^n p_k \int_0^1 f((2a-y_k)s + (2a-x_k)(1-s))ds \\
&\quad - \int_0^1 f\left(\frac{1}{P_n}\sum_{k=1}^n p_k(2a-y_k)s + \frac{1}{P_n}\sum_{k=1}^n p_k(2a-x_k)(1-s)\right) ds \\
&= \frac{1}{P_n}\sum_{k=1}^n p_k F(2a-x_k, 2a-y_k) - F(2a-\bar{x}, 2a-\bar{y}).
\end{aligned}$$

Now, if we put $n = 2, x_1 = x, x_2 = x + \frac{3h}{2}, y_1 = y_2 = y, p_1 = 1, p_2 = 2, 2a = 2x + 3h = 2y$, then the inequality (10) reduces to

$$\frac{1}{3}F(x,y) - F(x+h,y) \leq \frac{1}{3}F(x+3h,y) - F(x+2h,y).$$

Using the definition in (4) we get

$$2h^3(F[x, x+h, x+2h, x+3h])[y] \geq 0.$$

It is known that if this holds for all possible $x, h > 0$ then F is $(3,0)$-convex (see [**13**]).

If we put $n = 2, x_1 = x, x_2 = x + 2h_1, y_1 = y_2 = y, p_1 = p_2 = 1, 2a = 2x + 2h_1 = 2y + h_2$ then the inequality (10) reduces to

$$\begin{aligned}
&\frac{1}{2}\left(F(x,y) + F(x+2h_1,y)\right) - F(x+h_1,y) \\
\leq\ &\frac{1}{2}\left(F(x+2h_1,y+h_2) + F(x,y+h_2)\right) - F(x+h_1,y+h_2).
\end{aligned}$$

Using the definition in (4) we get

$$h_1^2 h_2(F[x, x+h_1, x+2h_1])[y, y+h_2] \geq 0.$$

As before, since this holds for all possible $x, h_1, y, h_2 > 0$, F is $(2,1)$-convex.

The proofs for $(0,3)$-convexity and $(1,2)$-convexity are similar. □

Remark 1. *Theorem 5 is a generalization of the Levinson inequality because the inequality (10) for* $x_k = y_k,\ k = 1, \ldots, n$ *recaptures the Levinson inequality (7).*

Remark 2. *The inequality (10) is strict if* f *is a strictly 3-convex function unless* $x_1 = x_2 = \cdots = x_n = y_1 = y_2 = \cdots = y_n$.

The following theorem is the integral version of Theorem 5.

Theorem 6. *Let $(\Omega, \mathbf{A}, \mu)$ be a probability space, $\alpha, \beta : \Omega \to [0, a]$ be functions from $L_1(\mu)$ and let f be a 3-convex function on $[0, 2a]$ and let F be defined as in (1). Then*

(11)
$$\int_\Omega F(\alpha(u), \beta(u)) d\mu(u) - F(\bar{\alpha}, \bar{\beta}) \leq \int_\Omega F(2a - \alpha(u), 2a - \beta(u)) d\mu(u) - F(2a - \bar{\alpha}, 2a - \bar{\beta}),$$

where $\bar{\alpha} = \int_\Omega \alpha(u) d\mu(u)$ and $\bar{\beta} = \int_\Omega \beta(u) d\mu(u)$.

If f is strictly 3-convex there is equality in (11) if and only if α and β are the same constant function.

Proof. By using the integral version of Levinson inequality we get

$$\begin{aligned}
& \int_\Omega F(\alpha(u), \beta(u)) d\mu(u) - F(\bar{\alpha}, \bar{\beta}) \\
= & \int_0^1 \left[\int_\Omega f(s\beta(u) + (1-s)\alpha(u)) d\mu(u) - f\left(\int_\Omega (s\beta(u) + (1-s)\alpha(u))\, d\mu(u) \right) \right] ds \\
\leq & \int_0^1 \left[\int_\Omega f(2a - s\beta(u) - (1-s)\alpha(u)) d\mu(u) \right. \\
& \left. - f\left(\int_\Omega (2a - s\beta(u) - (1-s)\alpha(u))\, d\mu(u) \right) \right] ds \\
= & \int_\Omega F(2a - \alpha(u), 2a - \beta(u)) d\mu(u) - F(2a - \bar{\alpha}, 2a - \bar{\beta}).
\end{aligned}$$

□

3. Inequalities for divided differences

In the following theorem we prove the Levinson inequality for divided differences.

Theorem 7. *Let f be an $(n+3)$-convex function on $[0, 2a]$ and $\mathbf{x} \in [0, a]^{n+1}$. Then ("$i$" is an upper index)*

(12)
$$\begin{aligned}
& \sum_{i=0}^m a_i f[x_0^i, \ldots, x_n^i] - f\left[\sum_{i=0}^m a_i x_0^i, \ldots, \sum_{i=0}^m a_i x_n^i \right] \\
\leq & \sum_{i=0}^m a_i f[2a - x_0^i, \ldots, 2a - x_n^i] - f\left[2a - \sum_{i=0}^m a_i x_0^i, \ldots, 2a - \sum_{i=0}^m a_i x_n^i \right]
\end{aligned}$$

holds for all $a_i \geq 0$ such that $\sum_{i=0}^m a_i = 1$. Consequently,

$$G(\mathbf{x}) = f[x_0, x_1, x_2]$$

is an (l_1, l_2, l_3)-convex function of the vector $\mathbf{x} = (x_0, x_1, x_2)$, when $l_1 + l_2 + l_3 = 3$.

Proof. Using the Levinson inequality for the 3-convex function $f^{(n)}$, we have

$$\begin{aligned}
&\sum_{i=0}^{m} a_i f[x_0^i,\dots,x_n^i] - f\left[\sum_{i=0}^{m} a_i x_0^i,\dots,\sum_{i=0}^{m} a_i x_n^i\right] \\
= &\sum_{i=0}^{m} a_i \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1} \\
&-\int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j \sum_{i=0}^{m} a_i x_j^i\right) du_0\dots du_{n-1} \\
= &\int_{\Delta_n}\left[\sum_{i=0}^{m} a_i f^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) - f^{(n)}\left(\sum_{i=0}^{m} a_i \sum_{j=0}^{n} u_j x_j^i\right)\right] du_0\dots du_{n-1} \\
\leq &\int_{\Delta_n}\left[\sum_{i=0}^{m} a_i f^{(n)}\left(2a-\sum_{j=0}^{n} u_j x_j^i\right)\right. \\
&\left.-f^{(n)}\left(\sum_{i=0}^{m} a_i\left(2a-\sum_{j=0}^{n} u_j x_j^i\right)\right)\right] du_0\dots du_{n-1} \\
= &\sum_{i=0}^{m} a_i \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j (2a-x_j^i)\right) du_0\dots du_{n-1} \\
&-\int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j \sum_{i=0}^{m} a_i (2a-x_j^i)\right) du_0\dots du_{n-1} \\
= &\sum_{i=0}^{m} a_i f[2a-x_0^i,\dots,2a-x_n^i] - f\left[2a-\sum_{i=0}^{m} a_i x_0^i,\dots,2a-\sum_{i=0}^{m} a_i x_n^i\right].
\end{aligned}$$

Now, if we put $n=2, m=1$,

$$\begin{aligned}
&x_0^0 = y_0, \quad x_0^1 = y_0 + \frac{3h}{2}, \\
&x_1^0 = x_1^1 = y_1, \\
&x_2^0 = x_2^1 = y_2, \\
&a_0 = \frac{1}{3}, \quad a_1 = \frac{2}{3},
\end{aligned}$$

and $2a = 2y_0 + 3h = 2y_1 = 2y_2$, then the inequality (12) reduces to

$$\begin{aligned}
&\frac{1}{3} G(y_0,y_1,y_2) - G(y_0+h,y_1,y_2) \\
\leq &\frac{1}{3} G(y_0+3h,y_1,y_2) - G(y_0+2h,y_1,y_2).
\end{aligned}$$

Using the generalization of definition (4) we get

$$2h^3\left(\left(G[y_0, y_0+h, y_0+2h, y_0+3h]\right)[y_1]\right)[y_2] \geq 0.$$

As in the proof of Theorem 5, since this holds for all possible $y_0, y_1, y_2, h > 0$, G is $(3, 0, 0)$-convex.

If we put $n = 2, m = 1$,

$$\begin{aligned} &x_0^0 = y_0, \quad x_0^1 = y_0 + 2h_0, \\ &x_1^0 = x_1^1 = y_1, \\ &x_2^0 = x_2^1 = y_2, \\ &a_0 = a_1 = \frac{1}{2}, \end{aligned}$$

and $2a = 2y_0 + 2h_0 = 2y_1 + h_1 = 2y_2$ then the inequality (12) reduces to

$$\begin{aligned} &\frac{1}{2}G(y_0, y_1, y_2) + \frac{1}{2}G(y_0 + 2h_0, y_1, y_2) - G(y_0 + h_0, y_1, y_2) \\ \leq\ &\frac{1}{2}G(y_0 + 2h_0, y_1 + h_1, y_2) + \frac{1}{2}G(y_0, y_1 + h_1, y_2) - G(y_0 + h_0, y_1 + h_1, y_2). \end{aligned}$$

Using the generalization of definition (4) we get

$$h_0^2 h_1\left(\left(G[y_0, y_0+h_0, y_0+2h_0]\right)[y_1, y_1+h_1]\right)[y_2] \geq 0.$$

As before, since this holds for all possible $y_0, y_1, y_2, h_0, h_1 > 0$, G is $(2, 1, 0)$-convex.

If we put $n = 2, m = 3$,

$$\begin{aligned} &x_0^0 = x_0^2 = y_0, \quad x_0^1 = x_0^3 = y_0 + h_0, \\ &x_1^0 = x_1^3 = y_1, \quad x_1^1 = x_1^2 = y_1 + h_1 \\ &x_2^0 = x_2^1 = y_2, \quad x_2^2 = x_2^3 = y_2 + h_2, \\ &a_0 = a_1 = a_2 = a_3 = \frac{1}{4}, \end{aligned}$$

and $2a = 2y_0 + h_0 = 2y_1 + h_1 = 2y_2 + h_2$ then the inequality (12) reduces to

$$\begin{aligned} &\frac{1}{4}G(y_0, y_1, y_2) + \frac{1}{4}G(y_0 + h_0, y_1 + h_1, y_2) \\ &+\frac{1}{4}G(y_0, y_1 + h_1, y_2 + h_2) + \frac{1}{4}G(y_0 + h_0, y_1, y_2 + h_2) \\ \leq\ &\frac{1}{4}G(y_0 + h_0, y_1 + h_0, y_2 + h_0) + \frac{1}{4}G(y_0, y_1, y_2 + h_2) \\ &+\frac{1}{4}G(y_0 + h_0, y_1, y_2) + \frac{1}{4}G(y_0, y_1 + h_1, y_2). \end{aligned}$$

Using the generalization of definition (4) we get

$$\frac{1}{4}h_0 h_1 h_2\left(\left(G[y_0, y_0+h_0]\right)[y_1, y_1+h_1]\right)[y_2, y_2+h_2] \geq 0.$$

As before, since this holds for all possible $y_0, y_1, y_2, h_0, h_1, h_2 > 0$, G is $(1,1,1)$-convex.

The proofs for $(0,3,0)$-convexity, $(0,0,3)$-convexity, $(1,2,0)$-convexity, $(0,2,1)$-convexity, $(0,1,2)$-convexity, $(2,0,1)$-convexity and $(1,0,2)$-convexity are similar. □

The integral version of the Levinson inequality for divided differences is given by the following theorem.

Theorem 8. *Let $p, g_i : \Omega \to [0,a], (i = 0, \dots, n)$ be functions from $L_1(\mu)$ and let f be an $(n+3)$-convex function on $[0, 2a]$. Then*

(13)
$$\begin{aligned}
&\int_\Omega p(x) f[g_0(x), \dots, g_n(x)] d\mu(x) - f\left[\int_\Omega p(x) g_0(x) d\mu(x), \dots, \int_\Omega p(x) g_n(x) d\mu(x)\right] \\
\le\ & \int_\Omega p(x) f[2a - g_0(x), \dots, 2a - g_n(x)] d\mu(x) \\
& - f\left[2a - \int_\Omega p(x) g_0(x) d\mu(x), \dots, 2a - \int_\Omega p(x) g_n(x) d\mu(x)\right]
\end{aligned}$$

holds for all $p(x) \ge 0$ such that $\int_\Omega p(x) d\mu(x) = 1$.

Proof. Using the integral Levinson inequality for the 3-convex function $f^{(n)}$, we have the following conclusion

$$\begin{aligned}
&\int_\Omega p(x) f[g_0(x), \dots, g_n(x)] d\mu(x) - f\left[\int_\Omega p(x) g_0(x) d\mu(x), \dots, \int_\Omega p(x) g_n(x) d\mu(x)\right] \\
=\ & \int_{\Delta_n} \left(\int_\Omega p(x) f^{(n)} \left(\sum_{i=0}^n u_i g_i(x) \right) d\mu(x) \right. \\
& \left. - f^{(n)} \left(\int_\Omega p(x) \sum_{i=0}^n u_i g_i(x) d\mu(x) \right) \right) du_0 \dots du_{n-1} \\
\le\ & \int_{\Delta_n} \left(\int_\Omega p(x) f^{(n)} \left(2a - \sum_{i=0}^n u_i g_i(x) \right) d\mu(x) \right. \\
& \left. - f^{(n)} \left(\int_\Omega p(x) \left(2a - \sum_{i=0}^n u_i g_i(x) \right) d\mu(x) \right) \right) du_0 \dots du_{n-1} \\
=\ & \int_\Omega p(x) f[2a - g_0(x), \dots, 2a - g_n(x)] d\mu(x) \\
& - f\left[2a - \int_\Omega p(x) g_0(x) d\mu(x), \dots, 2a - \int_\Omega p(x) g_n(x) d\mu(x)\right].
\end{aligned}$$

□

4. Applications to exponential convexity

Motivated by inequalities (10), (11), (12) and (13), under the same assumptions, we define the following functionals:

$$\begin{aligned}\Phi_1(f) &= \frac{1}{P_n}\sum_{k=1}^{n} p_k \frac{1}{x_k - y_k}\int_{2a-x_k}^{2a-y_k} f(t)dt - \frac{1}{\bar{x}-\bar{y}}\int_{2a-\bar{x}}^{2a-\bar{y}} f(t)dt \\ &\quad -\frac{1}{P_n}\sum_{k=1}^{n} p_k \frac{1}{y_k - x_k}\int_{x_k}^{y_k} f(t)dt + \frac{1}{\bar{y}-\bar{x}}\int_{\bar{x}}^{\bar{y}} f(t)dt, \end{aligned} \tag{14}$$

$$\begin{aligned}\Phi_2(f) &= \int_\Omega \left(\frac{1}{\alpha(u)-\beta(u)}\int_{2a-\alpha(u)}^{2a-\beta(u)} f(t)dt\right) d\mu(u) - \frac{1}{\bar{\alpha}-\bar{\beta}}\int_{2a-\bar{\alpha}}^{2a-\bar{\beta}} f(t)dt \\ &\quad -\int_\Omega \left(\frac{1}{\beta(u)-\alpha(u)}\int_{\alpha(u)}^{\beta(u)} f(t)dt\right) d\mu(u) + \frac{1}{\bar{\beta}-\bar{\alpha}}\int_{\bar{\alpha}}^{\bar{\beta}} f(t)dt, \end{aligned} \tag{15}$$

$$\begin{aligned}\Phi_3(f) &= \sum_{i=0}^{m} a_i f[2a - x_0^i, \ldots, 2a - x_n^i] - f\left[2a - \sum_{i=0}^{m} a_i x_0^i, \ldots, 2a - \sum_{i=0}^{m} a_i x_n^i\right] \\ &\quad -\sum_{i=0}^{m} a_i f[x_0^i, \ldots, x_n^i] + f\left[\sum_{i=0}^{m} a_i x_0^i, \ldots, \sum_{i=0}^{m} a_i x_n^i\right], \end{aligned} \tag{16}$$

$$\begin{aligned}\Phi_4(f) &= \int_\Omega p(x) f[2a - g_0(x), \ldots, 2a - g_n(x)] d\mu(x) \\ &\quad -f\left[2a - \int_\Omega p(x) g_0(x) d\mu(x), \ldots, 2a - \int_\Omega p(x) g_n(x) d\mu(x)\right] \\ &\quad -\int_\Omega p(x) f[g_0(x), \ldots, g_n(x)] d\mu(x) \\ &\quad +f\left[\int_\Omega p(x) g_0(x) d\mu(x), \ldots, \int_\Omega p(x) g_n(x) d\mu(x)\right]. \end{aligned} \tag{17}$$

Similarly as in [**11**] we can construct new families of exponentially convex functions and Cauchy type means by looking at these linear functionals. Also, we can prove the monotonicity property of the generalized Cauchy means obtained via these functionals.

Here we present an example for such a family of functions.

Example 1. *Consider a family of functions*

$$\Omega = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x)=\begin{cases}\frac{x^s}{s(s-1)(s-2)}, & s\notin\{0,1,2\},\\ \frac{x^j\ln x}{(-1)^{2-j}j!(2-j)!}, & s=j\in\{0,1,2\}.\end{cases}$$

Here, $\frac{d^3 f_s}{dx^3}(x)=x^{s-3}=e^{(s-3)\ln x}>0$ *which shows that* f_s *is* 3*-convex for* $x>0$ *and* $s\mapsto\frac{d^3 f_s}{dx^3}(x)$ *is exponentially convex by definition. Arguing as in [***11***] we get that the mappings* $s\mapsto\Phi_i(f_s), i=1,2$ *are exponentially convex. Now we get*

$$\mu_{s,q}(\Phi_i,\Omega)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{2\Phi_i(f_0f_s)}{\Phi_i(f_s)}+\sum_{k=0}^{2}\frac{1}{k-s}\right), & s=q\notin\{0,1,2\},\\ \exp\left(\frac{\Phi_i(f_0f_s)}{\Phi_i(f_s)}+\sum_{\substack{k=0\\k\neq s}}^{2}\frac{1}{k-s}\right), & s=q\in\{0,1,2\},\end{cases}$$

where for $s\neq -1,0,1,2$

$$\begin{aligned}\Phi_1(f_s) &= \frac{1}{(s-2)(s^3-s)}\left[\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{(2a-y_k)^{s+1}-(2a-x_k)^{s+1}}{x_k-y_k}\right.\\ &\left.-\frac{(2a-\bar{y})^{s+1}-(2a-\bar{x})^{s+1}}{\bar{x}-\bar{y}}-\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{y_k^{s+1}-x_k^{s+1}}{y_k-x_k}+\frac{\bar{y}^{s+1}-\bar{x}^{s+1}}{\bar{y}-\bar{x}}\right],\end{aligned}$$

$$\begin{aligned}\Phi_1(f_{-1}) &= \frac{1}{6}\left[\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{\ln(2a-y_k)-\ln(2a-x_k)}{y_k-x_k}-\frac{\ln(2a-\bar{y})-\ln(2a-\bar{x})}{\bar{y}-\bar{x}}\right.\\ &\left.+\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{\ln y_k-\ln x_k}{y_k-x_k}-\frac{\ln\bar{y}-\ln\bar{x}}{\bar{y}-\bar{x}}\right],\end{aligned}$$

$$\begin{aligned}\Phi_1(f_0) &= \frac{1}{2}\left[\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{(2a-y_k)\ln(2a-y_k)-(2a-x_k)\ln(2a-x_k)}{x_k-y_k}\right.\\ &-\frac{(2a-\bar{y})\ln(2a-\bar{y})-(2a-\bar{x})\ln(2a-\bar{x})}{\bar{x}-\bar{y}}\\ &\left.-\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{y_k\ln y_k-x_k\ln x_k}{y_k-x_k}+\frac{\bar{y}\ln\bar{y}-\bar{x}\ln\bar{x}}{\bar{y}-\bar{x}}\right],\end{aligned}$$

$$\begin{aligned}\Phi_1(f_1) &= \frac{1}{2}\left[\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{(2a-y_k)^2\ln(2a-y_k)-(2a-x_k)^2\ln(2a-x_k)}{y_k-x_k}\right.\\ &-\frac{(2a-\bar{y})^2\ln(2a-\bar{y})-(2a-\bar{x})^2\ln(2a-\bar{x})}{\bar{y}-\bar{x}}\\ &\left.+\frac{1}{P_n}\sum_{k=1}^{n}p_k\frac{y_k^2\ln y_k-x_k^2\ln x_k}{y_k-x_k}-\frac{\bar{y}^2\ln\bar{y}-\bar{x}^2\ln\bar{x}}{\bar{y}-\bar{x}}\right],\end{aligned}$$

$$
\begin{aligned}
\Phi_1(f_2) &= \frac{1}{6}\left[\frac{1}{P_n}\sum_{k=1}^{n} p_k \frac{(2a-y_k)^3\ln(2a-y_k)-(2a-x_k)^3\ln(2a-x_k)}{x_k-y_k}\right.\\
&\quad -\frac{(2a-\bar{y})^3\ln(2a-\bar{y})-(2a-\bar{x})^3\ln(2a-\bar{x})}{\bar{x}-\bar{y}}\\
&\quad \left.-\frac{1}{P_n}\sum_{k=1}^{n} p_k \frac{y_k^3\ln y_k - x_k^3\ln x_k}{y_k-x_k}+\frac{\bar{y}^3\ln\bar{y}-\bar{x}^3\ln\bar{x}}{\bar{y}-\bar{x}}\right].
\end{aligned}
$$

*For similarly results for Jensen's inequality involving averages of convex functions see [**3**] and [**5**].*

For a family of functions

$$
\tilde{\Omega} = \left\{\tilde{f}_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\right\}
$$

defined by

$$
\tilde{f}_s(x) = \begin{cases} \frac{x^s}{s(s-1)\ldots(s-(n+2))}, & s \notin \{0,1,\ldots,n+2\}, \\ \frac{x^j \ln x}{(-1)^{n+2-j} j!(n+2-j)!}, & s = j \in \{0,1,\ldots,n+2\}, \end{cases}
$$

analogous as above it is easy to prove that $s \mapsto \Phi_i\left(\tilde{f}_s\right)$ $(i = 3,4)$ *are exponentially convex. In this case, we get* $\tilde{\mu}_{s,q}\left(\Phi_i, \tilde{\Omega}\right)$ $(i = 3,4)$ *as follows*

$$
\tilde{\mu}_{s,q}\left(\Phi_i,\tilde{\Omega}\right) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{(-1)^{n+2}(n+2)!\Phi_i(f_0 f_s)}{\Phi_i(f_s)} + \sum_{k=0}^{n+2}\frac{1}{k-s}\right), & s = q \notin \{0,1,\ldots,n+2\}, \\ \exp\left(\frac{(-1)^{n+2}(n+2)!\Phi_i(f_0 f_s)}{2\Phi_i(f_s)} + \sum_{\substack{k=0\\k\neq s}}^{n+2}\frac{1}{k-s}\right), & s = q \in \{0,1,\ldots,n+2\}. \end{cases}
$$

For $s \notin \{0,1,\ldots,n+2\}$

$$
\Phi_3(\tilde{f}_s) = \prod_{k=0}^{n+2}\frac{1}{s-k} \times
$$

$$
\left[\sum_{i=0}^{m} a_i \frac{V(2\bar{\mathbf{a}}-\mathbf{x}^i,s,0)}{V(2\bar{\mathbf{a}}-\mathbf{x}^i,n,0)} - \frac{V(2\bar{\mathbf{a}}-\mathbf{aX},s,0)}{V(2\bar{\mathbf{a}}-\mathbf{aX},n,0)} - \sum_{i=0}^{m} a_i \frac{V(\mathbf{x}^i,s,0)}{V(\mathbf{x}^i,n,0)} + \frac{V(\mathbf{aX},s,0)}{V(\mathbf{aX},n,0)}\right]
$$

and for $s = j \in \{0,1,\ldots,n+2\}$

$$
\Phi_3(\tilde{f}_j) = \prod_{\substack{k=0\\k\neq j}}^{n+2}\frac{1}{j-k} \times
$$

$$
\left[\sum_{i=0}^{m} a_i \frac{V(2\bar{\mathbf{a}}-\mathbf{x}^i,j,1)}{V(2\bar{\mathbf{a}}-\mathbf{x}^i,n,0)} - \frac{V(2\bar{\mathbf{a}}-\mathbf{aX},j,1)}{V(2\bar{\mathbf{a}}-\mathbf{aX},n,0)} - \sum_{i=0}^{m} a_i \frac{V(\mathbf{x}^i,j,1)}{V(\mathbf{x}^i,n,0)} + \frac{V(\mathbf{aX},j,1)}{V(\mathbf{aX},n,0)}\right],
$$

where

$$\mathbf{X}=\begin{bmatrix} x_0^0 & x_1^0 & x_2^0 & \dots & x_{n-1}^0 & x_n^0 \\ x_0^1 & x_1^1 & x_2^1 & \dots & x_{n-1}^1 & x_n^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^m & x_1^m & x_2^m & \dots & x_{n-1}^m & x_n^m \end{bmatrix}, \quad \mathbf{a}=\{a_0,\dots,a_m\}, \quad \bar{\mathbf{a}}=\underbrace{\{a,\dots,a\}}_{(n+1)-times}.$$

*For similar results for Jensen's inequality for divided differences see [**9**] and [**15**].*

Φ_i $(i=1,2,3,4)$ *are positive, so there exist* $\xi_i, \tilde{\xi}_i \in [0,2a]$ *such that*

$$\xi_i^{s-q}=\frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \quad i=1,2, \qquad \tilde{\xi}_i^{s-q}=\frac{\Phi_i\left(\tilde{f}_s\right)}{\Phi_i\left(\tilde{f}_q\right)}, \quad i=3,4.$$

Since the functions $\xi_i \mapsto \xi_i^{s-q}$ *and* $\tilde{\xi}_i \mapsto \tilde{\xi}_i^{s-q}$ *are invertible for* $s \neq q$*, we have*

$$0 \le \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \le 2a, \quad i=1,2, \qquad 0 \le \left(\frac{\Phi_i\left(\tilde{f}_s\right)}{\Phi_i\left(\tilde{f}_q\right)}\right)^{\frac{1}{s-q}} \le 2a, \quad i=3,4,$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega)$ *and* $\tilde{\mu}_{s,q}\left(\Phi_i,\tilde{\Omega}\right)$ *are continuous, symmetric and monotonous, shows that* $\mu_{s,q}(\Phi_i,\Omega)$ *and* $\tilde{\mu}_{s,q}\left(\Phi_i,\tilde{\Omega}\right)$ *are means.*

REFERENCES

1. K. E. Atkinson: *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989.
2. P. S. Bullen: *An inequality of N. Levinson*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **412-460** (1973), 109–112.
3. V. Čuljak, I. Franjić, R. Ghulam, J. Pečarić: *Schur-convexity of averages of convex functions*, J. Ineq. Appl. **2011** (2011), Article ID 581918.
4. R. Farwig, D. Zwick: *Some divided difference inequalities for n-convex functions*, J. Math. Anal. Appl. **108** (1985), 430–437.
5. J. Jakšetić, J. Pečarić, G. Roqia: *On Jensen's inequality involving averages of convex functions*, Sarajevo J. Math. **8(20)**, no. 1 (2012), 53–68.
6. S. Karlin: *Total Positivity*, Stanford Univ. Press, Stanford, 1968.
7. S. Karlin, W. J. Studden: *Tchebycheff systems: with applications in analysis and statistics*, Interscience, New York, 1966.
8. N. Levinson: *Generalization of an inequality of Ky Fan*, J. Math. Anal. Appl. **80** (1964), 133–134.

9. Z. Pavić, J. Pečarić, A. Vukelić: *Exponential convexity and Jensen inequality for divided differences*, J. Math. Inequal. **5(2)** (2011), 157–168.

10. J. E. Pečarić, F. Proschan, Y. L. Tong: *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering, vol. **187** Academic Press, 1992.

11. J. Pečarić, J. Perić: *Improvements of the Giaccardi and the Petrović inequality and related Stolarsky type means*, An. Univ. Craiova Ser. Mat. Inform. **39** (2012), 65–75.

12. T. Popoviciu: *Sur l'approximation des fonctions convexes d'ordre superieur*, Mathematica **10**, (1934), 49–54.

13. T. Popoviciu: *Les fonctions convexes*, Herman and Cie, Éditeurs, Paris (1944).

14. T. Popoviciu: *Sur certaines inégalités qui caracterisent les fonctions convexes* , An. Sti. Univ. Al. I. Cuza. Iasi Sect. I a Math. (N.S.) **11B** (1965), 155–164.

15. G. Roqia, J. Pečarić, A. Vukelić: *n-exponential convexity of divided differences and related Stolarsky type means*, Math. Ineq. Appl. **16(4)** (2013), 1043–1063.

16. D. E. Wulbert: *Favard's Inequality on Average Values of Convex Functions*, Math. Comput. Model. **37** (2003), 1383–1391.

17. X. M. Zhang and Y. M. Chu: *Convexity of the integral arithmetic mean of a convex function*, Rocky Mt. J. Math. **40(3)** (2010), 1061–1068.

Ana Vukelić
University of Zagreb
Faculty of Food
Technology and Biotechnology,
Section for Mathematics
Pierottijeva 6,
10000 Zagreb, Croatia
E-mail: *avukelic@pbf.hr*]

(Received 21.02.2019)
(Revised 24.07.2019)