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THE EFFECT OF EDGE AND VERTEX DELETION ON OMEGA INVARIANT

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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Recently the first and last authors defined a new graph characteristic called omega related to Euler characteristic to determine several topological and combinatorial properties of a given graph. This new characteristic is defined in terms of a given degree sequence as a graph invariant and gives a lot of information on the realizability, number of realizations, connectedness, cyclicness, number of components, chords, loops, pendant edges, faces, bridges etc. of the family of realizations.

In this paper, the effect of the deletion of vertices and edges from a graph on omega invariant is studied.

1. INTRODUCTION

We take $G = (V, E)$ as a graph with $\mid V(G) \mid= n$ vertices and $\mid E(G) \mid= m$ edges. For a vertex v, the degree of v is denoted by by d_v. In particular, a vertex with degree zero (one) will be called an isolated (pendant) vertex. As usual, the smallest and biggest vertex degrees in a graph will be denoted by δ and Δ, respectively. An edge e connecting two neighbouring vertices u and v will be denoted by $e = uv$ and the vertices u and v are called adjacent vertices while the edge e is said to be

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incident with u and v.

If there is a path between every pair of vertices in a graph G, then G is called connected, otherwise disconnected. A degree sequence D is a non-decreasing sequence of non-negative integers corresponding to the degrees of the vertices of a graph G. A degree sequence in general is written as

$$D = \{{d_1}^{(a_1)}, {d_2}^{(a_2)}, {d_3}^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\},$$

where a_i's are positive integers. It is also possible to state a degree sequence as

$$D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\},$$

where a_i's are non-negative integers.

Let D be a set of non-decreasing non-negative integers. We say that a graph G is a realization of the set D if the degree sequence of G is equal to D. For a realizable degree sequence, there is at least one graph having this degree sequence. Therefore some realizations of a given degree sequence could be connected and some could be disconnected. There are results to determine whether a given set is realizable or not, see [**1**], [**2**], [**3**], [**7**], [**8**].

A graph having no cycle will be called acyclic. For example, all trees are acyclic. The remaining graphs are called cyclic graphs. An edge connecting a vertex to itself is called a loop, and at least two edges connecting two vertices will be called multiple edges. When there are no loops nor multiple edges, the graph will be called simple.

The number a_1 of leaves of any tree T is given by

$$a_1 = 2 + a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta, \tag{1}$$

where a_i denotes the number of vertices of degree i. Note that equation (1) can be rearranged as

$$a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2. \tag{2}$$

The authors tried to determine the conditions which give some information about the topological and combinatorical properties of the given degree sequence (or of a graph which is the realization of the given degree sequence), and came up with similar sums. Trying to unify those sums resulted in discovering a specific number which gives more information than expected, see [**5**]:

Definition 1. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a set which also is the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as*

$$\begin{aligned} \Omega(G) = &\ a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 \\ = &\ \sum_{i=1}^{\Delta}(i-2)a_i. \end{aligned}$$

Ω is even by the definition. In [**5**], the authors gave a new result using Ω to determine the realizability of a degree sequence:

Corollary 1. *Let D be a set of non-negative integers. If $\Omega(D)$ is odd, then D is not realizable.*

Two of the authors recently determined some criteria on omega to determine the connectedness of the realizations G of a given degree sequence D, [**4**]. It is shown that all graphs with $\Omega(G) \leq -4$ are disconnected, and if $\Omega(G) \geq -2$, then the graph is potentially connected. As most of the degree sequences can be realizable in different ways, the number of ways the given degree sequence can be realizable as a connected/disconnected graph is determined by means of omega. Also it is shown that if the realization is a connected graph and $\Omega(G) = -2$, then certainly the graph should be acyclic. Similarly, it is shown that if the realization is a connected graph G and $\Omega(G) \geq 0$, then certainly the graph should be cyclic. Also, when $\Omega(G) \leq -4$, the components of the disconnected graph could not all be cyclic and if all the components of G are cyclic, then $\Omega(G) \geq 0$.

In [**6**], the conditions on omega invariant for the realizations of a given degree sequence to be acyclic, unicyclic, bicyclic and tricyclic are determined. In [**4**], the same authors obtained solutions of some extremal graph theory problems by studying the maximum number of componenets and the maximum number of loops in three types of realizations of a given degree sequence.

2. EFFECT OF VERTEX AND EDGE DELETION ON Ω

It is a widely used method in Mathematics to study smaller objects to obtain information on the larger object. In this section, we study edge-deleted, vertex-deleted, path-contracted and cycle-contracted graphs and calculate the change in Ω for these operations. Using these methods successively, it is possible to obtain all the information which can be obtained by means of omega for large graphs by means of some known graphs.

Given a graph G. Let $u, u_1, u_2, \cdots, u_k$ be some of the vertices of G and let $e, e_1, e_2, \cdots, e_t$ be some of the edges of G. The graph obtained by deleting the vertices $u_1, u_2, \cdots, u_k$ together with all the incident edges to these vertices is denoted by $G - \{u_1, u_2, \cdots, u_k\}$. Similarly, the graph obtained by deleting the edges $e_1, e_2, \cdots, e_t$ will be denoted by $G - \{e_1, e_2, \cdots, e_t\}$. We shall call these operations vertex deletion and edge deletion. When only one vertex u or one edge e is deleted from G, the resulting graph will shortly be denoted by $G-u$ and $G-e$, respectively.

When we delete a pendant edge e from a graph G, the pendant vertex in G at the end of this pendant edge becomes an isolated vertex in $G - e$. Therefore there are many occasions where we face with isolated vertices in a given graph. If these vertices had no effect, then we could easily omit them. When studying with Ω, the contribution of each isolated vertex of G to $\Omega(G)$ is fixed:

Lemma 1. *Let $G = \{v\}$ be a graph consisting of one vertex v and no edges. Then*

$$\Omega(G) = \Omega(\{v\}) = -2.$$

Proof. As the degree sequence of $G = \{v\}$ is $\{0^{(1)}\}$, the result follows from the definition of Ω. □

Recall that $\Omega(G) = -2$ for all connected acyclic graphs. In some sense, a graph consisting of a single vertex can be counted as acyclic. So this result is expected.

Let G be a graph and let $u_1, u_2, \cdots, u_n \in V(G)$. We use the following notations in the rest of the paper:

$$\Delta(G, u_1) = \Omega(G) - \Omega(G - u_1)$$

and

$$\Delta(G, \{u_1, u_2, \cdots, u_n\}) = \Omega(G) - \Omega(G - \{u_1, u_2, \cdots, u_n\}).$$

Then we have

Theorem 1. *Let the graph G have no loops. Deleting a vertex $v \in G$ of degree d_v reduces $\Omega(G)$ by $2d_v - 2$. That is*

$$\Delta(G, v) = 2d_v - 2.$$

Proof. Let the neighbours of v be $v_1, v_2. \cdots, v_k$ where $k = d_v$, see Fig. 1. Let $d_1, d_2, \cdots, d_k$ be the degrees of $v_1, v_2, \cdots, v_k$ in G, respectively. The contribution of v and its neighbours in G to $\Omega(G)$ is

$$d_1 - 2 + d_2 - 2 + \cdots + d_k - 2 + d_v - 2 = d_1 + d_2 + \cdots + d_k + d_v - 2k - 2$$

and the contribution of the neighbours of v in $G - v$ is

$$d_1 - 3 + d_2 - 3 + \cdots + d_k - 3 = d_1 + d_2 + \cdots + d_k - 3k.$$

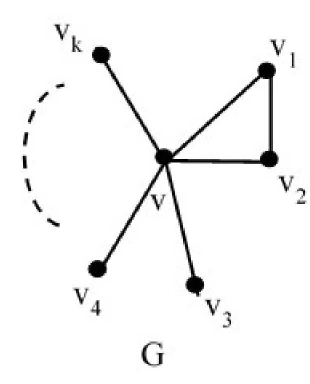

Figure 1 The neighbours of v

So the decrease in $\Omega(G)$ is

$$\begin{aligned}\Omega(G)-\Omega(G-v) &= d_v-2k-2+3k\\ &= d_v-2d_v-2+3d_v\\ &= 2d_v-2.\end{aligned}$$

□

Theorem 2. *Let v be a vertex in a graph G which is incident to $l \geq 1$ loops. Let $d_v = k$ be the degree of v. Then deleting v from G reduces $\Omega(G)$ by $\Delta(G,v) = 2(k-l-1)$.*

Proof. Let G be a graph with loops at a vertex $v \in V(G)$, see Fig. 2.

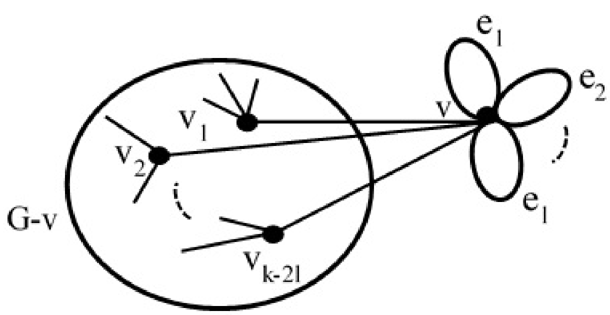

Figure 2 A graph G with l loops at the same vertex

The contribution of v and its neighbours $v_1, v_2, \cdots, v_{k-2l}$ to $\Omega(G)$ is $d_{v_1}-2+d_{v_2}-2+\cdots+d_{v_{k-2l}}-2+d_v-2$. Now delete v from G, see Fig. 3:

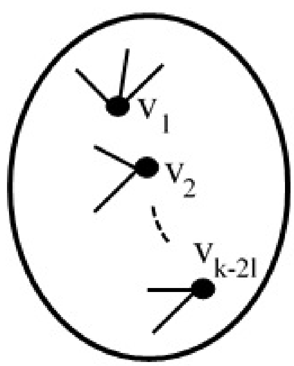

Figure 3 $G-v$

The contribution of the neighbours of v to $\Omega(G-v)$ is $d_{v_1}-3+d_{v_2}-3+\cdots+d_{v_{k-2l}}-3$ as the degree of each neighbour is reduced by 1 in $G-v$. Therefore the decrease in

$\Omega(G)$ will be $\Delta(G, v) = [d_{v_1} + d_{v_2} + \cdots + d_{v_{k-2l}} + k - 2(k - 2l + 1)] - [d_{v_1} + d_{v_2} + \cdots + d_{v_{k-2l}} - 3(k - 2l)] = 2(k - l - 1)$. □

Remark 1. *Note that by successively applying Theorem 2 to a graph G, we can calculate $\Omega(G - \{u_1, u_2, \cdots, u_n\})$. It is important to note that if we want to delete n vertices at one step, no two of these vertices must be adjacent.*

Example 1. *Consider the graph G in Fig. 4.*

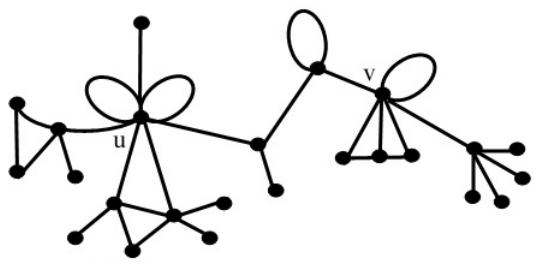

Figure 4 *A graph G*

Note that $\Omega(G) = 16$. We want to delete the vertices u and v and see the decrease $\Delta(G, \{u, v\})$ in $\Omega(G)$. First we delete u, see Fig. 5:

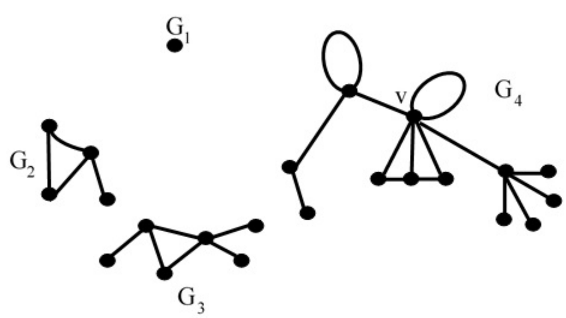

Figure 5 *$G - u$*

Note that $G - u$ is a disconnected graph with four components. These components have $\Omega(G_1) = -2$, $\Omega(G_2) = \Omega(G_3) = 0$ and $\Omega(G_4) = 6$. Using the additivity of Ω, we get $\Omega(G - u) = -2 + 0 + 0 + 6 = 4$. Indeed, by Theorem 2, $\Delta(G, u) = 2(9 - 2 - 1) = 12$ as $d_v = 9$ and there are two loops incident to v. Now delete v from $G - u$, see Fig. 6.

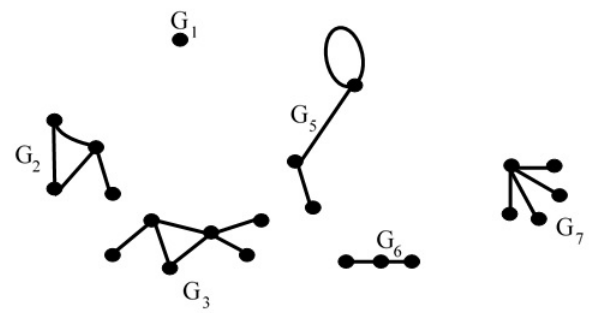

Figure 6 $G - \{u, v\}$

As above we find that $\Omega((G-u)-v) = \Omega(G-\{u,v\}) = -6$ *as* $\Omega(G_6) = \Omega(G_7) = -2$. *Indeed, by Theorem 2, we get* $\Delta(G-u,v) = 10$ *which is* $4-(-6)$. *If we want to delete both vertices* u *and* v *at the same time, then by Remark 1, they must be non-adjacent. As* u *and* v *are not adjacent, we can write*

$$\begin{aligned}\Delta(G,\{u,v\}) &= 2(9-2-1)+2(7-1-1)\\ &= 22.\end{aligned}$$

As $\Omega(G) = 16$, *we find that* $\Omega(G-\{u,v\}) = -6$. *That is the reduction process in Theorem 2 is additive:*

Lemma 2. *If* $u_1, u_2, \cdots, u_n$ *are the vertices of* G *no two of them are adjacent, then*

$$\Delta(G,\{u_1,u_2,\cdots,u_n\}) = \sum_{i=1}^{n} \Delta(G,u_i).$$

That is, $\Delta(G,u)$ *is additive on every set of non-adjacent vertices.*

But if two vertices are adjacent, then Δ *is not additive as in the following example. Let* G *be as in Fig. 7:*

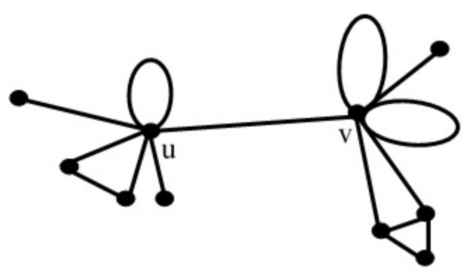

Figure 7 *A graph* G

We find that $\Omega(G) = 10$. *Deleting* u *and* v *at the same time gives a graph with five components, see Fig. 8.*

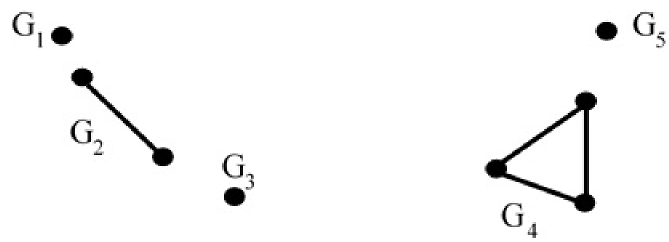

Figure 8 $G - \{u, v\}$

As $\Omega(G_1) = \Omega(G_2) = \Omega(G_3) = \Omega(G_5) = -2$ *and* $\Omega(G_4) = 0$*, we have* $\Omega(G - \{u, v\}) = -8$ *by Lemma 1. That is* $\Delta(G, \{u, v\}) = 18$*. But* $\Delta(G, u) = 10$ *and* $\Delta(G, v) = 10$.

Summarizing the results about vertex deletion gives the following:

Theorem 3. *Let* G *be a graph and let* $v \in V(G)$*. Then the size, order, number of closed regions and number of components of the vertex deleted graph* $G - v$ *are given explicitly as below:*

a. $m(G - v) = m(G) - d_v$,

b. $n(G - v) = n(G) - 1$,

c. $r(G - v) = r(G) - y$,

d. $c(G - v) = c(G) + d_v - 1 - y$,
where y *denotes the number of closed regions that have* v *as a vertex.*

The following result gives the change in omega when an edge is deleted:

Theorem 4. *Let* G *be a graph. Deleting an edge reduces* $\Omega(G)$ *by 2.*

Proof. Let $e = uv$ be an edge and d_u and d_v be the degrees of u and v, respectively. When the edge e is deleted, the numbers of d_u's and d_v's in $DS(G)$ decrease by 1 and the numbers of $d_u - 1$'s and $d_v - 1$'s increase by 1. Therefore the Ω decreases by 2. □

As a special case, we have

Theorem 5. *Let* G *be a graph. Deleting a loop reduces* $\Omega(G)$ *by 2.*

Proof. Let L be a loop in G with its unique vertex is of degree d. Deleting L reduces the number of d's in $DS(G)$ by 1 and increases $d - 2$'s in $DS(G)$ by 1. As a result, $\Omega(G)$ is reduced by 2. □

Theorem 6. *Contracting a path which does not belong to a cycle preserves* $\Omega(G)$*.*

Proof. Contracting a path which does not belong to a cycle does not change the number $r(G)$ of regions bounded by the edges of the graph G. So $\Omega(G)$ does not change as $\Omega(G) = 2(r(G) - c(G))$ where $c(G)$ is the number of components of G. □

The following special case follows directly:

Corollary 2. *Contracting bridges and pendant edges preserves* $\Omega(G)$.

Proof. It is clear from Theorem 6 as both a pendant edge and a bridge are paths of length one which does not belong to any cycle. □

Note that although $\Omega(G)$ is preserved when the bridges and pendant edges are contracted, the $DS(G)$ does not have to stay unchanged. Because of Theorem 6 and Corollary 2, instead of calculating the Ω of any graph G, we can first contract all bridges, paths which do not belong to any cycle, and pendant edges to obtain a smaller graph G' and we could calculate $\Omega(G')$ as $\Omega(G) = \Omega(G')$.

Example 2. *Consider the graph* G *with degree sequence* $D = \{1^{(2)}, 2^{(11)}, 3^{(5)}, 4^{(2)}\}$, *see Fig. 9.*

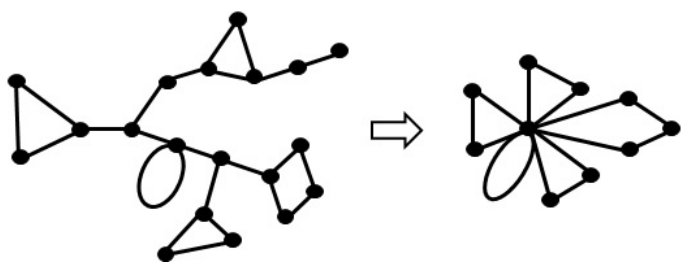

Figure 9

Then $\Omega(G) = 8$. *As in Fig. 9, we contract all bridges and pendant edges to obtain a smaller graph* G' *with degree sequence* $\{2^{(9)}, 10^{(1)}\}$. *Note that* $\Omega(G') = 8$. *That is* Ω *remains unchanged.*

Theorem 7. *Contracting a chord leaves* Ω *invariant.*

Proof. Let a chord $e = uv$ belong to two neighbor cycles C_n and C_m in a graph G, see Fig. 10.

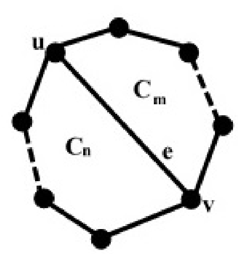

Figure 10 Before contraction

Contracting e gives the graph G' in Fig. 11.

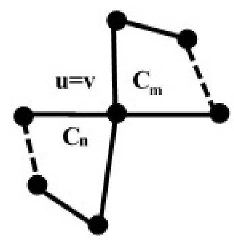

Figure 11 After contraction

Compare two degree sequences now. After the contraction, the vertices u and v will coincide, so that two 3's will be omitted from $DS(G)$ and a new vertex $u = v$ will appear in G' which will add a 4 to $DS(G)$. None of the other vertices will be effected by this contraction. So $\Omega(G') - \Omega(G) = 2 + 2 \cdot (-1) = 0$. □

Now we define a new operation which is useful in proving some results. Let $e = uv$ be an edge of a graph G. The operation of cutting this edge into two pieces and adding two new vertices w_1 and w_2 at both ends of this pieces so that we have two new edges uw_1 and vw_2 will be called opening up the edge e, see Fig. 12:

Figure 12 Opening up an edge e

The following result gives the effect of this operation on Ω:

Lemma 3. *Opening up an edge decreases Ω by 2.*

Proof. Let e be an edge of a graph G. After opening up this edge, there will be two new 1's in the degree sequence of the graph G. Calculating both Ω's gives the result. □

Lemma 4. *Let G be a connected graph with $\Omega(G) \geq 0$ and let $r(G)$ denote the number of closed regions that are bounded by the edges of G. After $1 + \frac{\Omega(G)}{2}$ opening ups, G becomes acyclic.*

Proof. By Theorem 1, G has $\frac{\Omega(G)}{2} + 1$ closed regions. As each opening up destroys one closed region, the result follows. □

By means of this new operation, we can give a second proof of Theorem 1 as follows: Let G be connected. If G is acyclic, then G is a tree and $\Omega(G) = -2$. So G, as a tree, has $\frac{\Omega(G)}{2} + 1 = \frac{-2}{2} + 1 = 0$ closed regions as expected. Therefore we can assume that G is cyclic. Then $\Omega(G) \geq 0$. Let G have $r(G)$ closed regions. To decrease the number of these regions by one, we need an opening up. Each opening up will decrease the Ω of the graph by 2 by Lemma 3. If the resulting graph after $r(G)$ of opening ups is denoted by G', then $\Omega(G') = \Omega(G) - 2r(G)$. As G' will be acyclic by Lemma 4, $\Omega(G') = -2$, giving the result.

REFERENCES

1. M. Aigner, E. Triesch, *Realizability and uniqueness in graphs,* Discrete Math. **136**(1994), 3-20.
2. M. D. Barrus, E. Donovan, *Neighborhood degree lists of graphs,* Discrete Math. **341(1)** (2018), 175-183.
3. S. A. Choudum, *On forcibly connected graphic sequences,* Discrete Math. **96**(1991), 175-181.
4. S. Delen, I. N. Cangul, *Extremal Problems on Components and Loops in Graphs,* Acta Mathematica Sinica English Series **34**(2018), 1-11.
5. S. Delen, I. N. Cangul, *A New Graph Invariant,* Turkish Journal of Analysis and Number Theory **6(1)**(2018), 30-33.
6. S. Delen, A. Yurttas, M. Togan, U. Ana, I. N. Cangul, *Omega Invariant of Graphs and Cyclicness,* Applied Sciences (2019) (preprint)
7. S. L. Hakimi, *On the realizability of a set of integers as degrees of the vertices of a graph,* J. SIAM Appl. Math. **10**(1962), 496-506.
8. V. Havel, *A remark on the existence of finite graphs (Czech),* Časopic Pěst. Mat. **80**(1955), 477-480.

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