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ON SOME NEW SEQUENCE SPACES OF ORDER α DEFINED BY INFINITE MATRIX

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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The goal of this paper is to study some new sequence spaces of order α that are defined using modulus function and infinite matrix A.

1. INTRODUCTION AND BACKGROUND

Following Ruckle [**10**] and Maddox [**6**], we recall that a modulus f is a function $f : [0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(x)$ for all $x, y \geq 0$,

(iii) f increasing,

(iv) f is continuous from the right at zero.

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Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$.

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, for $0 < p \leq 1$ is unbounded, but $f(x) = \dfrac{x}{1+x}$ is bounded.

Ruckle used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}$$

The space $L(f)$ is closely related to the space l_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

Maddox [**6**] introduced and examined some properties of the sequence spaces $w_0(f)$, $w(f)$ and $w_\infty(f)$ defined using a modulus f, which generalized the well-known spaces w_0, w and w_∞ of strongly summable sequences.

Recently E. Savas [**11**] generalized the concept of strong convergence by using a modulus f and examined some properties of the corresponding new sequence spaces.
Subsequently a lot of interesting investigations have been done by various authors on several related notions of modulus function (see for example [**7, 8, 9, 11, 13**].

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman at al [**4**] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

There is a strong connection between N_θ and the space w of strongly Cesàro summable sequences which is defined below,

$$w = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=0}^{n} |x_k - L| = 0, \text{ for some } L \right\}.$$

In the special case where $\theta = (2^r)$, we have $N_\theta = w$, (see, [**5**]).

Let $A = (a_{nk})$ be a nonnegative regular matrix summability method. Maddox's results are extended by Connor [**2**] as follows:

Definition 1. *Let f be a modulus and A be a nonnegative regular summability method. We let*

$$w(A,f)=\left\{x:\lim_n\sum_{k=1}^{\infty}a_{nk}f(|x_k-L|)=0\right\}$$

and

$$w(A,f)_0=\left\{x:\lim_n\sum_{k=1}^{\infty}a_{nk}f(|x_k|)=0\right\}.$$

By a φ-function we understood a continuous non-decreasing function $\varphi(u)$ defined for $u\geq 0$ and such that $\varphi(0)=0, \varphi(u)>0$, for $u>0$ and $\varphi(u)\to\infty$ as $u\to\infty$.
A φ-function φ is called non weaker than a φ-function ψ if there are constants $c,b,k,l>0$ such that $c\psi(lu)\leq b\varphi(ku)$, (for all large u) and we write $\psi\prec\varphi$.
A φ-function φ and ψ are called equivalent if there are positive constant b_1,b_2,c,k_1,k_2,l such that $b_1\varphi(k_1u)\leq c\psi(lu)\leq b_2\varphi(k_2u)$, (for all large u) and we write $\varphi\sim\psi$.
A φ-function φ is said to satisfy the condition (Δ_2), (for all large u) if for some constant $k>1$ there is satisfied the inequality $\varphi(2u)\leq k\varphi(u)$, (see, [**12**], [**14**]).

On the other hand in [**3**] a different direction was given to the study of Cesàro-type summability spaces of order α, $0<\alpha\leq 1$ and lacunary statistical convergence of order α where the notion of lacunary statistical convergence was introduced by replacing h_r by h_r^{α} in the denominator in the definition of lacunary statistical convergence. Also the notion of statistical convergence of order α was studied in [**1**]

The idea of lacunary strong (A,φ) with respect to a modulus function was introduced and studied by Waszak [**14**]. In the present paper, we introduce and study some properties of the following sequence spaces of order α which is defined using the φ- function, infinite matrix and modulus.

2. MAIN RESULTS

Let φ and f be given φ-function and modulus function, respectively and $p=(p_k)$ be a sequence of positive real numbers. Moreover, let $A=(a_{nk})$ be the infinite matrix, a lacunary sequence $\theta=(k_r)$ and $0<\alpha\leq 1$ be given. Then we define the following sequence spaces,

$$N_{\theta}^{\alpha}(A,\varphi,f,p)_0=\left\{x=(x_k):\lim_r\frac{1}{h_r^{\alpha}}\sum_{n\in I_r}f\Big(\left|\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\right|\Big)^{p_k}=0\},$$

where h_r^{α} denote the αth power $(h_r)^{\alpha}$ of h_r, that is $h^{\alpha}=(h_r^{\alpha})=(h_1^{\alpha},h_2^{\alpha},h_3^{\alpha},....)$.

If $x\in N_{\theta}^{\alpha}(A,\varphi,f,p)_0$, the sequence x is said to be lacunary strong (A,φ)-convergent to zero with respect to a modulus f. If we take $\varphi(x)=x$ for all x, we write

$$N_{\theta}^{\alpha}(A,f,p)_0=\left\{x=(x_k):\lim_r\frac{1}{h_r}\sum_{n\in I_r}f(\left|\sum_{k=1}^{\infty}a_{nk}x_k\right|)^{p_k}=0\right\}.$$

If we take $f(x)=x$, we write

$$N_\theta^\alpha(A,\varphi,p)_0=\left\{x=(x_k):\lim_r \frac{1}{h_r}\sum_{n\in I_r}\left|\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\right|^{p_k}=0\right\}.$$

If we take $p_k=p$,for all k, we have

$$N_\theta^\alpha(A,\varphi,f)_0=\left\{x=(x_k):\lim_r \frac{1}{h_r^\alpha}\sum_{n\in I_r}f\Big(\left|\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\right|\Big)^{p}=0\right\}.$$

If we take $A=I$ and $\varphi(x)=x$ respectively, then we have

$$(N_\theta^\alpha)_0=\left\{x=(x_k):\lim_r \frac{1}{h_r^\alpha}\sum_{k\in I_r}f(|x_k|)^{p_k}=0\right\}.$$

If we consider the matrix $A=(a_{nk})$ as follows:

$$a_{nk}:=\begin{cases}\frac{1}{n}, & \text{if} \quad n\geq k,\\ 0, & \text{otherwise.}\end{cases}$$

then we write,

$$N_\theta^\alpha(\mathbf{C},\varphi,f)_0=\left\{x=(x_k):\lim_r \frac{1}{h_r^\alpha}\sum_{n\in I_r}f\Big(\left|\frac{1}{n}\sum_{k=1}^{n}\varphi(|x_k|)\right|\Big)^{p_k}=0\right\}.$$

We now have

Theorem 1. *Let us assume that the matrix $A=(a_{nk})$ satisfies the condition*

$$a_{n1}+a_{n2}+...\leq K$$

for $n=1,2,...$ then let φ-functions φ and ψ satisfy the condition (Δ_2) for large u. If $\psi\prec\varphi$ then $N_\theta^\alpha(A,\varphi,f,p)\subset N_\theta^\alpha(A,\psi,f,p)$,

Proof. Let $x=(x_k)\in N_\theta^\alpha(A,\varphi,f,p)$. By assumption we have

(1) $$\varphi(|x_k|)\leq b\varphi(c|x_k|)$$

for b, c, u_0 and $|x_k| > u_0$. Let us denote $x = x^1 + x^2$, where $x^1 = (x_k^1)$ and $x_k^1 = (x_k)$ for $|x_k| < u_0$ for remaining values of k. It is easily seen that $x^1 \in N_\theta^\alpha(A, \varphi, f, p)$. Furthermore, by assumption and inequality (2.1), we get

$$\begin{aligned} \frac{1}{h_r^\alpha} \sum_{n \in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk} \psi(|x_k^2|) \Big| \Big)^{p_n} &\leq \frac{1}{h_r^\alpha} \sum_{n \in I_r} f\Big(b \Big| \sum_{k=1}^{\infty} a_{nk} \varphi(c|x_k^2|) \Big| \Big)^{p_n} \\ &\leq \frac{L}{h_r^\alpha} \sum_{n \in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk} \varphi(|x_k^2|) \Big| \Big)^{p_n}. \end{aligned}$$

where the constant L is connected with properties of f and φ. Hence we get $x^2 \in N_\theta^\alpha(A, \psi, f, p)$ and finally $x \in N_\theta^\alpha(A, \psi, f, p)$. □

Theorem 2. *Let the φ- function $\varphi(u)$ satisfies the condition (Δ_2) and let the matrix has the property*

$$a_{n1} + a_{n2} + ... \leq K$$

for $n = 1, 2, ...$ Then the following conditions are true.
(a) If $x = (x_k) \in N_\theta^\alpha(A, \varphi, f, p)$ and α is an arbitrary number, then $\alpha x \in N_\theta^\alpha(A, \varphi, f, p)$.
(b) If $x, y \in N_\theta^\alpha(A, \varphi, f, p)$ where $x = (x_k)$, $y = (y_k)$ and α, β are given numbers, then $\alpha x + \beta y \in N_\theta^\alpha(A, \varphi, f, p)$.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 3. *If $lim\ p_k > 0$ and $x = (x_k)$ is strongly $N_\theta^\alpha(A, \varphi, f, p)$ -summable to L with respect to the modulus function f, then $N_\theta^\alpha(A, \varphi, f, p) - lim\ x_k$ uniquely.*

Proof. Let $lim\ p_k = s > 0$. Suppose that $N_\theta^\alpha(A, \varphi, f, p) - lim\ x_k = L_1$ and $N_\theta^\alpha(A, \varphi, f, p) - lim\ x_k = L_2$. Then

$$\lim_r \frac{1}{h_r^\alpha} \sum_{n \in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk} \varphi(|x_k - L|) \Big| \Big)^{p_n} = 0,$$

and

$$\lim_r \frac{1}{h_r^\alpha} \sum_{n \in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk} \varphi(|x_k - L_1|) \Big| \Big)^{p_n} = 0$$

.

Definition of f, we write;

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|L-L_1|)\Big|\Big)^{p_n} &\\ &\leq \frac{D}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k-L|)\Big|\Big)^{p_n}\\ &+ \frac{D}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k-L_1|)\Big|\Big)^{p_n},\end{aligned}$$

where $sup\ p_k = H$ and $D = max(1, 2^{H-1})$. Hence

$$\frac{1}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|L-L_1|)\Big|\Big)^{p_k} = 0.$$

Since $\lim_k\ p_k = s$, it is easy to see that the limit is unique. □

Theorem 4. *Let f be a any modulus function and let φ-function φ, infinite matrix A, $p = (p_k)$ be a sequence of positive real numbers and the sequence θ be given. If*

$$w^\alpha(A,\varphi,f,p)_0 = \left\{x=(x_k) : \lim_m \frac{1}{m}\sum_{n=1}^{m} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{k_L}|)\Big|\Big)^{p_k} = 0,\right\}.$$

then the following relations are true :
(a) *If* $\liminf_r q_r > 1$ *then we have* $w^\alpha(A,\varphi,f,p)_0 \subseteq N_\theta^\alpha(A,\varphi,f,p)_0$,
(b) *If* $\sup_r q_r < \infty$, *then we have* $N_\theta^\alpha(A,\varphi,f,p)_0 \subseteq w^\alpha(A,\varphi,f,p)_0$,
(c) $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$,*then we have* $N_{\theta_0}^\alpha(A,\varphi,f,p)_0 = w^\alpha(A,\varphi,f)_0$.

Proof. (a) Let us suppose that $x \in w^\alpha(A,\varphi,f,p)$. There exists $\delta > 0$ such that $q_r > 1+\delta$ for all $r \geq 1$ and we have $h_r/k_r \geq \delta/(1+\delta)$ for sufficiently large r. Then, for all i,

$$\begin{aligned}\frac{1}{k_r^\alpha}\sum_{n=1}^{k_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} &\\ &\geq \frac{1}{k_r^\alpha}\sum_{n\in I_r} f\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_k}\\ &= \frac{h_r^\alpha}{k_r^\alpha}\frac{1}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n}\\ &\geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n}.\end{aligned}$$

Hence, $x \in N_\theta^\alpha(A,\varphi,f,p)_0$.

(b) If $\limsup_r q_r < \infty$ then there exist $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $x \in N_\theta^\alpha(A,\varphi,f,p)_0$ and ε is an arbitrary positive number, then there exists an index j_0 such that for every $j \geq j_0$ and all i,

$$R_j = \frac{1}{h_j^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} < \varepsilon$$

Thus, we can also find $K > 0$ such that $R_j \leq K$ for all $j = 1, 2, \ldots$. Now let m be any integer with $k_{r-1} \leq m \leq k_r$, then we obtain, for all i

$$I = \frac{1}{m^\alpha}\sum_{n=1}^{m} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} \leq \frac{1}{k_{r-1}^\alpha}\sum_{n=1}^{k_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} = I_1 + I_2$$

where

$$I_1 = \frac{1}{k_{r-1}^\alpha}\sum_{j=1}^{j_0}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n}$$

$$I_2 = \frac{1}{k_{r-1}^\alpha}\sum_{j=j_0+1}^{m}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n}$$

It is easy to see that,

$$\begin{aligned} I_1 &= \frac{1}{k_{r-1}^\alpha}\sum_{j=1}^{j_0}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} \\ &= \frac{1}{k_{r-1}^\alpha}\Big(\sum_{n\in I_1} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} + \ldots + \sum_{n\in I_{j_0}} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n}\Big) \\ &\leq \frac{1}{k_{r-1}^\alpha}(h_1R_1 + \ldots + h_{j_0}R_{j_0}), \\ &\leq \frac{1}{k_{r-1}^\alpha} j_0 k_{j_0}^\alpha sup_{1\leq i\leq j_0} R_i, \\ &\leq \frac{j_0 k_{j_0}^\alpha}{k_{r-1}^\alpha} K. \end{aligned}$$

Moreover, we have for all i

$$\begin{aligned} I_2 &= \frac{1}{k_{r-1}^{\alpha}} \sum_{j=j_0+1}^{m} \sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} \\ &= \frac{1}{k_{r-1}^{\alpha}} \sum_{j=j_0+1}^{m} \Big(\frac{1}{h_j} \sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} h_j \\ &\leq \varepsilon \frac{1}{k_{r-1}^{\alpha}} \sum_{j=j_0+1}^{m} h_j, \\ &\leq \varepsilon \frac{k_r^{\alpha}}{k_{r-1}^{\alpha}}, \\ &= \varepsilon q_r^{\alpha} < \varepsilon.M. \end{aligned}$$

Thus $I \leq \frac{j_o k_{j_o}^{\alpha}}{k_{r-1}^{\alpha}} K + \varepsilon.M$. Finally, $x \in w^{\alpha}(A,\psi,f,p)$.
The proof of (c) follows from (a) and (b). This completes the proof. □

Theorem 5. *Let* $0 < \alpha \leq \beta \leq 1$ *and* p *be a positive real number, then* $N_{\theta}^{\alpha}(A,\varphi,f)_0 \subseteq N_{\theta}^{\beta}(A,\varphi,f)_0$.

Proof. Let $x = (x_k) \in N_{\theta}^{\alpha}(A,\varphi,f)_0$. Then given α and β such that $< \alpha \leq \beta \leq 1$ and a positive real number p, we write

$$\frac{1}{h_r^{\beta}} \sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p} \leq \frac{1}{h_r^{\alpha}} \sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p}$$

and we get that $N_{\theta}^{\alpha}(A,\varphi,f)_0 \subseteq N_{\theta}^{\beta}(A,\varphi,f)_0$. □

The proof of the following result is a consequence of Theorem 2.2.

Corollary 6. *Let* $0 < \alpha \leq \beta \leq 1$ *and* p *be a positive real number. Then*

i) *If* $\alpha = \beta$, *then* $N_{\theta}^{\alpha}(A,\varphi,f)_0 = N_{\theta}^{\beta}(A,\varphi,f)_0$.

ii) $N_{\theta}^{\alpha}(A,\varphi,f)_0 \subseteq N_{\theta}(A,\varphi,f)_0$ *for each* $\alpha \in (0,1]$ *and* $0 < p < \infty$.

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