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A STUDY OF MÖBIUS-BERNOULLI NUMBERS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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Let k be a non-negative integer. We define the Möbius-Bernoulli numbers which is denoted by $M_k(n)$ and double Möbius-Bernoulli numbers $M_k(n,n')$ for some $n,n' \in \mathbb{N}$. In this article, we find formula of $M_k(n,n')$ and examples.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The Bernoulli polynomial $B_k(x)$ is usually defined by means of the following generating functions:

$$\frac{ue^{ux}}{e^u-1}=\sum_{k=0}^{\infty}B_k(x)\frac{u^k}{k!}.$$

Note that $B_k(x)$ are monic polynomials with rational coefficients and Bernoulli numbers $B_k := B_k(0)$. More basic results of Bernoulli polynomials are in [**6**]. The Bernoulli numbers $B_k^{(n)}$ of order n are defined by

$$\left(\frac{u}{e^u-1}\right)^n=\sum_{k=0}^{\infty}B_k^{(n)}\frac{u^k}{k!}. \tag{1.1}$$

For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $k \geq 0$ the number $M_k(n)$ is defined as follows:

$$\sum_{k=0}^{\infty}M_k(n)\frac{t^k}{k!}=\sum_{d|n}\mu(d)\frac{t}{e^{dt}-1}, \quad |t|<\frac{2\pi}{n}.$$

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The Möbius-Bernoulli numbers $M_k(n)$ are analogue Bernoulli numbers with $M_k(1) = B_k$, where $\mu(n)$ is the Möbius function. Let n' be a positive integer, we investigate the double Möbius-Bernoulli numbers $M_k(n,n')$ given by

$$M_k(n,n') = \sum_{j=0}^{k} \binom{k}{j} M_j(n) M_{k-j}(n'). \tag{1.2}$$

Lemma 1. *Let k be any non-negative integer, and n be a positive integer. Then*

$$M_k(n) = B_k \prod_{p|n} (1 - p^{k-1}). \tag{1.3}$$

Here, p are prime numbers with $p|n$.

Proof. Since $\mu(d) = 0$ if d is not a square-free integer, we have

$$\sum_{d|n} \mu(d) d^{k-1} = \sum_{d|n^*} \mu(d) d^{k-1}, \tag{1.4}$$

where $n^* = p_1 \cdots p_r$ with p_i distinct prime factors of n. Consider

(1.5)
$$\sum_{k=0}^{\infty} M_k(n) \frac{t^k}{k!} = \sum_{d|n} \frac{\mu(d)}{d} \frac{dt}{e^{dt}-1} = \sum_{d|n} \frac{\mu(d)}{d} \sum_{k=0}^{\infty} B_k \frac{d^k t^k}{k!} = \sum_{k=0}^{\infty} B_k (\sum_{d|n} \mu(d) d^{k-1}) \frac{t^k}{k!}.$$

By (1.4) and (1.5), we get

$$\begin{aligned} M_k(n) &= B_k \sum_{d|n^*} \mu(d) d^{k-1} = B_k \sum_{\{i_1,\cdots,i_s\}\subset I} \mu(p_{i_1}\cdots p_{i_s})(p_{i_1}\cdots p_{i_s})^{k-1} \\ &= B_k \sum_{\{i_1,\cdots,i_s\}\subset I} (-1)^s (p_{i_1}\cdots p_{i_s})^{k-1} = B_k \prod_{p|n} (1-p^{k-1}), \end{aligned}$$

where $I = \{1, 2, \cdots, r\}$. This completes the proof of Lemma 1. □

Remark 2. General results of Lemma 1 are in [**4**, Theorem 1].

By Lemma 1 and (1.2), we get

$$M_k(n,n') = \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} \prod_{p|n} (1-p^{j-1}) \prod_{q|n'} (1-q^{k-j-1}). \tag{1.6}$$

Consider the generating function :

(1.7)
$$\sum_{k=0}^{\infty} M_k(n,n') \frac{t^k}{k!} = (\sum_{k=0}^{\infty} M_k(n) \frac{t^k}{k!} \cdot)(\sum_{k=0}^{\infty} M_k(n') \frac{t^k}{k!}) = \sum_{\substack{d|n \\ d'|n'}} \mu(d)\mu(d') \frac{t^2}{(e^{dt}-1)(e^{d't}-1)}.$$

Note that, by definition,

$$\frac{t^2}{(e^{dt}-1)(e^{d't}-1)} = \sum_{k=0}^{\infty} B_k((d,d'))\frac{t^k}{k!}, \tag{1.8}$$

where $B_k((d,d'))$ are Bernoulli-Barnes numbers (for the general definition, see (2.12)). By (1.7) and (1.8), we have the following result:

Lemma 3. *Let n and n' be positive integers. Then by* (1.6)*,*

$$M_k(n,n') = \sum_{\substack{d|n,\\ d'|n'}} \mu(d)\mu(d')B_k((d,d')), \tag{1.9}$$

where $B_k((d,d'))$ are Bernoulli-Barnes numbers defined by (1.8)*.*
In particular, if n, n' are relative prime then, $\mu(dd') = \mu(d)\mu(d')$.

Example 4. We consider the following examples :
If $n' = 1$ and $n = p$ is a prime, then

$$\begin{aligned} M_k(p,1) &= \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}\prod_{p|n}(1-p^{j-1}) \\ &= \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}(1-p^{j-1}) = B_k^{(2)} - \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}p^{j-1}. \end{aligned}$$

It is well known [**2**, (2)] that

$$B_k^{(2)} = \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j} = -kB_{k-1} - (k-1)B_k. \tag{1.10}$$

Hence, $k \geq 1$ then

$$M_k(p,1) = -kB_{k-1} - (k-1)B_k - \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}p^{j-1}. \tag{1.11}$$

Example 5. Let α and β be positive integers. Applying the same method as Example 4, we get

$$\begin{aligned} M_k(2^\alpha,2^\beta) &= \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}(1-2^{j-1})(1-2^{k-j-1}) \\ &= \frac{k}{2}B_{k-1} - \frac{k(k-1)}{4}E_{k-2}(0) - 2^{k-2}(kB_{k-1}+(k-1)B_k). \end{aligned}$$

Lemma 6. *Let n, n' be positive integers and n^*, n'^* be their square free parts, respectively. Precisely, let $p_1,\cdots,p_r$ and $q_1,\cdots,q_s$ be the distinct prime factors of n and n', respectively. Then, clearly, $n^* = p_1\cdots p_r$ and $n'^* = q_1\cdots q_s$. Then we have $M_k(n,n') = M_k(n^*,n'^*)$ and $M_k(n) = M_k(n^*)$.*

Similarly, as Example 5, we get

$$
\begin{aligned}
M_k(p_1^\alpha, p_2^\beta) &= B_k^{(2)} - \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} (p_1^{j-1} + p_2^{k-j-1}) + \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} p_1^{j-1} p_2^{k-j-1} \\
&= B_k^{(2)} - \frac{1}{p_1} \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} p_1^{j} - \frac{1}{p_2} \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} p_2^{k-j} \\
&\quad + \frac{p_2^{k-1}}{p_1} \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} (\frac{p_1}{p_2})^{j},
\end{aligned}
$$

where p_1, p_2 are primes and α, β are positive integers. Hence, to compute $M_k(p_1^\alpha, p_2^\beta)$ explicitly, we need to compute $\sum_{j=0}^{k} \lambda^j \binom{k}{j} B_j B_{k-j}$ with $\lambda \in \mathbb{R}$.

2. MÖBIUS DIVISOR FUNCTIONS

Let $n^* := \mathrm{rad}(n) = \prod_{p|n} p$. We can deduce that

$$\sum_{d|n} \mu(d) \frac{t}{e^{dt} - 1} = \sum_{d|n^*} \mu(d) \frac{t}{e^{dt} - 1}$$

and

$$
\begin{aligned}
&\left(\sum_{d|n} \mu(d) \frac{t}{e^{dt} - 1} \right)^2 = \left(\sum_{d|n^*} \mu(d) \frac{t}{e^{dt} - 1} \right)^2 \\
&= \sum_{d_1, d_2 | n^*} \mu(d_1)\mu(d_2) \frac{t^2}{(e^{d_1 t} - 1)(e^{d_2 t} - 1)} \\
&= \sum_{D|n^*} \sum_{\substack{d_1, d_2 | n^* \\ (d_1, d_2) = D}} \mu(d_1)\mu(d_2) \frac{t^2}{(e^{d_1 t} - 1)(e^{d_2 t} - 1)} \\
&= \sum_{D|n^*} \sum_{\substack{(d_1, d_2) = 1 \\ d_1, d_2 | n^*/D}} \mu(Dd_1)\mu(Dd_2) \frac{t^2}{(e^{Dd_1 t} - 1)(e^{Dd_2 t} - 1)} \\
&= \sum_{D|n^*} \sum_{\substack{(d_1, d_2) = 1 \\ d_1, d_2 | n^*/D}} \mu(d_1 d_2) \frac{t^2}{((e^{Dt})^{d_1} - 1)((e^{Dt})^{d_2} - 1)} \\
&= \sum_{D|n^*} \sum_{d_1, d_2 | n^*/D} \mu(d_1 d_2) \frac{t^2}{((e^{Dt})^{d_1} - 1)((e^{Dt})^{d_2} - 1)}.
\end{aligned}
$$

The Bernoulli-Barnes numbers $B_k(\mathfrak{a})$, defined for a fixed vector $\mathfrak{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$ through

$$\frac{z^n}{(e^{a_1 z}-1)\cdots(e^{a_n z}-1)} = \sum_{k=0}^{\infty} B_k(\mathfrak{a})\frac{z^k}{k!}. \tag{2.12}$$

Combining (1.1) and (2.12), we get the relation between Bernoulli-Barnes numbers and Bernoulli numbers:

$$B_k(\mathfrak{a}) = \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1, \cdots, m_n} a_1^{m_1-1}\cdots a_n^{m_n-1} B_{m_1}\cdots B_{m_n}, \tag{2.13}$$

where $\binom{k}{m_1,\cdots,m_n} = \frac{k!}{m_1!\cdots m_n!}$.

In particular, for any prime p, we have

$$B_k((p,1)) = \sum_{j=0}^{k}\binom{k}{j} p^{j-1} B_j B_{k-j}.$$

In the following, we want to compute $B_k((p,1))$. By definition of Bernoulli-Barnes numbers,

$$\frac{t^2}{(e^t-1)(e^{pt}-1)} = \sum_{k=0}^{\infty} B_k((p,1))\frac{t^k}{k!}.$$

Lemma 7. *Let p be a prime. Then*

$$\frac{1}{(x-1)(x^p-1)} = \frac{1}{p}\frac{1}{(x-1)^2} - \frac{p-1}{2p}\frac{1}{x-1} + \sum_{j=1}^{p-1}\frac{1}{p(\zeta_p^{-j}-1)}\cdot\frac{1}{\zeta_p^j x - 1},$$

where $\zeta_p = e^{\frac{2\pi i}{p}} \in \mathbb{C}$.

Proof. It is easily checked by the Chinese remainder theorem.

□

From Lemma 7, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k((p,1))\frac{t^k}{k!} &= \frac{t^2}{(e^t-1)(e^{tp}-1)} \\
&= \frac{1}{p}\frac{t^2}{(e^t-1)^2} - \frac{p-1}{2p}\frac{t^2}{e^t-1} + \sum_{j=1}^{p-1}\frac{1}{p(\zeta_p^{-j}-1)}\cdot\frac{t^2}{\zeta_p^j e^t-1} \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{p}B_k^{(2)} - \frac{p-1}{2p}kB_{k-1} + \sum_{j=1}^{p-1}\frac{1}{p(\zeta_p^{-j}-1)}kB_{k-1}((\zeta_p^j))\right)\frac{t^k}{k!},
\end{aligned} \tag{2.14}$$

where $B_k((\zeta_p^j))$, so called Apostol-Bernoulli numbers, are exactly defined this way. And we assume B_k and $B_k((\zeta_p^j))$ are both zero if $k<0$.
Extracting the coefficient of both side of $\frac{t^k}{k!}$ by (2.14) yields the following theorem.

Theorem 8. *Notations as above, for $k \geq 0$, we have,*

$$B_k((p,1)) = \frac{1}{p}B_k^{(2)} - \frac{p-1}{2p}kB_{k-1} + \sum_{j=1}^{p-1}\frac{k}{p(\zeta_p^{-j}-1)}B_{k-1}((\zeta_p^j)).$$

For $\omega \in \mathbb{C}-\{0\}$ and x being a variable, the n-th Apostol-Bernoulli polynomial $B_n(x;\omega)$ is defined by the generating function

$$\sum_{n=0}^{\infty} B_n(x;\omega)\frac{t^n}{n!} = \frac{te^{xt}}{\omega e^t - 1}, \quad (|t+\log(\omega)| < 2\pi),$$

where $\omega = |\omega|e^{i\theta}$, $-\pi \leq \theta < \pi$ and $\log(\omega) = \log|\omega| + i\theta$.

Let n be a positive integer and $\omega \neq 1$. It is well known that

$$B_n(x;\omega) = \sum_{k=0}^{n}\binom{n}{k}B_k(0;\omega)x^{n-k}$$

and where, by definition, the Apostol-Bernoulli numbers $B_k((\omega)) = B_k(0;\omega)$

(2.15)

$$B_k((\omega)) = B_k(0;\omega) = \frac{k}{\omega}\sum_{j=0}^{k-1}(-1)^j j!S(k-1,j)(\frac{\omega}{\omega-1})^{j+1}, \quad \text{for } k \geq 0 \text{ (see [\textbf{1}], [\textbf{3}])},$$

with $S(k,j)$ being Stirling numbers of the second kind.

It is well known that

$$B_k^{(2)} = -kB_{k-1} - (k-1)B_k, \text{ for } k \geq 0. \tag{2.16}$$

By Theorem 8, (2.15) and (2.16), we get

$$\begin{aligned} pB_k((p,1)) &= -(\frac{p-1}{2}+k)B_{k-1} - (k-1)B_k \\ &+ k(k-1)\sum_{j=1}^{p-1}\frac{1}{1-\zeta_p^j}\sum_{l=0}^{k-2}(-1)^l l!S(k-2,l)\left(\frac{\zeta_p^j}{\zeta_p^j-1}\right)^{l+1}, \quad \text{for } k \geq 0. \end{aligned}$$

3. FORMULA OF $M_K(N, N')$

In this section, we want to compute $M_k(n, n')$ with n, n' being positive integers. First, assume $(n, n') = 1$. By Lemma 3, we have

$$M_k(n, n') = \sum_{d|n, d'|n'} \mu(dd')B_k((d, d')).$$

Lemma 9. *Let $(d, d') = 1$. Then we have*

$$\frac{1}{(x^d-1)(x^{d'}-1)} = \frac{1}{dd'}\frac{1}{(x-1)^2} - \frac{d+d'-2}{2dd'}\frac{1}{(x-1)} + \sum_{j=1}^{d-1}\frac{1}{d(\zeta_d^{-jd'}-1)}\cdot\frac{1}{(\zeta_d^j x-1)} + \sum_{j=1}^{d'-1}\frac{1}{d'(\zeta_{d'}^{-jd}-1)}\cdot\frac{1}{(\zeta_{d'}^j x-1)},$$

where $\zeta_d = e^{\frac{2\pi i}{d}}$ and $\zeta_{d'} = e^{\frac{2\pi i}{d'}}$ are roots of unity.

Proof. It is easily checked by the Chinese remainder theorem. □

Remark 10. In Lemma 9, if $d = p$ and $d' = 1$, then we recover Lemma 7.

Theorem 11. *Let $n \geq 1$, $(d, d') = 1$ and $x, y \in \mathbb{R}$. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}d^k d'^{n-k}B_k(x)B_{n-k}(y)$$
$$=(1-n)B_n(dx+d'y) + (d(x-\frac{1}{2}) + d'(y-\frac{1}{2}))nB_{n-1}(dx+d'y)$$
$$+ nd'\sum_{j=1}^{d-1}\frac{1}{(\zeta_d^{-jd'}-1)}B_{n-1}(dx+d'y;\zeta_d^j) + nd\sum_{j=1}^{d'-1}\frac{1}{(\zeta_{d'}^{-dj}-1)}B_{n-1}(dx+d'y;\zeta_{d'}^j).$$

Proof. Using the generating function of Bernoulli polynomials, we get

$$\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\binom{n}{k}d^k d'^{n-k}B_k(x)B_{n-k}(y))\frac{t^n}{n!} = \frac{dte^{dtx}}{e^{dt}-1}\cdot\frac{d'te^{d'tx}}{e^{d't}-1}.$$

Substituting x by e^t in Lemma 9, we have

$$\begin{aligned}
&\frac{dd't^2e^{(dx+d'y)t}}{(e^{dt}-1)(e^{d't}-1)}\\
=&\frac{t^2e^{(dx+d'y)t}}{(e^t-1)^2}-(\frac{(d+d'-2)t}{2})(\frac{te^{(dx+d'y)t}}{e^t-1})+\sum_{j=1}^{d-1}\frac{d't}{(\zeta_d^{-jd'}-1)}\cdot(\frac{te^{(dx+d'y)t}}{\zeta_d^je^t-1})\\
&+\sum_{j=1}^{d'-1}\frac{dt}{(\zeta_{d'}^{-jd}-1)}\cdot(\frac{te^{(dx+d'y)t}}{\zeta_{d'}^je^t-1})\\
=&\sum_{n=0}^{\infty}B_n^{(2)}(dx+d'y)\frac{t^n}{n!}-\frac{(d+d'-2)}{2}\sum_{n=0}^{\infty}B_n(dx+d'y)\frac{t^{n+1}}{n!}\\
&+\sum_{j=1}^{d-1}\frac{d'}{(\zeta_d^{-jd'}-1)}\sum_{n=0}^{\infty}B_n(dx+d'y;\zeta_d^j)\frac{t^{n+1}}{n!}\\
&+\sum_{j=1}^{d'-1}\frac{d}{(\zeta_{d'}^{-jd}-1)}\sum_{n=0}^{\infty}B_n(dx+d'y;\zeta_{d'}^j)\frac{t^{n+1}}{n!}.
\end{aligned}$$

Combining the above two formulas, we get for $n\geq 0$,

(3.17)

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}d^kd'^{n-k}B_k(x)B_k(y)\\
=&B_n^{(2)}(dx+d'y)\frac{t^n}{n!}-(\frac{d+d'-2}{2dd'})nB_{n-1}(dx+d'y)\\
&+\sum_{j=1}^{d-1}\frac{d'}{(\zeta_d^{-jd'}-1)}nB_{n-1}(dx+d'y;\zeta_d^j)+\sum_{j=1}^{d'-1}\frac{d}{(\zeta_{d'}^{-jd}-1)}nB_{n-1}(dx+d'y;\zeta_{d'}^j).
\end{aligned}$$

Recall [**5**, p.149] that the generalized Bernoulli polynomials $B_n^{(2)}(x+y)$ of degree 2, is given by

(3.18)

$$B_n^{(2)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(y)=(1-n)B_n(x+y)+(x+y-1)nB_{n-1}(x+y).$$

Substituting $B_n^{(2)}(x+y)$ in (3.17) by (3.18), we get the desired result. □

Example 12. We consider some special cases of the Theorem 11.

1. If $d=2$, $d'=1$ [**5**, p.150]:

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}2^kB_k(x)B_{n-k}(y)\\
=&(1-n)B_n(2x+y)+(2x+y-\frac{3}{2})nB_{n-1}(2x+y)+\frac{n(n-1)}{4}E_{n-2}(2x+y).
\end{aligned}$$

Theorem 13. *Let* $(n,n')=1$, *and* $(d,d')=1$, *Then we have*

$$M_k(n,n') = \sum_{\substack{d|n\\ d'|n'}} \frac{\mu(dd')}{dd'}(1-k)B_k - \sum_{\substack{d|n\\ d'|n'}} \frac{\mu(dd')}{2}(\frac{1}{d}+\frac{1}{d'})kB_{k-1} + A_1 + A_2,$$

where

$$A_1 = k\sum_{\substack{d|n\\ d'|n'}} \frac{\mu(dd')}{d}\sum_{j=1}^{d-1}\frac{1}{\zeta_d^{-jd'}-1}B_{k-1}(0;\zeta_d^j),$$

$$A_2 = k\sum_{\substack{d|n\\ d'|n'}} \frac{\mu(dd')}{d'}\sum_{j=1}^{d'-1}\frac{1}{\zeta_{d'}^{-jd}-1}B_{k-1}(0;\zeta_{d'}^j).$$

Proof. By (1.8) and (1.9),

$$M_k(n,n') = \sum_{\substack{d|n\\ d'|n'}} \mu(dd')B_k((d,d')) \tag{3.19}$$

and

$$\sum_{k=0}^{\infty} B_k((d,d'))\frac{t^k}{k!} = \frac{t^2}{(e^{dt}-1)(e^{d'}-1)} = \frac{1}{dd'}(\frac{dt}{e^{dt}-1})(\frac{d't}{e^{d't}-1}).$$

By formula (2.13), we have

$$B_n((d,d')) = \sum_{k=0}^{n}\binom{n}{k}d^k {d'}^{n-k}B_kB_{n-k}.$$

In Theorem 11, putting $x=y=0$, we get

(3.20)

$$dd'B_k((d,d')) = (1-k)B_k - (\frac{d+d'}{2})kB_{k-1} + kd'\sum_{j=1}^{d-1}\frac{1}{(\zeta_d^{-jd'}-1)}B_{k-1}(0;\zeta_d^j) + kd\sum_{j=1}^{d'-1}\frac{1}{(\zeta_{d'}^{-dj}-1)}B_{k-1}(0;\zeta_{d'}^j).$$

By (3.19) and (3.20), we get the theorem. □

Example 14. We consider a special case of Theorem 13.

1. Case $d=2$, $d'=1$:

$$M_k(2,1) = \sum_{d|2}\frac{\mu(d)}{d}(1-k)B_k - \sum_{d|2}\frac{\mu(d)}{2}(1+\frac{1}{d})kB_{k-1} + k\frac{\mu(2)}{2}(\frac{1}{-2})B_{k-1}(0;-1)$$
$$= -\frac{1}{4}kB_{k-1} - \frac{1}{2}(k-1)B_k - \frac{k(k-1)}{8}E_{k-2}(0).$$

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