

**RATIO ASYMPTOTICS FOR BIORTHOGONAL
MATRIX POLYNOMIALS WITH UNBOUNDED
RECURRENCE COEFFICIENTS.**

*Dedicated to Academician Professor Gradimir Milovanović
on the occasion of his 70th birthday.*

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In this paper we study matrix biorthogonal polynomials sequences that satisfy a nonsymmetric three term recurrence relation with unbounded matrix coefficients. The outer ratio asymptotics for this family of matrix biorthogonal polynomials is derived under quite general assumptions. Some illustrative examples are considered.

1. INTRODUCTION

The study of the outer ratio asymptotics, i.e. the limit of the ratio of two consecutive polynomials p_n and p_{n+1} of a sequence of polynomials, $(p_n(x))_{n \in \mathbb{N}}$, orthogonal with respect to an inner product, outside the convex hull of the support of the measure of orthogonality has attracted the interest of many researchers in the last decades. Since the first study of Nevai in 1979 (cf. [14]) for orthogonal polynomials with respect to a measure supported on an infinite subset of the real line with convergent recurrence coefficients, more general situations have been considered, such as the case of asymptotically periodic recurrence coefficients with a finite number of accumulation points (cf. [16], [19], [20]) or the case of unbounded recurrence coefficients (cf. [19]).

In 1993 Durán (cf. [9]) gave the characterization of symmetric bilinear forms for which the multiplication operator by a polynomial is a symmetric. There,

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necessary and sufficient conditions were deduced, so that, a sequence of scalar polynomials $(p_n(x))_{n \in \mathbb{N}}$ satisfying a $(2N + 1)$ -term recurrence relation

$$h(x)p_n(x) = c_{n,0}p_n(x) + \sum_{k=1}^N (\bar{c}_{n,k}p_{n-k}(x) + c_{n+k,k}p_{n+k}(x)),$$

is orthogonal with respect to a symmetric bilinear form (generalization of the Favard's theorem). In particular, their attention was focused on the discrete Sobolev type inner products. In that work, the author gave a first idea to connect scalar orthogonal polynomials with respect to a bilinear form and matrix orthogonal polynomials with respect to a positive definite matrix of measures. From the above result, Durán and Van Assche [13] proved that if $(p_n(x))_{n \in \mathbb{N}}$ is a sequence of scalar polynomials satisfying a $(2N + 1)$ -term recurrence relation, then it is related to a matrix polynomial sequence $(P_n(x))_{n \in \mathbb{N}}$ satisfying a three-term recurrence relation with matrix coefficients

$$(1) \quad xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x), \quad n \in \mathbb{N},$$

and initial conditions $P_0(x) = \mathbf{I}_N$ and $P_{-1}(x) = \mathbf{0}_N$, where for each $n \in \mathbb{N}$, A_n is an upper triangular, nonsingular matrix and B_n is a Hermitian matrix.

The above results reawakened the interest on matrix orthogonal polynomials (cf. the survey paper [7]). So, in [12] Durán and López-Rodríguez studied properties for the zeros of a sequence of matrix polynomials $(P_n(x))_{n \in \mathbb{N}}$ which are orthonormal with respect to a positive definite matrix of measures W . Next, Durán in [10] showed two important results: the first one is a quadrature formula for matrix polynomials and the second one is the Markov theorem for matrix orthogonal polynomials when again the matrix of measures is positive definite. In [8], the author deals with the outer ratio asymptotics for matrix orthogonal polynomials. Therein, Durán obtained the asymptotic behavior of two consecutive polynomials belonging to the matrix Nevai class, i.e. these polynomials satisfy a three-term recurrence formula as in (1) where, again, A_n are nonsingular, upper triangular matrices and B_n are Hermitian matrices for all $n \in \mathbb{N}$, and such that $A_n \rightarrow A$ and $B_n \rightarrow B$. Later on, Durán and Daneri-Vias analyzed the above case but when the matrix sequences $(A_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$ diverge in a particular way (cf. [11] for details).

Recently, Yakhlef and Marcellán [18] have studied the outer relative asymptotics of sequences of matrix orthogonal polynomials for Uvarov perturbations in the degenerate case, i.e. given a positive definite matrix of measures α , and its corresponding sequences of matrix orthonormal polynomials, $(P_n^\alpha(x))_{n \in \mathbb{N}}$, satisfying a three-term recurrence relation as in (1), they define a new matrix of measures β as

$$d\beta(u) = d\alpha(u) + M\delta(u - c),$$

where M is a positive definite matrix, $\delta(u - c)$ is the Dirac measure supported at c that is located outside the support of $d\alpha$, and the sequence of matrix orthonormal

polynomials associated with $d\beta$, $(P_n^\beta(x))_{n \in \mathbb{N}}$. Then, they study the outer relative asymptotic between the sequences $(P_n^\beta(x))_{n \in \mathbb{N}}$ and $(P_n^\alpha(x))_{n \in \mathbb{N}}$ under quite general assumptions on the coefficients of the three-term recurrence relation $(A_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$.

When the matrix of measures, W , is no longer Hermitian, we can define a bilinear (respectively, sesquilinear form) in $\mathbb{R}^{N \times N}[x]$ (respectively, in $\mathbb{C}^{N \times N}[x]$) and deal with sequences of biorthogonal matrix polynomials, $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$, which play the role of the left and right-orthogonalities (cf. Definition 1). It can be proven that these sequences satisfy a three-term recurrence relation

$$(2) \quad x V_n(x) = \alpha_n V_{n+1}(x) + \beta_n V_n(x) + \gamma_n V_{n-1}(x),$$

$$(3) \quad x G_n(x) = a_n G_{n+1}(x) + b_n G_n(x) + c_n G_{n-1}(x),$$

where $a_n, \alpha_n, b_n, \beta_n$, are nonsingular matrices. Without loss of generality we can suppose that a_n, α_n are lower triangular matrices and c_n, γ_n are upper triangular matrices (cf. [13]).

As we have said above, matrix polynomials defined by recurrence formulas as (2) appear in a natural way in the literature and its study paid an increasing attention in the last decades. For example in [4], the authors give a matrix interpretation of the multiple orthogonality in terms of matrix orthogonal polynomials satisfying the same kind of recursion formulas. On the other hand, in [1], [2], [15] were studied perturbations of measures (Christoffel, Geronimus, and Geronimus-Uvarov) which yield to non-positive definite matrix of measures, and thus, to biorthogonality.

Example 1. Let μ be a scalar measure supported on the real line, and $(p_n(x))_{n \in \mathbb{N}}$ its corresponding sequence of monic orthogonal polynomials. If $W(x)$ is a matrix polynomial of degree M , then we can define a new measure $d\hat{\mu} = W(x) d\mu$ (Christoffel transformation of the measure) which clearly is non-positive definite as $W(x)$ needs not to be identical to $W^T(x)$. In particular, taking $W(x) = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$, then following the techniques developed in [1], it is easy to see that the sequence of matrix polynomials

$$V_n(x) = \begin{bmatrix} K_n(x, 0)/p_n(0) & \frac{p_n(0) K_n(x, 0) - W(p_n, p_{n+1})(0) p_n(x)}{p_n^2(0) x} \\ 0 & K_n(x, 0)/p_n(0) \end{bmatrix},$$

$$G_n(x) = - \begin{bmatrix} K_n(x, 0)/p_n(0) & 0 \\ -(K_n(x, 0) p'_n(0) + \frac{\partial K_n(x, 0)}{\partial y})/p_n(0) & K_n(x, 0)/p_n(0) \end{bmatrix}.$$

with $W(p_n, p_{n+1})(x) = p_n(x) p'_{n+1}(x) - p_{n+1}(x) p'_n(x)$, and

$$K_n(x, 0) = (p_n(0) p_{n+1}(x) - p_{n+1}(0) p_n(x))/x,$$

satisfies

$$\int V_m(x) W(x) G_n(x) d\mu(x) = \mathbf{I}_N \delta_{n,m},$$

i.e. $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ are sequences of biorthonormal polynomial with respect to $d\hat{\mu}$ and satisfy the three-term recurrence relations (2) and (3) with $a_n = \gamma_{n+1}$, $b_n = \beta_n$, and $c_n = \alpha_{n-1} = \mathbf{I}_N$ (cf. Theorem 1).

Outer ratio asymptotics for the class of matrix biorthogonal polynomials satisfying the three-term recurrence relation as in (2) was studied for the first time in [6]. There, the authors analyzed the case of convergent recurrence coefficients by introducing an analog of the Nevai class of matrix polynomials for the nonsymmetric case, the generalized matrix Nevai class.

In the present contribution our aim is to generalize those results when the coefficients of the nonsymmetric recurrence formula diverge in a particular way.

The structure of the manuscript is as follows. Section 2 provides the basic background about matrix biorthogonal polynomials $(V_n(x))_{n \in \mathbb{N}}$ and $(G_n(x))_{n \in \mathbb{N}}$ that satisfy dual nonsymmetric recurrence relations (see cf. [4, 5, 6]). Here, we establish the relations between the zeros of these two families of polynomials and discuss the most appropriate way of scaling these matrix polynomials in order to obtain its asymptotic behavior. Section 3 deals with the outer ratio asymptotics for left and right-orthogonal matrix polynomials with varying recurrence coefficients. We also deduce quadrature and Liouville-Ostrogradski formulas for the right-orthogonal polynomials. Section 4 is focused on our main result (Theorem 8), the outer ratio asymptotics of matrix orthogonal polynomials satisfying recurrence formulas with nonsymmetric and nonsingular recurrence coefficients diverging in a particular way. Section 5 is focused on the study of the case when certain matrix appearing in the recurrence formula is singular.

2. MATRIX BIORTHOGONAL POLYNOMIALS

Let us consider a *quasidefinite* $N \times N$ matrix of measures, W , i.e. $W = [w_{i,j}]_{i,j=0}^{N-1}$, with measures, $w_{i,j}$, $i, j \in \{0, \dots, N-1\}$, supported on the real line but not necessarily positive definite with matrix moments, $U_n = \int x^n dW(x)$, $n \in \mathbb{N}$, and such that the Hankel determinants satisfy

$$\det([U_{i+j}]_{i,j=0,\dots,n}) \neq 0, \quad n \in \mathbb{N}.$$

We will assume, without loss of generality, that the matrix of measures is normalized by $U_0 = \int dW(x) = \mathbf{I}_N$. If P and R are matrix polynomials in $\mathbb{C}^{N \times N}[x]$, then we introduce the following *sesquilinear form*,

$$\langle P, R \rangle = \int P(x) dW(x) R^T(x), \quad P, R \in \mathbb{C}^{N \times N}[x].$$

Definition 1. Let W be a quasidefinite $N \times N$ matrix of measures. The matrix polynomial sequences, $(V_n(x))_{n \in \mathbb{N}}$ (respectively, $(G_n(x))_{n \in \mathbb{N}}$), such that for every

$n, m \in \mathbb{N}$, $\deg V_n(x) = n$ (respectively, $\deg G_m(x) = m$) and

$$\int V_n(x) dW(x) x^m = \Omega_n^{(1)} \delta_{n,m}, \quad m = 0, \dots, n$$

(respectively, $\int x^n dW(x) G_m(x) = \Omega_m^{(2)} \delta_{n,m}, \quad n = 0, \dots, m$),

where $\delta_{n,m}$ is the Kronecker symbol and $\Omega_n^{(1)}$ (respectively, $\Omega_n^{(2)}$) are nonsingular matrices for $n \in \mathbb{N}$, are said to be *the left* (respectively, *right*) *biorthogonal polynomial sequences* with respect to W .

We also refer to $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ as biorthonormal polynomial sequences with respect to the quasidefinite matrix of measures W , when

$$\int V_n(x) dW(x) G_m(x) = \mathbf{I}_N \delta_{n,m}, \quad n, m \in \mathbb{N}.$$

Theorem 1 (cf. [5], Theorem 4). *Given a quasidefinite matrix of measures W , then its biorthogonal polynomial sequences, $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ satisfy the three-term recurrence relations*

$$(4) \quad x V_n(x) = A_n V_{n+1}(x) + B_n V_n(x) + C_n V_{n-1}(x), \quad n \in \mathbb{N},$$

$$(5) \quad x G_n(x) = G_{n-1}(x) A_{n-1} + G_n(x) B_n + G_{n+1}(x) C_{n+1}, \quad n \in \mathbb{N},$$

with $V_{-1}(x) = \mathbf{0}_N$, $V_0(x) = \mathbf{I}_N$, and $G_{-1}(x) = \mathbf{0}_N$, $G_0(x) = \mathbf{I}_N$. Here, A_n and C_n are nonsingular matrices for every $n \in \mathbb{N}$.

Without loss of generality we can suppose that $(A_n)_{n \in \mathbb{N}}$ (respectively, $(C_n)_{n \in \mathbb{N}}$) is a sequence of lower (respectively, upper) triangular matrices (cf. [13]).

In the same way as in the scalar case, the Favard's Theorem for matrix polynomials can be found in the literature (cf. [5], Theorem 7).

Theorem 2. *Given a quasidefinite matrix of measures W , then its biorthogonal polynomial sequences, $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$, are such that for each n , $V_n(x)$ and $G_n(x)$ have the same zeros.*

Proof. Notice that the N -block tridiagonal matrix associated with the recurrence relation (5) for the polynomials $G_n(x)$ is the transpose of the N -block tridiagonal matrix

$$J = \begin{bmatrix} B_0 & A_0 & \mathbf{0}_N & & \\ C_1 & B_1 & A_1 & \ddots & \\ \mathbf{0}_N & C_2 & B_2 & \ddots & \\ & \ddots & \ddots & \ddots & \end{bmatrix},$$

associated with the recurrence relation (4) for the polynomials $V_n(x)$. The result follows by taking into account that the zeros of $V_n(x)$ ($G_n(x)$, respectively) are the eigenvalues of J_n (J_n^T , respectively), where J_n is the truncated matrix of J , with dimension $nN \times nN$. \square

Definition 2. Let W be a quasidefinite matrix of measures and $(V_n(x))_{n \in \mathbb{N}}$ and $(G_n(x))_{n \in \mathbb{N}}$ the corresponding sequences of biorthogonal matrix polynomials. We define the first kind associated polynomial sequences $(V_n^{(1)}(x))_{n \in \mathbb{N}}$ and $(G_n^{(1)}(x))_{n \in \mathbb{N}}$, as follows,

$$V_{n-1}^{(1)}(x) = \int \frac{V_n(x) - V_n(y)}{x - y} dW(y), \quad G_{n-1}^{(1)}(x) = \int dW(y) \frac{G_n(x) - G_n(y)}{x - y}.$$

The first kind associated polynomial sequences $(V_n^{(1)}(x))_{n \in \mathbb{N}}$, $(G_n^{(1)}(x))_{n \in \mathbb{N}}$ also satisfy the three-term recurrence relations (4) and (5) with initial conditions, $V_{-1}^{(1)}(x) = \mathbf{0}_N$, $V_0^{(1)}(x) = A_0^{-1}$, and $G_{-1}^{(1)}(x) = \mathbf{0}_N$, $G_0^{(1)}(x) = C_1^{-1}$.

In order to obtain the outer ratio asymptotics for polynomials with varying recurrence coefficients we will need some auxiliary results such as quadrature and Liouville-Ostrogradsky type formulas for biorthogonal polynomials. For the left-orthogonal polynomials these results can be found in the literature (see [6]).

Lemma 1 (cf. [12], Lemma 2.2). *Let $A(x)$ be an $N \times N$ matrix polynomial and let \mathbf{a} be a zero of $A(x)$ of multiplicity p , i.e. \mathbf{a} is a zero of $\det A(x)$ of multiplicity p . We put $L(\mathbf{a}, A(x)) = \{v \in \mathbb{C}^N : \bar{v}^T A(\mathbf{a}) = \mathbf{0}_N\}$ and $R(\mathbf{a}, A(x)) = \{v \in \mathbb{C}^N : A(\mathbf{a})v = \mathbf{0}_N\}$. If $\dim L(\mathbf{a}, A(x)) = \dim R(\mathbf{a}, A(x)) = p$, then $(\text{Adj}(A(x)))^{(j)}(\mathbf{a}) = \mathbf{0}_N$, for $j = 0, \dots, p-2$, and $(\text{Adj}(A(x)))^{(p-1)}(\mathbf{a}) \neq \mathbf{0}_N$. Moreover, $\text{rank}(\text{Adj}(A(x)))^{(p-1)}(\mathbf{a}) = p$.*

Lemma 2 (cf. [3], Proposition 5.14). *Let $P_n(x)$ be a matrix polynomial of degree n with m different zeros $\{x_{n,1}, \dots, x_{n,m}\}$ and with $\{\ell_1, \dots, \ell_m\}$ as corresponding multiplicities. For any matrix polynomial $R(x)$ of degree less than or equal to $n-1$ and $x \in \mathbb{C} \setminus \{x_{n,1}, \dots, x_{n,m}\}$ we have*

$$R(x) (P_n(x))^{-1} = \sum_{k=1}^m \frac{C_{n,k}}{x - x_{n,k}}, \quad (P_n(x))^{-1} R(x) = \sum_{k=1}^m \frac{D_{n,k}}{x - x_{n,k}},$$

where

$$C_{n,k} = \frac{\ell_k}{(\det P_n)^{(\ell_k)}(x_{n,k})} R(x_{n,k}) (\text{Adj } P_n(x))^{(\ell_k-1)}(x_{n,k}),$$

$$D_{n,k} = \frac{\ell_k}{(\det P_n)^{(\ell_k)}(x_{n,k})} (\text{Adj } P_n(x))^{(\ell_k-1)}(x_{n,k}) R(x_{n,k}).$$

Theorem 3 (Quadrature formula). *Let $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ be the sequences of biorthogonal matrix polynomials with respect to a quasidefinite matrix of measures W , and let $(V_n^{(1)}(x))_{n \in \mathbb{N}}$, $(G_n^{(1)}(x))_{n \in \mathbb{N}}$ be its first kind associated polynomial sequences. Given the different zeros of $V_n(x)$, $\{x_{n,1}, \dots, x_{n,s}\}$, with multiplicities*

$\{\ell_1, \dots, \ell_s\}$, we define the matrices $\Gamma_{m,k}$, $\tilde{\Gamma}_{m,k}$, as

$$\Gamma_{n,k} = \frac{\ell_k}{(\det V_n)^{(\ell_k)}(x_{n,k})} (\text{Adj } V_n(x))^{(\ell_k-1)}(x_{n,k}) V_{n-1}^{(1)}(x_{n,k}),$$

$$\tilde{\Gamma}_{n,k} = G_{n-1}^{(1)}(x_{n,k}) \frac{\ell_k}{(\det G_n)^{(\ell_k)}(x_{n,k})} (\text{Adj } G_n(x))^{(\ell_k-1)}(x_{n,k}).$$

Then, for any polynomial $P(x)$ of degree less than or equal to $2n-1$ the following quadrature formula holds

$$\int P(x) dW(x) = \sum_{k=1}^s P(x_{n,k}) \Gamma_{n,k}, \quad \int dW(x) P(x) = \sum_{k=1}^s \tilde{\Gamma}_{n,k} P(x_{n,k}).$$

Proof. We will prove the quadrature formula for the right orthogonal polynomials, because the left one is already proved in [6]. Let $P(x)$ be a matrix polynomial of degree less than or equal to $2n-1$. Since $G_n(x)$ is a polynomial with nonsingular leading coefficient, then $P(x) = G_n(x) C(x) + R(x)$. Here $C(x)$ and $R(x)$ are matrix polynomials with degree of $R(x)$ less than or equal to $n-1$. Using Lemma 2 we get

$$G_n^{-1}(x) R(x) = \sum_{k=1}^s \frac{D_{n,k}}{x - x_{n,k}},$$

where the matrices $D_{n,k}$ are

$$D_{n,k} = \frac{\ell_k}{(\det G_n)^{(\ell_k)}(x_{n,k})} (\text{Adj } G_n(x))^{(\ell_k-1)}(x_{n,k}) R(x_{n,k}).$$

Taking into account that $R(x_{n,k}) = P(x_{n,k}) - G_n(x_{n,k}) C(x_{n,k})$ and

$$(6) \quad (\text{Adj } G_n)^{(\ell_k-1)}(x_{n,k}) G_n(x_{n,k}) = G_n(x_{n,k}) (\text{Adj } G_n)^{(\ell_k-1)}(x_{n,k}) = \mathbf{0}_N,$$

we get
$$D_{n,k} = \frac{\ell_k}{(\det G_n)^{(\ell_k)}(x_{n,k})} (\text{Adj } G_n(x))^{(\ell_k-1)}(x_{n,k}) P(x_{n,k}).$$

Thus,
$$P(x) = G_n(x) C(x) + \sum_{k=1}^s G_n(x) \frac{D_{n,k}}{x - x_{n,k}}.$$
 Using again (6), we have

$$P(x) = G_n(x) C(x) + \sum_{k=1}^s \frac{G_n(x) - G_n(x_{n,k})}{x - x_{n,k}} D_{n,k}.$$

and by the definition of the first kind associated polynomial, it follows that

$$\int dW(x) P(x) = \int dW(x) G_n(x) C(x) + \sum_{k=1}^s G_{n-1}^{(1)}(x_{n,k}) D_{n,k}.$$

So, from orthogonality we have
$$\int dW(x) P(x) = \sum_{k=1}^s G_{m-1}^{(1)}(x_{m,k}) D_{m,k},$$
 and the result follows. \square

Theorem 4 (Liouville-Ostrogradski formula). *Let $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ be the sequences of matrix biorthogonal polynomials with respect to a quasidefinite matrix of measures W and $(V_n^{(1)}(x))_{n \in \mathbb{N}}$, $(G_n^{(1)}(x))_{n \in \mathbb{N}}$ be, respectively, the first kind associated polynomial sequences. Then,*

$$(7) \quad V_n(z) G_n^{(1)}(z) - V_{n+1}^{(1)}(z) G_{n+1}(z) = A_n^{-1},$$

$$(8) \quad V_n(x) G_n^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n+1}(x) = C_{n+1}^{-1},$$

where A_n, C_n are the nonsingular matrices in (5).

Proof. Equation (7) was already proved in [6]. To prove (8) we proceed by induction on n . For $n = 0$ the result follows from the initial conditions. We assume that the formula $V_p(x) G_p^{(1)}(x) - V_{p-1}^{(1)}(x) G_{p+1}(x) = C_{p+1}^{-1}$ is true for $p = 1, \dots, n-1$.

First, we use the recurrence relation in $G_n^{(1)}(x)$ and $G_{n+1}(x)$ to obtain

$$\begin{aligned} V_n(x) G_n^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n+1}(x) &= (V_n(x) G_{n-1}^{(1)}(x) - V_{n-1}^{(1)}(x) G_n(x)) \\ &\quad \times (x\mathbf{I}_N - B_n) C_{n+1}^{-1} - (V_n(x) G_{n-2}^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n-1}(x)) A_{n-1} C_{n+1}^{-1}. \end{aligned}$$

Second, we prove that $V_n(x) G_{n-1}^{(1)}(x) - V_{n-1}^{(1)}(x) G_n(x) = \mathbf{0}_N$. Using the definition of the first kind associated polynomial, we get

$$\begin{aligned} V_n(x) G_{n-1}^{(1)}(x) - V_{n-1}^{(1)}(x) G_n(x) &= \int \frac{V_n(y) dW(y)}{x-y} G_n(x) - V_n(x) \int \frac{dW(y) G_n(y)}{x-y}. \end{aligned}$$

Adding and subtracting $\int \frac{V_n(y) dW(y) G_n(y)}{x-y}$ in the last relation and taking in account the left and right-orthogonalities, the result follows. With this in mind

$$(9) \quad \begin{aligned} V_n(x) G_n^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n+1}(x) &= -(V_n(x) G_{n-2}^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n-1}(x)) A_{n-1} C_{n+1}^{-1}. \end{aligned}$$

Now, using the recurrence relations for $V_n(x)$ and $V_{n-1}^{(1)}(x)$,

$$\begin{aligned} V_n(x) G_{n-2}^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n-1}(x) &= A_{n-1}^{-1}(x - B_{n-1})(V_{n-1}(x) G_{n-2}^{(1)}(x) - V_{n-2}^{(1)}(x) G_{n-1}(x)) \\ &\quad - A_{n-1}^{-1} C_{n-1} (V_{n-2}(x) G_{n-2}^{(1)}(x) - V_{n-3}^{(1)}(x) G_{n-1}(x)). \end{aligned}$$

Since $V_{n-1}(x) G_{n-2}^{(1)}(x) - V_{n-2}^{(1)}(x) G_{n-1}(x) = \mathbf{0}_p$, we deduce

$$\begin{aligned} V_n(x) G_{n-2}^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n-1}(x) &= -A_{n-1}^{-1} C_{n-1} (V_{n-2}(x) G_{n-2}^{(1)}(x) - V_{n-3}^{(1)}(x) G_{n-1}(x)). \end{aligned}$$

Using this relation in (9) we obtain

$$\begin{aligned} V_n(x) G_n^{(1)}(x) - V_{n-1}^{(1)}(x) G_{n+1}(x) \\ = A_{n-1}^{-1} C_{n-1} (V_{n-2}(x) G_{n-2}^{(1)}(x) - V_{n-3}^{(1)}(x) G_{n-1}(x)) A_{n-1} C_{n+1}^{-1}. \end{aligned}$$

According to the induction hypothesis the result follows. \square

In the sequel, we will assume that the matrix recurrence coefficients diverge in a particular way: we will suppose that there exists a sequence of positive definite matrices $(D_n)_{n \in \mathbb{N}}$ such that

$$(10) \quad \begin{cases} \lim_{n \rightarrow \infty} D_n^{-1/2} A_n D_n^{-1/2} = A, & \lim_{n \rightarrow \infty} D_n^{-1/2} B_n D_n^{-1/2} = B, \\ \lim_{n \rightarrow \infty} D_n^{-1/2} C_n D_n^{-1/2} = C, & \lim_{n \rightarrow \infty} D_n^{-1/2} D_{n-1}^{1/2} = \mathbf{I}_N. \end{cases}$$

When unbounded coefficients are considered in the scalar case (assuming the same hypothesis given by (10)), the outer ratio asymptotics is then obtained for the scaled polynomials $p_n(c_n z)$. However, in the matrix case there is a large range of possibilities to define the scaled matrix polynomial $P(Hx)$ (cf. [11]). From now on we will work with two notions of scaled matrix polynomials depending on the kind of orthogonality (left or right) that we will deal with. In the case of left-orthogonality, the suitable definition of scaled matrix polynomials was introduced by Durán in [11].

Definition 3 (cf. [11]). Given the sequences of recurrence coefficients $(A_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$, and $(C_n)_{n \in \mathbb{N}}$, we can define a sequence of matrix polynomials $(\mathbf{V}_n(x))_{n \in \mathbb{N}}$ in one matrix variable, as

$$T \mathbf{V}_n(T) = A_n \mathbf{V}_{n+1}(T) + B_n \mathbf{V}_n(T) + C_n \mathbf{V}_{n-1}(T),$$

with initial conditions $\mathbf{V}_{-1}(T) = \mathbf{0}_N$, $\mathbf{V}_0(T) = \mathbf{I}_N$. We define the *left-scaled matrix polynomials* as $V_n^H(x) = \mathbf{V}_n(Hx)$. On the other hand, the natural definition in order to scale the right-orthogonal polynomials is the following one. Using the recurrence coefficients we can define another matrix polynomial sequence of one matrix variable, $(\mathbf{G}_n)_{n \in \mathbb{N}}$, as

$$\mathbf{G}_n(T) T = \mathbf{G}_{n-1}(T) A_{n-1} + \mathbf{G}_n(T) B_n + \mathbf{G}_{n+1}(T) C_{n+1},$$

with initial conditions $\mathbf{G}_{-1}(x) = \mathbf{0}_N$ and $\mathbf{G}_0(x) = \mathbf{I}_N$. Now, we define the *right-scaled matrix polynomials* as $G_n^H(x) = \mathbf{G}_n(Hx)$.

Notice that, in particular, for each non-negative integer k the scaled polynomial sequences $(V_n^{D^k}(x))_{n \in \mathbb{N}}$ and $(G_n^{D^k}(x))_{n \in \mathbb{N}}$, are biorthogonal with respect to a certain varying matrix of measures W_k . We will say that $(V_n^{D^k}(x))_{n \in \mathbb{N}}$ and $(G_n^{D^k}(x))_{n \in \mathbb{N}}$ are matrix biorthogonal polynomials with varying recurrence coefficients. In this way, our main result, Theorem 8, will be a consequence of a more general theorem on outer ratio asymptotics cf. Theorem 6.

In the sequel, we associate with three given matrices A, C, B , where A, C , are nonsingular, the left-orthogonal Chebyshev matrix polynomials of second kind $(U_n^{C,B,A}(x))_{n \in \mathbb{N}}$ which are defined by the recurrence formula

$$(11) \quad x U_n^{C,B,A}(x) = C U_{n+1}^{C,B,A}(x) + B U_n^{C,B,A}(x) + A U_{n-1}^{C,B,A}(x), \quad n \in \mathbb{N},$$

with initial conditions $U_0^{C,B,A}(x) = \mathbf{I}_N$ and $U_{-1}^{C,B,A}(x) = \mathbf{0}_N$, as well as the right-orthogonal Chebyshev matrix polynomial of second kind $(T_m^{A,B,C}(x))_{m \in \mathbb{N}}$ given by

$$(12) \quad x T_n^{A,B,C}(x) = T_{n+1}^{A,B,C}(x) A + T_n^{A,B,C}(x) B + T_{n-1}^{A,B,C}(x) C, \quad n \in \mathbb{N},$$

where $T_0^{A,B,C}(x) = \mathbf{I}_N$ and $T_{-1}^{A,B,C}(x) = \mathbf{0}_N$. We denote by $W_{C,B,A}$ the matrix weight for which the polynomial sequences $(U_n^{C,B,A}(x))_{n \in \mathbb{N}}$, $(T_n^{A,B,C}(x))_{n \in \mathbb{N}}$ are biorthogonal.

3. OUTER RATIO ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS WITH VARYING RECURRENCE COEFFICIENTS

For each $k = 1, 2, \dots$, we consider orthogonal matrix polynomials $(R_{n,k}(x))_{n \in \mathbb{N}}$ and $(S_{n,k}(x))_{n \in \mathbb{N}}$, given by the recurrence relations

$$(13) \quad x R_{n,k}(x) = A_{n,k} R_{n+1,k}(x) + B_{n,k} R_{n,k}(x) + C_{n,k} R_{n-1,k}(x), \quad n \in \mathbb{N},$$

$$(14) \quad x S_{n,k}(x) = S_{n-1,k}(x) A_{n-1,k} + S_{n,k}(x) B_{n,k} + S_{n+1,k}(x) C_{n+1,k}, \quad n \in \mathbb{N},$$

with $R_{0,k}(x) = \mathbf{I}_N$, $R_{-1,k}(x) = \mathbf{0}_N$, and $S_{0,k}(x) = \mathbf{I}_N$, $S_{-1,k}(x) = \mathbf{0}_N$.

For a fixed k , these matrix polynomial sequences are biorthogonal with respect to a certain quasidefinite matrix of measures which we denote by W_k .

As far as we know, the only result on outer ratio asymptotics for matrix polynomials satisfying nonsymmetric recurrence relations is the following one.

Theorem 5 (cf. [6], Theorem 3). *Let W be a quasidefinite matrix of measures, and $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ biorthogonal polynomial sequences with respect to W , satisfying, respectively, the three-term recurrence relation (4), (5). Let us assume*

$$\lim_{n \rightarrow \infty} A_n = A, \quad \lim_{n \rightarrow \infty} B_n = B, \quad \lim_{n \rightarrow \infty} C_n = C,$$

with A, C nonsingular matrices. We denote by Δ_n the set of zeros of $(V_n(x))_{n \in \mathbb{N}}$, and $\Gamma = \bigcap_{N \geq 0} M_N$, where $M_N = \bigcup_{n \geq N} \Delta_n$. Then,

$$\lim_{n \rightarrow \infty} V_{n-1}(x) V_n^{-1}(x) A_{n-1}^{-1} = \int \frac{dW_{C,B,A}(y)}{x-y}, \quad x \in \mathbb{C} \setminus \Gamma,$$

$$\lim_{n \rightarrow \infty} C_n^{-1} G_n^{-1}(x) G_{n-1}(x) = \int \frac{dW_{C,B,A}(y)}{x-y}, \quad x \in \mathbb{C} \setminus \Gamma,$$

where $W_{C,B,A}$ is the matrix of measures associated with the second kind Chebyshev matrix polynomials. Moreover, the convergence is locally uniform on compact subsets of $\mathbb{C} \setminus \Gamma$.

In the previous case, the coefficients of the recurrence relation are assumed to be convergent.

The following result generalizes the previous one in two senses: we consider a case of varying recurrence coefficients and, for a fixed k , the recurrence coefficients will diverge in a particular way.

Theorem 6. *Let W_k be, for each $k \in \mathbb{N}$, a quasidefinite matrix of measures and $(R_{n,k}(x))_{n \in \mathbb{N}}$, $(S_{n,k}(x))_{n \in \mathbb{N}}$ be the sequences of biorthogonal matrix polynomials depending on a parameter k , $k = 1, 2, \dots$, satisfying (13), (14). Let $(n_m)_{m \in \mathbb{N}}$, $(k_m)_{m \in \mathbb{N}}$, be two increasing sequences of positive integers and we will assume that there exist three matrices A , B , and C , with A and B nonsingular, such that for all $l \in \mathbb{N}$,*

$$(15) \quad \lim_{m \rightarrow \infty} A_{n_m-l, k_m} = A, \quad \lim_{m \rightarrow \infty} B_{n_m-l, k_m} = B, \quad \lim_{m \rightarrow \infty} C_{n_m-l, k_m} = C.$$

We denote by $\tilde{\Delta}_{n,k}$ the set of zeros of S_{n_m, k_m} and $\tilde{\Gamma} = \bigcap_{N \geq 0} \tilde{M}_{N,k}$, where $\tilde{M}_{N,k} =$

$$\overline{\bigcup_{n \geq N} \tilde{\Delta}_{n,k}}. \text{ Then,}$$

$$(16) \quad \lim_{m \rightarrow \infty} R_{n_m-1, k_m}(x) R_{n_m, k_m}^{-1}(x) A_{n_m-1, k_m}^{-1} = \int \frac{dW_{C,B,A}(t)}{x-t}, \quad x \in \mathbb{C} \setminus \Gamma,$$

$$(17) \quad \lim_{m \rightarrow \infty} C_{n_m, k_m}^{-1} S_{n_m, k_m}^{-1}(x) S_{n_m-1, k_m}(x) = \int \frac{dW_{C,B,A}(y)}{x-y}, \quad x \in \mathbb{C} \setminus \tilde{\Gamma},$$

where $W_{C,B,A}(x)$ is the matrix weight for the generalized Chebyshev matrix polynomials defined in (12). Moreover, the convergence is locally uniform for x on compact subsets of $\mathbb{C} \setminus \Gamma$.

Proof. We will prove the asymptotic result (17). Notice that (16) follows by using similar arguments. First, we consider the sequence of discrete measures $(\mu_{n,k})_{n \in \mathbb{N}}$ defined by

$$\mu_{n,k} = \sum_{j=1}^s \delta_{x_{n,k,j}} R_{n-1,k}(x_{n,k,j}) \tilde{\Gamma}_{n,k,j} S_{n-1,k}(x_{n,k,j}), \quad n \in \mathbb{N},$$

where $x_{n,k,j}$, $j = 1, \dots, s$, are the different zeros of the matrix polynomial, $R_{n,k}(x)$, or, equivalently, the zeros of $S_{n,k}(x)$ (cf. Theorem 2) with multiplicities $\{\ell_1, \dots, \ell_s\}$, and

$$\tilde{\Gamma}_{n,k,j} = S_{n-1,k}^{(1)}(x_{n,k,j}) \frac{\ell_j (\text{Adj}(S_{n,k}(x)))^{(\ell_j-1)}(x_{n,k,j})}{(\det(S_{n,k}(x)))^{(\ell_j)}(x_{n,k,j})}.$$

Notice that from the quadrature formula

$$\begin{aligned} \int d\mu_{n,k}(x) &= \sum_{j=1}^s R_{n-1,k}(x_{n,k,j}) \tilde{\Gamma}_{n,k,j} S_{n-1,k}(x_{n,k,j}) \\ &= \int R_{n-1,k}(x) dW_k(x) S_{n-1,k}(x) = \mathbf{I}_N, \quad n = 1, 2, \dots \end{aligned}$$

According to Lemma 2, we get

$$(S_{n,k}(x))^{-1} S_{n-1,k}(x) = \sum_{k=1}^s \frac{D_{n,k,j}}{x - x_{n,k,j}},$$

where

$$(18) \quad D_{n,k,j} = \frac{\ell_j}{(\det S_{n,k})^{(\ell_j)}(x_{n,k,j})} (\text{Adj } S_{n,k}(x))^{(\ell_j-1)}(x_{n,k,j}) S_{n-1,k}(x_{n,k,j}).$$

Multiplying in the left hand side of (18) by $C_{n,k}^{-1}$

$$C_{n,k}^{-1} D_{n,k,j} = C_{n,k}^{-1} \frac{\ell_j}{(\det S_{n,k})^{(\ell_j)}(x_{n,k,j})} (\text{Adj } S_{n,k}(x))^{(\ell_j-1)}(x_{n,k,j}) S_{n-1,k}(x_{n,k,j}),$$

and applying the Liouville-Ostrogradski formula (8)

$$S_{n,k}(x_{n,k,j}) (\text{Adj } S_{n,k})^{(\ell_k-1)}(x_{n,k,j}) = \mathbf{0}_N,$$

we get $C_{n,k}^{-1} D_{n,k,j} = R_{n-1,k}(x_{n,k,j}) \tilde{\Gamma}_{n,k} S_{n-1,k}(x_{n,k,j})$. From the definition of the matrices $\tilde{\Gamma}_{n,k}$, we have $C_{n,k}^{-1} S_{n,k}^{-1}(x) S_{n-1,k}(x) = \int \frac{d\mu_{n,k}(y)}{x-y}$, $x \in \mathbb{C} \setminus \tilde{\Gamma}$. For two given nonnegative integers n, k , let us consider the generalized Chebyshev matrix polynomials of the second kind, $(T_n^{A,B,C}(x))_{n \in \mathbb{N}}$, defined in (12). We can prove by induction that

$$(19) \quad \lim_{m \rightarrow \infty} \int d\mu_{n_m, k_m}(x) T_l^{A,B,C}(x) = \mathbf{I}_N \delta_{l,0}.$$

To this end, we can write

$$(20) \quad S_{n-1,k}(x) T_l^{A,B,C}(x) = S_{n,k}(x) K_{l,n-1,k}(x) + \sum_{i=1}^n S_{n-i,k}(x) \Delta_{i,l,n-1,k},$$

where $K_{l,n-1,k}(x)$ is a matrix polynomial with degree less than or equal to $n-1$. Thus,

$$\begin{aligned} \int d\mu_{n,k}(x) T_l^{A,B,C}(x) &= \sum_{j=1}^s R_{n-1,k}(x_{n,k,j}) \tilde{\Gamma}_{n,k,j} S_{n-1,k}(x_{n,k,j}) T_l^{A,B,C}(x_{n,k,j}) \\ &= \sum_{j=1}^s R_{n-1,k}(x_{n,k,j}) \tilde{\Gamma}_{n,k,j} (S_{n,k}(x_{n,k,j}) K_{l,n-1,k}(x_{n,k,j}) + \sum_{i=1}^n S_{n-i,k}(x_{n,k,j}) \Delta_{i,l,n-1,k}). \end{aligned}$$

According to the definition of the matrices $\tilde{\Gamma}_{n,k}$ and taking into account that

$$\begin{aligned} (\text{Adj}(S_{n,k})(x))^{(\ell_k-1)}(x_{n,k,j}) S_{n,k}(x_{n,k,j}) \\ = S_{n,k}(x_{n,k,j}) (\text{Adj}(S_{n,k})(x))^{(\ell_k-1)}(x_{n,k,j}) = \mathbf{0}_N, \end{aligned}$$

we get $\int d\mu_{n,k}(x) T_l^{A,B,C}(x) = \sum_{j=1}^s R_{n-1,k}(x_{n,k,j}) \tilde{\Gamma}_{n,k,j} \left(\sum_{i=1}^n S_{n-i,k}(x_{n,k,j}) \Delta_{i,l,n-1,k} \right)$.

Using the quadrature formula given in Theorem 3, we conclude

$$\int d\mu_{n,k}(x) T_l^{A,B,C}(x) = \sum_{i=1}^n \int R_{n-1,k}(x) dW_k(x) S_{n-i,k}(x) \Delta_{i,l,n-1,k} = \Delta_{1,l,n-1,k}.$$

So, (19) follows when $\lim_{m \rightarrow \infty} \Delta_{j,l,n_m-1,k_m} = \mathbf{I}_N \delta_{j,l+1}$, holds. We use induction on l .

When $l = 0$ the result is immediate. Now assuming that the result is valid up to l , the three-term recurrence relation for the matrix polynomials $(T_n^{A,B,C}(x))_{n \in \mathbb{N}}$ yields

$$S_{n-1,k}(x) T_{l+1}^{A,B,C}(x) = S_{n-1,k}(x) (x T_l^{A,B,C}(x) - T_l^{A,B,C}(x) B - T_{l-1}^{A,B,C}(x) C) A^{-1}.$$

Using (20) and the three-term recurrence relation for $(S_{n,k}(x))_{n \in \mathbb{N}}$

$$\begin{aligned} \Delta_{j,l+1,n-1,k} &= A_{n-j,k} \Delta_{j-1,l,n-1,k} A^{-1} + B_{n-j,k} \Delta_{j,l,n-1,k} A^{-1} \\ &\quad + C_{n-j} \Delta_{j+1,l,n-1,k} A^{-1} - \Delta_{j,l,n-1,k} B A^{-1} - \Delta_{j,l-1,n-1,k} C A^{-1}. \end{aligned}$$

For $j \geq l+3$ or $j \leq l-1$ the induction hypothesis shows that

$$\lim_{m \rightarrow \infty} \Delta_{j,l+1,n_m-1,k_m} = \mathbf{0}_N.$$

We study the cases $j = l$, $j = l+1$, and $j = l+2$ separately:

Case 1. $j = l$.

$$\begin{aligned} \lim_{m \rightarrow \infty} \Delta_{j,l+1,n_m-1,k_m} &= \lim_{m \rightarrow \infty} (A_{n_m-l,k_m} \Delta_{l-1,l,n_m-1,k_m} A^{-1} \\ &\quad + B_{n_m-l,k_m} \Delta_{l,l,n_m-1,k_m} A^{-1} + C_{n_m-l} \Delta_{l+1,l,n_m-1,k_m} A^{-1} - \Delta_{l,l,n_m-1,k_m} B A^{-1} \\ &\quad - \Delta_{l,l-1,n_m-1,k_m} C A^{-1}) = (C - C) A^{-1} = \mathbf{0}_N. \end{aligned}$$

Case 2. $j = l+1$.

$$\begin{aligned} \lim_{m \rightarrow \infty} \Delta_{l+1,l+1,n_m-1,k_m} &= \lim_{m \rightarrow \infty} (A_{n_m-l-1,k_m} \Delta_{l,l,n_m-1,k_m} A^{-1} \\ &\quad + B_{n_m-l-1,k_m} \Delta_{l+1,l,n_m-1,k_m} A^{-1} + C_{n_m-l-1} \Delta_{l+2,l,n_m-1,k_m} A^{-1} - \Delta_{l+1,l,n_m-1,k_m} B A^{-1} \\ &\quad - \Delta_{l+1,l-1,n_m-1,k_m} C A^{-1}) = (B - B) A^{-1} = \mathbf{0}_N. \end{aligned}$$

Case 3. $j = l+2$.

$$\begin{aligned} \lim_{m \rightarrow \infty} \Delta_{l+2,l+1,n_m-1,k_m} &= \lim_{m \rightarrow \infty} (A_{n_m-l-2,k_m} \Delta_{l+1,l,n_m-1,k_m} A^{-1} \\ &\quad + B_{n_m-l-2,k_m} \Delta_{l+2,l,n_m-1,k_m} A^{-1} + C_{n_m-l-2} \Delta_{l+3,l,n_m-1,k_m} A^{-1} - \Delta_{l+2,l,n_m-1,k_m} B A^{-1} \\ &\quad - \Delta_{l+2,l-1,n_m-1,k_m} C A^{-1}) = A A^{-1} = \mathbf{I}_N. \end{aligned}$$

Now, in the same way that was done in [11], one can prove

$$\lim_{m \rightarrow \infty} \int \frac{d\mu_{n_m, k_m}(y)}{x - y} = \int \frac{dW_{C, B, A}(y)}{x - y}, \quad x \in \mathbb{C} \setminus \Gamma,$$

by using the so called *method of moments*. □

4. MAIN RESULT

Definition 4 (cf. [17] Section 7.7). Let $A, B \in \mathbb{C}^{N \times N}$ be Hermitian matrices. We write $A \geq B$ if the matrix $A - B$ is positive semi-definite. Similarly, $A > B$ means that $A - B$ is positive definite.

It is easy to see that the relation \geq (respectively, $>$) is reflexive and transitive.

Definition 5. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of Hermitian matrices. We say that $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence if $A_{n+1} \geq A_n$ for every $n \in \mathbb{N}$.

Theorem 7. Let the matrix sequence $(D_n)_{n \in \mathbb{N}}$ be increasing. Then, for each $n \in \mathbb{N}$, the polynomials $V_n^{D_n}(x)$ and $G_n^{D_n}(x)$ have the same zeros. Moreover, denoting by $\tilde{x}_{n,j}$ the zeros of $V_n^{D_n}(x)$ or $G_n^{D_n}(x)$, then there exists a positive constant M (independent of n) such that the zeros $\tilde{x}_{n,j}$ are contained in a disk $D = \{z \in \mathbb{C} : |z| < M\}$.

Proof. The N -block tridiagonal matrix associated with $G_n^{D_n}(x)D_k^{1/2}$ is the transpose of the matrix

$$\tilde{J}^{(k)} = \begin{bmatrix} D_k^{-1/2} B_0 D_k^{-1/2} & D_k^{-1/2} A_0 D_k^{-1/2} & \mathbf{0}_N & & \\ D_k^{-1/2} C_1 D_k^{-1/2} & D_k^{-1/2} B_1 D_k^{-1/2} & D_k^{-1/2} A_1 D_k^{-1/2} & \ddots & \\ \mathbf{0}_N & D_k^{-1/2} C_2 D_k^{-1/2} & D_k^{-1/2} B_2 D_k^{-1/2} & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

associated with $D_k^{1/2} V_n^{D_n}(x)$. The first statement is a straightforward consequence of the fact that the zeros of $V_n^{D_n}(x)$ are the eigenvalues of $\tilde{J}_{nN}^{(k)}$ (truncated N -block tridiagonal matrix of dimension nN) and the zeros of $G_n^{D_n}(x)D_k^{1/2}$ are the eigenvalues of $\tilde{J}_{nN}^{(k)T}$ (the transpose of the previous one). Using the Gershgorin disk theorem for the location of eigenvalues, it is enough to show that the entries of the matrix \tilde{J}_{nN} are bounded (independently of n). But the entries of this matrix are

$$\begin{aligned} D_k^{-1/2} A_n D_k^{-1/2} &= D_k^{-1/2} D_n^{1/2} D_n^{-1/2} A_n D_n^{-1/2} D_n^{1/2} D_k^{-1/2}, \\ D_k^{-1/2} B_n D_k^{-1/2} &= D_k^{-1/2} D_n^{1/2} D_n^{-1/2} B_n D_n^{-1/2} D_n^{1/2} D_k^{-1/2}, \\ D_k^{-1/2} C_n D_k^{-1/2} &= D_k^{-1/2} D_n^{1/2} D_n^{-1/2} C_n D_n^{-1/2} D_n^{1/2} D_k^{-1/2}. \end{aligned}$$

Since the matrices $D_n^{-1/2} A_n D_n^{-1/2}$, $D_n^{-1/2} B_n D_n^{-1/2}$, and $D_n^{-1/2} C_n D_n^{-1/2}$ converge and $\mathbf{I}_N \geq D_k^{-1/2} D_n^{1/2}$ if $n \leq k$ since $(D_n)_{n \in \mathbb{N}}$ is an increasing sequence, then the result follows. \square

The theory of matrix orthogonal polynomials with varying recurrence coefficients studied in the previous section allows us to prove the main result of this paper, i.e. the outer ratio asymptotics for matrix biorthogonal polynomials satisfying three-term recurrence relations as in (4) and (5) with unbounded coefficients reads as follows.

Theorem 8 (Outer ratio asymptotics). *Let $(V_n(x))_{n \in \mathbb{N}}$ and $(G_n(x))_{n \in \mathbb{N}}$ be the sequences of biorthogonal matrix polynomials with respect to a quasidefinite matrix of measures, W , satisfying the recurrence relations (4), (5), respectively. Let assume that there exists a sequence of positive definite matrices $(D_n)_{n \in \mathbb{N}}$ such that (10) holds with A and C nonsingular matrices, and consider the scaled matrix polynomials sequences $(V_n^{D_n}(x))_{n \in \mathbb{N}}$, $(G_n^{D_n}(x))_{n \in \mathbb{N}}$ given in Definition 3. We denote by Δ_n the set of zeros of $V_n^{D_n}(x)$ and by $\Gamma = \bigcap_{N \geq 0} M_N$, where $M_N = \bigcup_{n \geq N} \Delta_n$.*

Then,

- (a) *If we assume that the matrix sequence $(D_n)_{n \in \mathbb{N}}$ is increasing, then Γ is a compact set.*
 (b) *The following outer ratio asymptotics hold*

$$\lim_{n \rightarrow \infty} D_n^{1/2} V_{n-1}^{D_n}(z) (V_n^{D_n}(z))^{-1} A_{n-1}^{-1} D_n^{1/2} = \int \frac{dW_{C,B,A}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

$$\lim_{n \rightarrow \infty} D_n^{1/2} C_n^{-1} (G_n^{D_n}(z))^{-1} G_{n-1}^{D_n}(z) D_n^{1/2} = \int \frac{dW_{C,B,A}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

where $W_{C,B,A}$ is the matrix weight for the generalized Chebyshev matrix polynomials of the second kind defined by (11). Moreover, the convergence is locally uniform for z on compact subsets of $\mathbb{C} \setminus \Gamma$.

Proof. Let $(D_n)_{n \in \mathbb{N}}$ be a sequence of $N \times N$ positive definite matrices. In order to apply Theorem 6, we consider the scaled matrix polynomials $V_n^{D_n}(x)$, $G_n^{D_n}(x)$ associated with the parameters $(A_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$, $(C_n)_{n \in \mathbb{N}}$. Taking into account their definitions, we have

$$(21) \quad x D_k V_n^{D_k}(x) = A_n V_{n+1}^{D_k}(x) + B_n V_n^{D_k}(x) + C_n V_{n-1}^{D_k}(x),$$

$$(22) \quad G_m^{D_k}(x) D_k x = G_{m-1}^{D_k}(x) A_{m-1} + G_m^{D_k}(x) B_m + G_{m+1}^{D_k}(x) C_{m+1},$$

and so

$$\begin{aligned}
 xD_k^{1/2} V_n^{D_k}(x) &= D_k^{-1/2} A_n D_k^{-1/2} D_k^{1/2} V_{n+1}^{D_k}(x) \\
 &\quad + D_k^{-1/2} B_n D_k^{-1/2} D_k^{1/2} V_n^{D_k}(x) + D_k^{-1/2} C_n D_k^{-1/2} D_k^{1/2} V_{n-1}^{D_k}(x), \\
 xG_n^{D_k}(x) D_k^{1/2} &= G_{n-1}^{D_k}(x) D_k^{1/2} D_k^{-1/2} A_{n-1} D_k^{-1/2} \\
 &\quad + G_n^{D_k}(x) D_k^{1/2} D_k^{-1/2} B_n D_k^{-1/2} + G_{n+1}^{D_k}(x) D_k^{1/2} D_k^{-1/2} C_{n+1} D_k^{-1/2},
 \end{aligned}$$

respectively. For each k , taking

$$\begin{cases} R_{n,k}(x) = D_k^{1/2} V_n^{D_k}(x), & S_{n,k}(x) = G_n^{D_k}(x) D_k^{1/2}, \\ A_{n,k} = D_k^{-1/2} A_n D_k^{-1/2}, & B_{n,k} = D_k^{-1/2} B_n D_k^{-1/2}, & C_{n,k} = D_k^{-1/2} C_n D_k^{-1/2}, \end{cases}$$

the matrix polynomial sequences $(R_{n,k}(x))_{n \in \mathbb{N}}$, $(S_{n,k}(x))_{n \in \mathbb{N}}$ satisfy the three-term recurrence relations, (13), (14), respectively, with initial conditions $R_{0,k}(x) = G_{0,k}(x) = D_k^{1/2}$ and $R_{-1,k}(x) = G_{-1,k}(x) = \mathbf{0}_N$. Then, the matrix polynomial sequences $(R_{n,k}(x))_{n \in \mathbb{N}}$, $(S_{n,k}(x))_{n \in \mathbb{N}}$ are orthogonal with respect to a certain varying matrix of measures W_k . Under the assumptions (10) it is easy to see that the limit conditions (15) are satisfied for $n_m = k_m = m$. Then, Theorem 8 holds.

Finally, from Theorem 7 we have that if the matrix sequence $(D_n)_{n \in \mathbb{N}}$ is increasing, then the zeros of $V_n^{D_n}(x)$ and $G_n^{D_n}(x)$ are bounded (and so Γ is a compact set), as we wanted to prove. \square

Example 2. In Example 1 let us take the scalar measure $d\mu = x^\alpha e^{-x} dx$, $\alpha > -1$, supported on $]0, \infty[$ and $(L_n^\alpha)_{n \in \mathbb{N}}$ its corresponding sequence of monic orthogonal polynomial (Laguerre polynomials of parameter α). The biorthogonal sequences of matrix polynomials with respect to the new measure $\begin{bmatrix} x & -1 \\ 0 & x \end{bmatrix} x^\alpha e^{-x} dx$ are

$$V_n(x) = \begin{bmatrix} L_n^{\alpha+1}(x) & \frac{L_n^{\alpha+1}(x) - \frac{\alpha+n+1}{\alpha+1} L_n^\alpha(x)}{x} \\ 0 & L_n^{\alpha+1}(x) \end{bmatrix} = \begin{bmatrix} L_n^{\alpha+1}(x) & -\frac{n}{\alpha+1} L_{n-1}^{\alpha+2}(x) \\ 0 & L_n^{\alpha+1}(x) \end{bmatrix},$$

$$G_n(x) = - \begin{bmatrix} (-1)^n \frac{\Gamma(\alpha+1) K_n^\alpha(x,0)}{\Gamma(\alpha+n+1)} & 0 \\ \frac{n K_n^\alpha(x,0)}{(\alpha+1)} + (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \frac{\partial K_n^\alpha(x,0)}{\partial y} & (-1)^n \frac{\Gamma(\alpha+1) K_n^\alpha(x,0)}{\Gamma(\alpha+n+1)} \end{bmatrix}.$$

Moreover, the sequences $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ satisfy the three-term recurrence relations (4), (5) with $A_n = \mathbf{I}_N$,

$$B_n = \begin{bmatrix} 2n + \alpha + 2 & -2/(\alpha + 1) \\ 0 & 2n + \alpha + 2 \end{bmatrix}, \quad C_n = \begin{bmatrix} n(\alpha + n + 1) & 0 \\ 0 & n(\alpha + n + 1) \end{bmatrix}.$$

On the other hand, taking the sequence of positive definite matrices $(D_n)_{n \in \mathbb{N}}$, where $D_n = \begin{bmatrix} n^2 & 0 \\ 0 & n^2 \end{bmatrix}$, we find that $D_n^{1/2} = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$, $D_n^{-1/2} = \begin{bmatrix} 1/n & 0 \\ 0 & 1/n \end{bmatrix}$ and

$D_n^{-1/2} D_{n-1}^{1/2} = \mathbf{I}_N$. Thus,

$$D_n^{-1/2} B_n D_n^{-1/2} = \begin{bmatrix} (2n + \alpha + 2)/n^2 & -2/((\alpha + 1)n^2) \\ 0 & (2n + \alpha + 2)/n^2 \end{bmatrix}, \text{ and so } D_n^{-1/2} B_n D_n^{-1/2} \rightarrow \mathbf{0}_N,$$

$$D_n^{-1/2} C_n D_n^{-1/2} = \begin{bmatrix} \frac{\alpha+n+1}{n} & 0 \\ 0 & \frac{\alpha+n+1}{n} \end{bmatrix}, \text{ and so } D_n^{-1/2} C_n D_n^{-1/2} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From Theorem 8 we get

$$\lim_{n \rightarrow \infty} D_n^{1/2} V_{n-1}^{D_n}(z) (V_n^{D_n}(z))^{-1} D_n^{1/2} = \int \frac{dW_{I,0,I}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

$$\lim_{n \rightarrow \infty} D_n^{1/2} C_n^{-1} (G_n^{D_n}(z))^{-1} G_{n-1}^{D_n}(z) D_n^{1/2} = \int \frac{dW_{I,0,I}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

where $W_{I,0,I}$ is the matrix of measures associated with the sequence of orthogonal polynomials $(U_n^{I,0,I}(x))_{n \in \mathbb{N}}$ satisfying

$$xU_n^{I,0,I}(x) = U_{n+1}^{I,0,I}(x) + U_{n-1}^{I,0,I}(x), \quad n \in \mathbb{N},$$

with initial conditions $U_0^{I,0,I}(x) = \mathbf{I}_N$, $U_{-1}^{I,0,I}(x) = \mathbf{0}_N$.

Notice that it is an interesting situation, because $W_{I,0,I}(x)$ is a positive definite matrix of measures. Moreover, using Corollary 2.3 in [8] we obtain

$$\int \frac{dW_{I,0,I}(t)}{z-t} = \frac{zI}{2} - \frac{\sqrt{(z^2-4)I}}{2}.$$

Example 3. Let $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ be matrix polynomial sequences satisfying the recurrence relations (4) and (5), respectively, with

$$A_n = \begin{bmatrix} 2n^2 & 0 \\ 7n^2 + 1 & 5n^2 \end{bmatrix}, \quad B_n = \begin{bmatrix} 3/n & 4/n \\ n & 8n^2 \end{bmatrix}, \quad C_n = \begin{bmatrix} 2n^2 & 2n \\ 0 & n \end{bmatrix}.$$

If we take the sequence of positive definite matrices, $(D_n)_{n \in \mathbb{N}}$,

$$D_n = \frac{n^8}{(n^2-1)^2} \begin{bmatrix} 1/n^2 + 1/n^4 & -2/n^3 \\ -2/n^3 & 1/n^2 + 1/n^4 \end{bmatrix},$$

then $D_n^{1/2} = \frac{n^4}{n^2-1} \begin{bmatrix} 1/n & -1/n^2 \\ -1/n^2 & 1/n \end{bmatrix}$, $D_n^{-1/2} = \begin{bmatrix} 1/n & 1/n^2 \\ 1/n^2 & 1/n \end{bmatrix}$, $D_n^{-1/2} D_{n-1}^{1/2} \rightarrow \mathbf{I}_N$.

From here

$$D_n^{-1/2} A_n D_n^{-1/2} = \begin{bmatrix} 2 + 1/n^3 + 5/n^2 + 7/n & (7n^3 + 7n^2 + 1)/n^4 \\ 7 + 1/n^2 + 7/n & 5 + 1/n^3 + 2/n^2 + 7/n \end{bmatrix},$$

and so, $D_n^{-1/2} A_n D_n^{-1/2} \rightarrow \begin{bmatrix} 2 & 0 \\ 7 & 5 \end{bmatrix}$,

$$D_n^{-1/2} B_n D_n^{-1/2} = \begin{bmatrix} (9n^2 + 3n + 4)/n^4 & (8n^3 + 5n + 3)/n^4 \\ (9n^4 + 3n + 4)/n^5 & (8n^5 + n^3 + 4n + 3)/n^5 \end{bmatrix},$$

$$D_n^{-1/2} B_n D_n^{-1/2} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix},$$

$$D_n^{-1/2} C_n D_n^{-1/2} = \begin{bmatrix} 2 + 3/n^2 & 5/n \\ (3n^2 + 2)/n^3 & 1 + 4/n^2 \end{bmatrix}, \quad D_n^{-1/2} C_n D_n^{-1/2} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 8 yields

$$\begin{aligned} \lim_{n \rightarrow \infty} D_n^{1/2} V_{n-1}^{D_n}(z) (V_n^{D_n}(z))^{-1} A_{n-1}^{-1} D_n^{1/2} &= \int \frac{dW_{C,B,A}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma, \\ \lim_{n \rightarrow \infty} D_n^{1/2} C_n^{-1} (G_n^{D_n}(z))^{-1} G_{n-1}^{D_n}(z) D_n^{1/2} &= \int \frac{dW_{C,B,A}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma, \end{aligned}$$

where $dW_{C,B,A}(x)$ is the matrix of measures of the sequence of biorthogonal polynomials satisfying for all $n \in \mathbb{N}$,

$$\begin{aligned} xU_n^{C,B,A}(x) &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} U_{n+1}^{C,B,A}(x) + \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} U_n^{C,B,A}(x) + \begin{bmatrix} 2 & 0 \\ 7 & 5 \end{bmatrix} U_{n-1}^{C,B,A}(x), \\ xT_n^{A,B,C}(x) &= T_{n+1}^{A,B,C}(x) \begin{bmatrix} 2 & 0 \\ 7 & 5 \end{bmatrix} + T_n^{A,B,C}(x) \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} + T_{n-1}^{A,B,C}(x) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

with initial conditions $U_0^{C,B,A}(x) = T_0^{C,B,A}(x) = \mathbf{I}_N$, $U_{-1}^{C,B,A}(x) = T_{-1}^{C,B,A}(x) = \mathbf{0}_N$.

5. THE SINGULAR CASE

In this section, we study the case when the limit matrices A or C are singular. In [11], for the case of symmetric recurrence coefficients the authors proved that the outer ratio asymptotic also exists in the singular case, although they cannot compute explicitly the degenerate positive definite matrix of measures appearing in the limit. A similar argument can be applied for obtaining the existence of outer ratio asymptotics for matrix polynomials satisfying recurrence relations with nonsymmetric coefficients.

Theorem 9. *Let $(V_n(x))_{n \in \mathbb{N}}$, $(G_n(x))_{n \in \mathbb{N}}$ be the sequences of biorthogonal matrix polynomials with respect to a quasidefinite matrix of measures, W , satisfying the recurrence relations (4), (5). Let us assume that $(D_n)_{n \in \mathbb{N}}$ is an increasing sequence of matrices. Under the hypotheses of Theorem 8, if limit matrix A is singular, then there exists a matrix of measures, ν_1 , for which*

$$\lim_{n \rightarrow \infty} D_n^{1/2} V_{n-1}^{D_n}(z) (V_n^{D_n}(z))^{-1} A_{n-1}^{-1} D_n^{1/2} = \int \frac{d\nu_1(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma.$$

Finally, if the matrix C is singular, then there exists a matrix of measures ν_2 , such that

$$\lim_{n \rightarrow \infty} D_n^{1/2} C_n^{-1} (G_n^{D_n}(z))^{-1} G_{n-1}^{D_n}(z) D_n^{1/2} = \int \frac{d\nu_2(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma.$$

Moreover, if we write $F_A(z) = \int \frac{d\nu_1(t)}{z-t}$ and $F_C(z) = \int \frac{d\nu_2(t)}{z-t}$, with $z \notin \text{supp}(\nu)$, then these analytic matrix functions satisfy the matrix equation

$$(23) \quad C F(z) A F(z) + (B - zI) F(z) + I = \mathbf{0}_N.$$

Proof. Following the technique given in Section 4 of [11], it is enough to reduce the result to the case of varying recurrence coefficients and to use the matrix polynomials $x^l \mathbf{I}_N$, instead of $U_n^{C,B,A}(x)$ and $T_n^{A,B,C}(x)$, in the proof of Theorem 6.

In order to prove that the Hilbert transforms of the measures ν_1 and ν_2 satisfy the matrix equation (23), let us remind that the polynomials $V_n^{D_k}(x)$ and $G_n^{D_k}(x)$ satisfy (21), (22), i.e.

$$\begin{aligned} x D_k^{1/2} V_n^{D_k}(x) &= D_k^{-1/2} A_n V_{n+1}^{D_k}(x) + D_k^{-1/2} B_n V_n^{D_k}(x) + D_k^{-1/2} C_n V_{n-1}^{D_k}(x), \\ x G_n^{D_k}(x) D_k^{1/2} &= G_{n-1}^{D_k}(x) A_{n-1} D_k^{-1/2} + G_n^{D_k}(x) B_n D_k^{-1/2} + G_{n+1}^{D_k}(x) C_{n+1} D_k^{-1/2}. \end{aligned}$$

Let us multiply the right hand side of the first one by $(V_n^{D_n})^{-1}(x) D_n^{-1/2}$, and the left hand side of the second one by $D_n^{-1/2} (G_n^{D_n})^{-1}(x)$. Now, we put $k = n$ and take limit as n tends to infinity. Then, from Theorem 8

$$\begin{aligned} \lim_{n \rightarrow \infty} D_n^{-1/2} A_n V_{n+1}^{D_k}(x) (V_n^{D_n})^{-1}(x) D_n^{-1/2} &= F_A^{-1}, \\ \lim_{n \rightarrow \infty} D_n^{-1/2} (G_n^{D_n})^{-1}(x) G_{n+1}^{D_n}(x) C_{n+1} D_n^{-1/2} &= F_C^{-1}, \end{aligned}$$

and the result follows. \square

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