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APPROXIMATION BY THE K^λ MEANS OF FOURIER SERIES AND CONJUGATE SERIES OF FUNCTIONS IN $H_{\alpha,p}$

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In this paper, we consider two operators based on the K^λ means of the Fourier series of and conjugate series of functions of class L^p, $p > 1$. We study the degree of convergence of these operators to the functions whose Fourier series and Allied series are used.

1. INTRODUCTION

Various summation methods have been used to make sense of divergent series (see, for example, [**5, 6**]). These summation methods transform the series into a convergent series and the sum of the divergent series is then defined to be the sum of this associated series. The method is called *regular* if that method agrees with the standard sum on convergent series. Applying a summation method may still be advantageous for a convergent sum as it may speed up the convergence. Here we shall determine the degree of convergence of certain means of the Fourier series and conjugate series of a function f to itself in $H_{\alpha,p}$. We define numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ by

$$\prod_{\nu=0}^{n-1}(x+\nu) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m,$$

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where,

$$\prod_{\nu=0}^{n-1}(x+\nu)=\frac{\Gamma(x+n)}{\Gamma(x)}.$$

We will use the conventions $\begin{bmatrix}0\\0\end{bmatrix}=1$. The numbers $\begin{bmatrix}n\\m\end{bmatrix}$ are known as the absolute values of the Stirling numbers of the first kind. Let $\lambda>0$. The K^λ mean $K_n^\lambda(t,x)$ of a sequence $\{t_n(x)\}$ is defined by

$$K_n^\lambda(t,x):=\frac{\gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k t_k(x).$$

If $K_n^\lambda(t,x)\to s$ as $n\to\infty$, we say that the sequence $\{t_n(x)\}$ is summable K^λ to s. The method K^λ is regular for $\lambda>0$ and this case will be supposed in this paper.

Let $C_{2\pi}$ be the space of all 2π-periodic functions defined on $[0,2\pi]$ and for $0<\alpha\le 1$ define the Hölder continuous functions

$$H_\alpha:=\{f\in C_{2\pi}:\exists M>0\,\forall x,y,|f(x)-f(y)|\le M|x-y|^\alpha\}.$$

For $0<\alpha\le 1$, H_α is a Banach space [**10**] under the norm

$$\|f\|_\alpha:=\|f\|_\infty+\sup_{x\neq y}\Delta^\alpha f(x,y),$$

where,

$$\Delta^\alpha f(x,y):=\frac{|f(x)-f(y)|}{|x-y|^\alpha},\quad x\neq y$$

and

$$\Delta^0 f(x,y):=0.$$

Alextis [**1**] studied the degree of approximation of the functions in H_α class by the Cesáro means of their Fourier series in the sup-norm. It was Prössodoorf [**11**] who initiated the work on the degree of approximation of functions in H_α by the Féjer mean of the Fourier series in the Hölder metric. This result has been generalized by Chandra [**2**], Mohapatra and Chandra [**9**], Singh [**13**], [**14**] using different methods. Chandra [**2**], [**3**] has studied the degree of approximation problem in the Hölder metric using Borel and Euler means. The degree of approximation of a function f in H_α has been studied by Das, Ghosh, and Ray [**4**] by using (e,c) means of Fourier Series in the Hölder metric.

Let

$$K_n^\lambda(f,x)=\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k S_k(f,x).$$

and

$$\tilde{K}_n^\lambda(f,x)=\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k \tilde{S}_k(f,x)$$

where, $S_n(f,x)$ and $\tilde{S}_n(f,x)$, respectively, denote the n th partial sums of the series

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

and the conjugate

$$f \sim \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx).$$

The K^{λ} means were first introduced by Karamata [**7**]. Lototsky [**8**] reintroduced the special case $\lambda = 1$. Vuckovic [**15**] first studied the K^{λ} summability of the Fourier series. Sadangi [**12**] proved the following two theorems:

Theorem 1.1 (A). *Let $K_n^{\lambda}(f,x)$ be the K^{λ} mean of the Fourier series of f at x. If*

$0 \le \beta < \alpha \le 1$ *and* $f \in H_{\alpha}$ *then*

$$\left\|K_n^{\lambda}(f,\cdot) - f\right\|_{\beta} = O(1)\frac{\log\log n}{(\log n)^{\alpha-\beta}}$$

Theorem 1.2 (B). *Let $h = \frac{\pi}{l(n)}$, where $l(n) = \frac{3}{2} + \lambda \sum_{k=1}^{n-1} \frac{1}{\lambda+k}$. Let $\tilde{K}_n(f,x)$ be the K^{λ} mean of the series conjugate to the Fourier series of f at x. If $0 \le \beta < \alpha \le 1$ and $f \in H_{\alpha}$, then*

$$\left\|\tilde{K}_n^{\lambda}(f,\cdot) - \tilde{f}(\cdot,h)\right\|_{\beta} = O(1)\frac{\log\log n}{(\log n)^{\alpha-\beta}}, 0 \le \beta < \alpha \le 1$$

Here

$$\tilde{f}(x,\varepsilon) = -\frac{2}{\pi}\int_{\varepsilon}^{\pi}\psi_x(t)\frac{1}{2}\cot\frac{1}{2}t\,dt$$

$$\tilde{f}(x) = \lim_{\varepsilon\to 0^+}\tilde{f}(x,\varepsilon),$$

and

$$\psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

2. PRELIMINARY DEFINITIONS

Let $L_p[0,2\pi]$ be the space of all 2π-periodic integrable functions.

$$H_{\alpha,p} := \left\{ f \in L_p[0,2\pi] : \left(\int_0^{2\pi}|f(x+t) - f(x)|^p\,dx\right)^{\frac{1}{p}} \le K|t|^{\alpha} \right\},$$

where K is a positive constant. The space $H_{\alpha,p}(p > 1, \alpha \leq 0 < 1)$ is a Banach space under the norm $\| \cdot \|_{\alpha,p}$:

$$\|f\|_{\alpha,p} := \|f\|_p + \sup_{t\neq 0} \frac{\|f(y+t) - f(y)\|_p}{|t|^{\alpha}}.$$

The metric induced by the norm $\|\cdot\|_{\alpha,p}$ on $H_{\alpha,p}$ is called Holder continuous with degree p. It can be seen that

$$\|f\|_{\beta,p} \leq (2\pi)^{\alpha-\beta} \|f\|_{\alpha,p}.$$

Since $f \in H_{\alpha,p}$ if and only if $\|f\|_{\alpha,p} < \infty$, we have,

$$L_p[0,2\pi] \supseteq H_{\beta,p} \supseteq H_{\alpha,p}, \quad p > 1, \quad 0 \leq \beta < \alpha \leq 1.$$

3. MAIN RESULTS

In this paper we will prove the following two theorems:

Theorem 3.3. *Let $K_n^{\lambda}(f,x)$ be the K^{λ} mean of the Fourier series of f at x. If $p > 1$, $0 \leq \beta < \alpha \leq 1$, and $f \in H_{\alpha,p}$, then*

$$\left\|K_n^{\lambda}(f,\cdot) - f\right\|_{\beta,p} = O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right) \tag{1}$$

Theorem 3.4. *Let $h = \frac{\pi}{l(n)}$, where $l(n) = \frac{3}{2} + \lambda \sum_{k=1}^{\lambda-1} \frac{1}{\lambda+k}$. Let $\tilde{K}_n(f,x)$ be the K^{λ} mean of the series conjugate to the Fourier series of f at x. If $0 \leq \beta < \alpha \leq 1$ and $f \in H_{\alpha,p}$, then*

$$\left\|\tilde{K}_n(f,x) - \tilde{f}(\cdot,h)\right\|_{\beta,p} = O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right) \tag{2}$$

4. ADDITIONAL NOTATIONS AND LEMMAS

We use the following additional notations:

$$\begin{aligned}
G(t) &= \varphi_{y+u}(t) - \varphi_y(t) \\
\tilde{G}(t) &= \psi_{y+u}(t) - \psi_y(t) \\
L_n(y) &= K_n^\lambda(f,y) - f(y) \\
\tilde{L}_n(y) &= \tilde{K}_n^\lambda(f,y) - \tilde{f}(y,h) \\
\varphi_x(t) &= \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \\
\tilde{E}_n(t) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \tilde{D}_k(t) \\
\rho_k(t) &= (\lambda^2 + 2\lambda k \cos t + k^2)^{1/2} \\
\tan\theta_k &= \frac{\lambda \sin t}{\lambda \cos t + k} \\
R(n,t) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} \rho_k(t) \\
l(n) &= \frac{3}{2} + \lambda \sum_{0}^{n-1} \frac{1}{\lambda + k} \\
h &= \frac{\pi}{l(n)}
\end{aligned}$$

We need the following lemmas for the proof of our theorems.

Lemma 4.5. *Let $0 \le \beta < \alpha \le 1$. If $f \in H_{\alpha,p}$ then for $0 < t \le \pi$ and $u \ne 0$*

$$G(t) = O(1) \begin{cases} t^\alpha \\ |u|^a \\ |u|^\beta t^{\alpha-\beta} \end{cases} , \quad \tilde{G}(t) = O(1) \begin{cases} t^\alpha \\ |u|^a \\ |u|^\beta t^{\alpha-\beta} \end{cases}$$

Proof. The first and second estimates of (1) follow from the definition of $H_{a,p}$ and $\varphi_x(t)$. Writing

$$|G(t)| = |G(t)|^{1-\frac{\beta}{\alpha}} |G(t)|^{\frac{\beta}{\alpha}}$$

and using the first two estimates of (1) we can derive the third one. Proof of (2) is similar to that of (1). □

Lemma 4.6. *Let $0 \le \beta < \alpha \le 1$. If $f \in H_{\alpha,p}$ then for $x \ne y$*

1.

$$G\left(t+h\right)-G\left(t\right)=O(1)\begin{cases}h^{\alpha}\\|u|^{a}\\|u|^{\beta}h^{\alpha-\beta}\end{cases}$$

2.

$$\tilde{G}\left(t+h\right)-\tilde{G}\left(t\right)=O(1)\begin{cases}h^{\alpha}\\|u|^{a}\\|u|^{\beta}h^{\alpha-\beta}\end{cases}$$

Proof. Writing

$$G\left(t+h\right)-G\left(t\right)=\frac{1}{2}\left[f\left(y+u+t+h\right)-f\left(y+u+t\right)\right]+\left[f\left(y+u-t-h\right)-f\left(y+u-t\right)\right]$$
$$+\left[f\left(y+t+h\right)-f\left(y+t\right)\right]+\left[f\left(y-t-h\right)-f\left(y-t\right)\right]$$

and using the fact that $f\in H_{\alpha,p}$ we obtain the first estimate of (1). The remaining part of the proof is similar to that of Lemma 4.5 and hence it is omitted. □

Lemma 4.7. *Suppose that A and c are both positive constants. Let γ be any real number. Then as $\mu\to\infty$,*

$$\int_{\frac{\pi}{\mu}}^{c}t^{\gamma}e^{-A\mu t^{2}}dt=O(1)\begin{cases}\mu^{-\gamma-1}, & \gamma<-1\\ \log\mu, & \gamma=-1\\ \frac{1}{\mu^{k}}, & 2k-1<\gamma\le 2k\\ \frac{1}{\mu^{\gamma-k}}, & 2k\le\gamma\le 2k+1\end{cases},k=0,1,2,3,\ldots$$

Lemma 4.8. *We have*

$$K_{n}\left(t\right)=R\left(n,t\right)\sin\left(\frac{3}{2}t+\sum_{k=1}^{n-1}\theta_{k}\right)$$
$$\tilde{K}_{n}\left(t\right)=R(n,t)\cos\left(\frac{3}{2}t+\sum_{k=1}^{n-1}\theta_{k}\right)$$

Proof. By simple computation, we obtain

$$\begin{aligned}\tilde{K}_n(t) + iK_n(t) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(\lambda e^{it}\right)^k e^{\frac{1}{2}it} \\ &= \frac{\Gamma(\lambda) e^{\frac{1}{2}it}}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} (\lambda e^{it} + k) \\ &= \frac{\Gamma(\lambda) e^{\frac{1}{2}it}}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} [(\lambda \cos t + k) + i\lambda \sin t] \\ &= \frac{\Gamma(\lambda) e^{\frac{1}{2}it}}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} \rho_k(t) e^{i\theta_k} \\ &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} R(n,t) \exp\left\{ i \left(\frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) \right\}\end{aligned}$$

from which the lemma follows. □

Lemma 4.9. *Let $0 < t < \pi$. Then for some positive constant A,*

1. $R(n,t) = \begin{cases} O(1) \\ O(1)e^{-At^2 \log n} \end{cases}$

2. $K_n(t) = \begin{cases} O(1) \\ O(1)e^{-At^2 \log n} \end{cases}$

3. $\tilde{K}_n(t) = \begin{cases} O(1) \\ O(1)e^{-At^2 \log n} \end{cases}$

4. $\tilde{E}_n(t) = O\left(t^{-1}\right)$

Proof. $R(n,t)$ attains its maximum value for $t = 0$ and it is easy to see that $R(n,0) = 1$ and this ensures the first estimate of 1. Now

$$R(n,t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} (\lambda^2 + 2\lambda k \cos t + k^2)^{\frac{1}{2}} \tag{3}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} (\lambda + k) \left[1 - \frac{4k \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right]^{\frac{1}{2}} \tag{4}$$

$$= \prod_{k=0}^{n-1} \left[1 - \frac{4\lambda \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right]^{1/2} \tag{5}$$

$$= \exp\left[-\frac{1}{2}\sum_{k=1}^{n-1} \log\left\{1 - \frac{4\lambda \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right\}^{-1}\right] \tag{6}$$

At this stage, we observe that

$$0 < \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2} < 1$$

for $k = 1, 2, 3, \ldots$ and $0 < t < \pi$. As $\log(1-\theta)^{-1} \geq \theta$ for $0 < \theta < 1$ and $\sin \geq \frac{2x}{\pi}, 0 \leq x \leq \frac{\pi}{2}$, we have

$$\begin{aligned} \sum_{k=1}^{n-1} \log\left\{1 - \frac{4\lambda \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right\}^{-1} &\geq \sum_{k=1}^{n-1} \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2} \\ &\geq \frac{2\lambda t^2}{\pi^2} \sum_{k=1}^{n-1} \frac{k}{(\lambda+k)^2} \\ &\geq At^2 \log n \end{aligned} \tag{7}$$

where A is some positive constant. Using (6)and (7) we obtain the second estimate of Lemma 4.9, and 3 follows. As $\tilde{D}_k(t) = O(t^{-1})$ 4 follows at once. □

Lemma 4.10. *Let* $0 < t \leq \frac{\pi}{4}$. *Then*

1. $\sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k\right) - \sin l(n)\, t = 0(t^3 \log n)$
2. $\cos\left(\frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k\right) - \cos l(n)\, t = 0(t^3 \log n)$

Proof. We have

$$\left|\sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k\right) - \sin l(n)\, t\right| \leq \left|\left(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k\right) - l(n)\, t\right|. \tag{8}$$

Next, we note that

$$0 < \frac{\lambda \sin t}{\lambda \cos t + k} < 1$$

whenever $0 < t \leq \frac{\pi}{4}$ and $k \geq 1$. Thus for $0 < t \leq \frac{\pi}{4}, 1 \leq k \leq n-1$

$$\begin{aligned}
\theta_k &= \left[\tan^{-1} \frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda \sin t}{\lambda \cos t + k}\right] + \left[\frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda \cos t + k}\right] \\
&\quad + \left[\frac{\lambda t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda + k}\right] + \frac{\lambda t}{\lambda + k} \\
&= O\left[\left(\frac{\lambda \sin t}{\lambda \cos t + k}\right)^3\right] + O\left[\frac{t^3}{\lambda \cos t + k}\right] + O\left[\frac{t^3}{(\lambda \cos t + k)(\lambda + k)}\right] + \frac{\lambda t}{\lambda + k} \\
&= O\left(\frac{t^3}{k^3}\right) + O\left(\frac{t^3}{k^2}\right) + O\left(\frac{t^3}{k}\right) + \frac{\lambda t}{\lambda + k} \\
&= \frac{\lambda t}{\lambda + k} + O\left(\frac{t^3}{k}\right)
\end{aligned}$$

It follows that

$$\text{(9)} \qquad \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k = \frac{3}{2}t + \lambda t \sum_{k=1}^{n-1} \frac{1}{\lambda + k} + O(t^3) \sum_{k=1}^{n-1} \frac{1}{k} = tl(n) + O(t^3 \log n)$$

Using (9) in (8), we obtain Lemma 4.10 part (1). We omit the proof of (2) as it is similar to that of (1). □

Lemma 4.11. *We have*

1. $R'(n,t) = O(1)\, t \log n R(n,t), 0 < t \leq \frac{\pi}{2}$

2. $R(n, t+h) - R(n,t) = O(1)\, t e^{-At^2 \log n}, h \leq t \leq \frac{\pi}{2}$

Proof. We have

$$R(n,t) = \lambda \prod_{k=1}^{n-1} \rho_k(t)$$

and so by logarithmic differentiation,

$$\begin{aligned}
R'\left(n,t\right) &= R(n,t)\sum_{k=1}^{n-1}\frac{\rho_k'(t)}{\rho_k(t)}\\
&= R(n,t)\sum_{k=1}^{n-1}\frac{(-\lambda k\sin t)}{(\rho_k\left(t\right))^2}\\
&= O\left(1\right)tR\left(n,t\right)\sum_{k=1}^{n-1}\frac{1}{k}(\text{ because } \rho_k(t)\geq k, 0<t\leq\frac{\pi}{2})\\
&= O\left(1\right)t\log nR(n,t)
\end{aligned}$$

and this completes the proof of (1). By the Mean Value Theorem, for some ξ with $0<\xi<1$,

$$\begin{aligned}
R\left(n,t+h\right)-R\left(n,t\right) &= hR'(n,t+\xi h)\\
&= O\left(1\right)h\left(t+\xi h\right)R\left(n,t+\xi h\right)\log n\\
&= O\left(1\right)te^{-At^2\log n},
\end{aligned}$$

for $h\leq t\leq\frac{\pi}{2}$ □

The following lemma is due to Hardy, Littlewood, and Pólya [13].

Lemma 4.12. *If $h(y,t)$ is a function of two variables defined for $0\leq t\leq\pi, 0\leq y\leq 2\pi$,*
then

$$\left\|\int h\left(y,t\right)dt\right\|_p\leq\int\left\|h\left(y,t\right)\right\|_p dt,\quad p>1$$

Finally, it is known ([**16**], p.50) that

$$S_n\left(f,x\right)-f\left(x\right)=\frac{2}{\pi}\int_0^{\pi}\varphi_x\left(t\right)D_n\left(t\right)dt \tag{10}$$

and

$$\tilde{S}_n\left(f,x\right)=\frac{-2}{\pi}\int_0^{\pi}\psi_x\left(t\right)\tilde{D}_n\left(t\right)dt, \tag{11}$$

where

$$D_n\left(t\right)=\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t},$$

and

$$\tilde{D}_n(t) = \frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}.$$

5. PROOF OF THEOREM 3.3

Using (10) we have

$$\begin{aligned}
K_n^{\lambda}(f,x) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k S_k(f,x) \\
&= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \left(\frac{2}{\pi} \int_0^{\pi} \varphi_x(t) D_k(t)\,dt \right) + f(x) \frac{\Gamma(\lambda)}{(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \\
&= \frac{2}{\pi} \frac{\int_0^{\pi} \varphi_x(t)}{2\sin\frac{1}{2}t} K_n(t)\,dt + f(x),
\end{aligned}$$

from which it follows that

$$L_n(x) = K_n^{\lambda}(f,x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\varphi_x(t)}{2\sin\frac{1}{2}t} K_n(t)\,dt$$

Now, for fixed δ with $0 < \delta < \frac{\pi}{4}$, we write for $u \neq 0$

$$\begin{aligned}
L_n(y+u) - L_n(y) &= \frac{2}{\pi} \int_0^{\pi} \frac{G(t)}{2\sin\frac{1}{2}t} K_n(t)\,dt \\
&= \frac{2}{\pi} \left(\int_0^{h} + \int_h^{\delta} + \int_{\delta}^{\pi} \right) \frac{G(t)}{2\sin\frac{1}{2}t} K_n dt \\
&= \frac{2}{\pi}(P+Q+R) = I_1 + I_2 + I_3
\end{aligned}$$

By Minkowski's inequality, we have,

$$\|L_n(y+u) - L_n(y)\|_p \leq \|I_1\|_p + \|I_2\|_p + \|I_3\|_p$$

And by Lemma 4.12, we may write,

$$\begin{aligned}
\|I_1\|_p &\leq \frac{2}{\pi} \int_0^{h} \|G(t)\|_p \left| \frac{K_n(t)}{2\sin\frac{1}{2}t} \right| dt, \\
\|I_2\|_p &\leq \frac{2}{\pi} \int_h^{\delta} \|G(t)\|_p \left| \frac{K_n(t)}{2\sin\frac{1}{2}t} \right| dt, \qquad (12) \\
\|I_3\|_p &\leq \frac{2}{\pi} \int_{\delta}^{\pi} \|G(t)\|_p \left| \frac{K_n(t)}{2\sin\frac{1}{2}t} \right| dt.
\end{aligned}$$

By Lemma 4.5 (1) and Lemma 4.9 (2), we obtain

$$\|I_1\|_p = O\left(|u|^\beta \int_0^h t^{\alpha-\beta-1} dt\right) = O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right). \tag{13}$$

and

$$\begin{aligned} \|I_3\|_p &= O\left(|u|^\beta \int_\delta^\pi t^{\alpha-\beta-1} e^{-At^2 \log n} dt\right) \\ &= O\left(|u|^\beta e^{-A\delta^2} \log n\right) \\ &= O\left(\frac{|u|^\beta}{(\log n)^\Delta}\right), \Delta \text{ positive however large.} \end{aligned} \tag{14}$$

Using Lemma 4.8, we may write

(15)

$$\begin{aligned} \|I_2\|_p &\le \int_h^\delta \|G(t)\|_p \left|\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right| |K_n(t)|\, dt + \int_h^\delta \frac{\|G(t)\|_p}{t} R(n,t) \sin l(n) dt \\ &+ \int_h^\delta \frac{\|G(t)\|_p}{t} R(n,t) \left(\sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k\right) - \sin l(n)\, t\right) dt \\ &= I + J + K \end{aligned}$$

As $\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right) = O(t)$, using Lemma 4.5 (1) and Lemma 4.9 (2), we obtain

$$\begin{aligned} I &= O\left(|u|^\beta \int_h^\delta t^{\alpha-\beta+1} e^{-At^2 \log n} dt\right) \\ &= O\left(\frac{|u|^\beta}{\log n}\right) \end{aligned} \tag{16}$$

by Lemma 4.7 (replacing μ by $\log n$). Now

(17)

$$J = \int_h^\delta \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)\, t|\, dt = \left(\int_h^{2h} + \int_{2h}^{\delta+h} - \int_\delta^{\delta+h}\right) \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)\, t|\, dt$$

Replacing t by $t+h$ in the second integral of the above line, we obtain,

$$J=\int_{h}^{2h}\frac{\left\|G\left(t\right)\right\|_{p}}{\left|t\right|}\left|R\left(n,t\right)\sin l\left(n\right)t\right|dt$$
$$-\int_{h}^{\delta}\frac{\left\|G\left(t+h\right)\right\|_{p}}{\left|t+h\right|}\left|R\left(n,t+h\right)\sin l\left(n\right)t\right|dt$$
$$-\int_{\delta}^{\delta+h}\frac{\left\|G\left(t\right)\right\|_{p}}{\left|t\right|}\left|R\left(n,t\right)\sin l\left(n\right)t\right|dt$$

This together with (3.23) gives

$$\begin{aligned}2J=&\int_{h}^{\delta}\left[\frac{\left\|G\left(t\right)\right\|_{p}}{\left|t\right|}\left|R\left(n,t\right)\right|-\frac{\left\|G\left(t+h\right)\right\|_{p}}{\left|t+h\right|}\left|R(n,t+h)\right|\right]\left|\sin l\left(n\right)t\right|dt\\&+\int_{h}^{2h}\frac{\left\|G\left(t\right)\right\|_{p}}{\left|t\right|}\left|R\left(n,t\right)\sin l\left(n\right)t\right|dt\\&-\int_{\delta}^{\delta+h}\frac{\left\|G\left(t\right)\right\|_{p}}{\left|t\right|}\left|R\left(n,t\right)\sin l\left(n\right)t\right|dt\\&=L+M-N\end{aligned}\tag{18}$$

By Lemma 4.5 (1) and Lemma 4.9 (1), we get

$$M=O\left(\left|u\right|^{\beta}\int_{h}^{2h}t^{\alpha-\beta-1}dt\right)=O\left(\frac{\left|u\right|^{\beta}}{\left(\log n\right)^{\alpha-\beta}}\right)\tag{19}$$

and

$$\begin{aligned}N&=O\left(\left|u\right|^{\beta}\int_{\delta}^{\delta+h}t^{\alpha-\beta-1}e^{-At^{2}\log n}dt\right)\\&=O\left(\left|u\right|^{\beta}e^{-A\delta^{2}\log n}\right)\\&=O\left(\frac{\left|u\right|^{\beta}}{\left(\log n\right)^{\Delta}}\right),\Delta\text{ positive however large}\end{aligned}\tag{20}$$

We rewrite

$$
\begin{aligned}
L &= \int_h^\delta \frac{\|G(t)\|_p - \|G(t+h)\|_p}{t} R(n,t)\sin l(n)\, tdt \\
&+ \int_h^\delta \left(R(n,t) - R(n,t+h)\right) \frac{\|G(t+h)\|_p}{t} \sin l(n)\, tdt \\
&+ \int_h^\delta \left(\frac{1}{t} - \frac{1}{t+h}\right) \|G(t+h)\|_p R(n,t+h)\sin l(n)\, tdt \\
&= L_1 + L_2 + L_3
\end{aligned}
\tag{21}
$$

By Lemma 4.5(1) and Lemma 4.6 (1), we obtain

$$
L_1 = O\left(|u|^\beta h^{\alpha-\beta} \int_h^\delta \frac{dt}{t}\right) = O\left(|u|^\beta \frac{\log\log n}{(\log n)^{\alpha-\beta}}\right) \tag{22}
$$

By Lemma 4.5 (2) and Lemma 4.11 (2), we have

$$
\begin{aligned}
L_2 &= O\left(|u|^\beta \int_h^\delta (t+h)^{\alpha-\beta} e^{-At^2 \log n} dt\right) \\
&= O\left(|u|^\beta \int_h^\delta t^{\alpha-\beta} e^{-At^2 \log n} dt\right) \\
&= O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right)
\end{aligned}
\tag{23}
$$

using Lemma 3 (μ is replaced by $\log n$).

Using Lemma 4.5 (2), Lemma 4.9 (1), and Lemma 4.7 (replacing μ by $\log n$), we have

$$
\begin{aligned}
L_3 &= h \int_h^\delta \frac{\|G(t+h)\|_p}{t(t+h)} R(n,t+h)\sin l(n)\, tdt \\
&= O\left(h|u|^\beta \int_h^\delta \frac{(t+h)^{\alpha-\beta-1}}{t} e^{-At^2 \log n} dt\right) \\
&= O\left(h|u|^\beta \int_h^\delta t^{\alpha-\beta-2} e^{-At^2 \log n} dt\right) \\
&= O|u|^\beta \begin{cases} \frac{1}{(\log n)^{\alpha-\beta}}, & \alpha-\beta \neq 1 \\ \frac{\log\log n}{\log n}, & \alpha-\beta = 1 \end{cases}
\end{aligned}
\tag{24}
$$

Collecting the results (18) through (24), we obtain,

$$J = O\left(|u|^{\beta}\,\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1 \tag{25}$$

Now, using Lemma 4.5 (1), Lemma 4.9 (1), and Lemma 4.10 (1), we have

$$\begin{aligned} K &= \int_h^{\delta} \frac{\|G(t)\|_p}{t} R(n,t) \left(\sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k\right) - \sin l(n)\, t\right) dt \\ &= O\left(|u|^{\beta} \log n \int_h^{\delta} t^{\alpha-\beta+2} e^{-At^2 \log n} dt\right) \\ &= O\left(\frac{|u|^{\beta}}{(\log n)^{\alpha-\beta}}\right), \end{aligned} \tag{26}$$

by applying Lemma 4.7.

Collecting the results from (12) – (16), (25), and (26), we obtain

$$|L_n(y+u) - L_n(y)| = O\left(|u|^{\beta}\,\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1$$

which ensures that

$$\sup_{u\neq 0} \frac{|L_n(y+u) - L_n(y)|}{|u|^{\beta}} = O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1 \tag{27}$$

Again $f \in H_{\alpha,p} \implies \|\varphi_x(t)\|_p = O(t^{\alpha})$ and so proceeding as above, we obtain

$$\|L_n(\cdot)\|_p = O\left(\frac{\log\log n}{(\log n)^{\alpha}}\right), 0 < \alpha \le 1 \tag{28}$$

combining (27) and (28) we obtain (1) and this completes the proof of Theorem 3.3.

6. PROOF OF THEOREM 3.4

Using (11) , we obtain

$$\begin{aligned}\tilde{K}_n^{\lambda}(f,x) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k\tilde{S}_k(f,x)\\ &= -\frac{2}{\pi}\int_0^{\pi}\psi_x(t)\left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k\tilde{D}_k(t)\right)dt\\ &= -\frac{2}{\pi}\int_0^{h}\psi_x(t)\left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k\tilde{D}_k(t)\right)dt\\ &\quad -\frac{2}{\pi}\int_h^{\pi}\psi_x(t)\left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k\right)\frac{1}{2}\cot\frac{1}{2}tdt\\ &\quad +\frac{2}{\pi}\int_h^{\pi}\frac{\psi_x(t)}{2\sin\frac{1}{2}t}\left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k\cos\left(k+\frac{1}{2}\right)t\right)dt\\ &= -\frac{2}{\pi}\int_0^{h}\psi_x(t)\tilde{E}_n(t)\,dt+\tilde{f}(x,h)+\frac{2}{\pi}\int_h^{\pi}\frac{\psi_x(t)}{2\sin\frac{1}{2}t}\tilde{K}_n(t)\,dt\end{aligned}$$

which further ensures that

$$\begin{aligned}\tilde{L}_n(x) &= \tilde{K}_n^{\lambda}(f,x)-\tilde{f}(x,h)\\ &= -\frac{2}{\pi}\int_0^{h}\psi_x(t)\tilde{E}_n(t)\,dt+\frac{2}{\pi}\int_h^{\pi}\frac{\psi_x(t)}{2\sin\frac{1}{2}t}\tilde{K}_n(t)\,dt\end{aligned} \tag{29}$$

Now for fixed δ with $0<\delta<\frac{\pi}{4}$ and $x\neq y$, we write

$$\begin{aligned}\tilde{L}_n(y+u)-\tilde{L}_n(y) &= \frac{-2}{\pi}\int_0^{h}\tilde{G}(t)\tilde{E}_n(t)\,dt+\frac{2}{\pi}\int_h^{\delta}\frac{\tilde{G}(t)}{2\sin\frac{1}{2}t}\tilde{K}_n(t)\,dt\\ &\quad +\frac{2}{\pi}\int_{\delta}^{\pi}\frac{\tilde{G}(t)}{2\sin\frac{1}{2}t}\tilde{K}_n(t)\\ &= -\tilde{P}+\tilde{Q}+\tilde{R}\end{aligned} \tag{30}$$

By Minkowski's inequality, we have

$$\|L_n(y+u)-L_n(y)\|_p\le\|\tilde{P}\|_p+\|\tilde{Q}\|_p+\|\tilde{R}\|_p.$$

By Lemma 4.12,

$$(31)\qquad \begin{aligned} \|\tilde{P}\|_p &\leq \frac{2}{\pi}\int_0^h \|\tilde{G}\left(t\right)\|_p \left|\tilde{E}_n\left(t\right)\right| dt \\ \|\tilde{Q}\|_p &\leq \frac{2}{\pi}\int_h^\delta \|\tilde{G}\left(t\right)\|_p \left|\frac{\tilde{K}_n(t)}{2\sin\frac{1}{2}t}\right| dt \\ \|\tilde{R}\|_p &\leq \frac{2}{\pi}\int_\delta^\pi \|\tilde{G}\left(t\right)\||_p \left|\frac{\tilde{K}_n\left(t\right)}{2\sin\frac{1}{2}t}\right| dt \end{aligned}$$

By Lemma 4.5 (2) and Lemma 4.9 (4), we have

$$(32)\qquad \|\tilde{P}\|_p = O\left(|u|^\beta \int_0^h t^{\alpha-\beta-1}dt\right) = O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right)$$

Using Lemma 4.5 (2) and Lemma 4.9 (3), we have

$$(33)\qquad \begin{aligned} \|\tilde{R}\|_p &= O\left(|u|^\beta \int_\delta^\pi t^{\alpha-\beta-1}e^{-At^2\log n}dt\right) \\ &= O\left(|u|^\beta e^{-A\delta^2\log n}\right) \\ &= O\left(\frac{|u|^\beta}{(\log n)^\Delta}\right), \Delta \text{ positive however large} \end{aligned}$$

Now, adopting the lines of arguments similar to those used in estimating Q in the proof of Theorem 3.3, we can obtain

$$(34)\qquad \|\tilde{Q}\|_p = O\left(|u|^\beta \frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1.$$

Combining the results of (31) – (34), we obtain

$$(35)\qquad \sup_{u\neq 0}\frac{\|\tilde{L}_n\left(y+u\right)-\tilde{L}_n\left(y\right)\|_p}{|u|^\beta} = O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1.$$

Similarly,

$$(36)\qquad \|\tilde{L}_n\left(\cdot\right)\|_p = O\left(\frac{\log\log n}{(\log n)^{\alpha}}\right), 0 < \alpha \leq 1.$$

(2) follows from (35) and (36) and this completes the proof of Theorem 3.4.

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