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ACCESSIBLE SPECTRUM OF GRAPHS

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This paper computes eigenvalues of discrete complete hypergraphs and partitioned hypergraphs. We define positive equivalence relation on hypergraphs that establishes a connection between hypergraphs and graphs. With this regards it makes a connection between spectrum of graphs and spectrum of quotient of any hypergraphs. Finally, this study tries to construct spectrum of path trees via quotient of partitioned hypergraphs.

1. INTRODUCTION

Graph theory is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operation research, optimization, economics, networking routing, transportation and computer science. Spectral graph theory studies the relation between graph properties and the spectrum of the adjacency matrix or Laplace matrix and the theory of association schemes and coherent configurations studies the algebra generated by associated matrices. Spectral graph theory is a widely studied and highly applicable subject in combinatorics, computer science and the social sciences. Broadly speaking, one first encodes the structure of a graph in a matrix and then pursues connections between graph properties and the eigenvalues [**5**].

The notion of hypergraph has been introduced by Berge as a generalization of graph around 1960 and one of the initial concerns was to extend some classical results of graph theory and the notion of hypergraph has been considered as a useful tool to analyze the structure of a system. Further materials regarding spectrum

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graphs and eigenvalue of hypergraphs are available in the literature too [**1, 2, 4, 8, 11, 12, 13**]. Hypergraphs make it possible to more compactly describe many proofs in graph theory, and may also unify several theorems in ordinary graph theory. Today, some futures of hypergraphs are used in computer science, notably in machine learning, and there has been a lot of research about using hypergraphs in relational databases, which might be viewed as a sort of data mining. There is also research about networks where matroids and hypergraphs are used together in the demonstrations[**6**].

Recently, M. Hamidi et al. enumerated the number of partitioned hypergraphs and discrete complete hypergraphs. They defined a positive equivalence relation on hypergraphs which this relation establishes a connection between hypergraphs and graphs. Moreover, they defined the concept of (extended) derivable graph and considered a connection between of hypergraphs and (extended) derivable graphs. Via the positive equivalence relation on hypergraphs, they showed that some special trees are derivable graph and complete graphs are self derivable graphs [**10**].

Regarding these points, the aim of this paper is to introduce special weak adjacency matrix of partitioned hypergraphs and discrete complete hypergraphs. We compute eigenvalues of partitioned hypergraphs and discrete complete hypergraphs constructed on any arbitrary set and proved some theorems. Indeed, we study the relationship between the topological properties of a hypergraph with the spectral (algebraic) properties of the weak adjacency matrices associated with the hypergraph. Originally, in study of spectral hypergraph theory, we analyzed the weak adjacency matrix of a hypergraph, especially its eigenvalues. A central goal of our research is to deduce the main properties and the structure of a hypergraph from its invariants, whence eigenvalues are strongly connected to almost all key invariants of a hypergraph. It is a natural question what is the relationships between spectrum of hypergraphs and spectrum of graphs. Our original motivation comes the concept of spectrum of graphs, in which we generalize the concept of spectrum of graphs to concept of spectrum of hypergraphs and in this regard the notation of accessible of spectrum of graphs is presented. So we have to compute eigenvalues of partitioned and discrete complete hypergraphs and classify them. Indeed, we consider some graphs which are accessible spectrum graph. We show that path special trees are accessible spectrum trees and compare their eigenvalues with eigenvalues of quotient of corresponding of hypergraphs.

2. PRELIMINARIES

In this section, we recall some definitions and results are indispensable to our research paper.

Definition 1. [**3**] *A graph G is a finite nonempty set V of objects called vertices (the singular is vertex) together with a set E of 2-element subsets of V called edges and is shown with $G = (V, E)$. Two graphs G and H are isomorphic (they have* *the same structure) if there exists a bijective function $\varphi : V(G) \longrightarrow V(H)$ so that two vertices u and v are adjacent in G if and only if $\varphi(u)$ and $\varphi(v)$ are adjacent in H. The function φ is then called an isomorphism. If G and H are isomorphic, we write $G \cong H$.*

Definition 2. [7] *Let $G = (V, E)$ be a graph. An elementary subdivision of G results when an edge $e = \{u, w\}$ is removed from G and then the edges $\{u, v\}, \{v, w\}$ are added to $G \smallsetminus e$, where $v \notin V$. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they are isomorphic or if they can both are obtained from the same graph $H = (V_3, E_3)$ by a sequence of elementary subdivision.*

Definition 3. [2] *Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A hypergraph on G is a pair $H = (G, \{E_i\}_{i=1}^{m})$ such that*

(i) for all $1 \le i \le m, \emptyset \neq E_i \subseteq G$;

(ii) $\bigcup_{i=1}^{m} E_i = G$.

The elements $x_1, x_2, \ldots, x_n$ of G are called vertices, and the sets $E_1, E_2, \ldots, E_m$ are the edges (hyperedges) of the hypergraph. For any $1 \le k \le m$ if $|E_k| \ge 2$, then E_k is represented by a solid line surrounding its vertices, if $|E_k| = 1$ by a cycle on the element (loop). If for all $1 \le k \le m$ $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph.

In any hypergraph, two vertices x and y are said to be adjacent if there exists a hyperedge E_i which contains the two vertices ($x, y \in E_i$). The degree of a vertex is the number of hyperedges which contains the vertex and is showed by $deg(x)$ ($deg(x) = |\{E_i \mid x \in E_i\}|$). In a hypergraph its incidence matrix is a matrix $M_G = (m_{ij})_{n\times m}$, with m columns representing the hyperedges $E_1, E_2, \ldots, E_m$ and n rows representing the vertices $x_1, x_2, \ldots, x_n$, where

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \in E_i, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix $A_n(G) = (a_{ij})_{n\times n}$ of G is defined in this way: it is a square matrix which rows and columns are indexed by the vertices of G, where

$$a_{ij} = \begin{cases} 0 & \text{if } a_i = a_j, \\ |\{E_i \mid a_i, a_j \in E_i\}| & \text{if } a_i \neq a_j. \end{cases}$$

Definition 4. [9] *Let $(G, \{E_x\}_{x\in G})$ be a hypergraph. Then $(G, \{E_x\}_{x\in G})$ is called a complete hypergraph, if for any $x, y \in G$ there exits a hyperedge E such that $\{x, y\} \subseteq E$. Let $(G, \{E_i\}_{i=1}^{n+1})$ be a complete hypergraph.*

(i) $(G, \{E_i\}_{i=1}^{n})$ is called a joint complete hypergraph, if for any $1 \le i \le n$, $|E_i| = i$, $E_i \subseteq E_{i+1}$;

(ii) $(G, \{E_i\}_{i=1}^{n+1})$ *is called a discrete complete hypergraph, if for any* $1 \leq i \neq j \leq n$, $|E_i| = |E_j|$, $E_i \cap E_j = \emptyset$ *and* $|E_{n+1}| = n$;

(iii) $(G, \{E_i\}_{i=1}^{n+1})$ *is called an ordinary complete hypergraph, if* $n = 0$ *and* $E_i = G$.

Definition 5. [**10**] *Let* $(G, \{E_i\}_{i=1}^{n})$ *be a hypergraph. Then define a binary relation* η *on* G *as follows:* $\eta_1 = \{(x, x) \mid x \in G\}$ *and for every integer* $k \geq 2$,

$$x\ \eta_k\ y \Longleftrightarrow \exists\ \ E_i^s,\ \text{such that } \{x, y\} \subseteq E_i^s,\ \text{where } k = |E_i^s| = \min\{|E_t|;\ x, y \in E_t\}$$

and for all $1 \leq i, j \leq n, \nexists\ E_i \neq E_i^s$, *or* $E_j \neq E_i^s$, *such that* $x \in E_i, y \in E_j$ *and* $|E_i| < k, |E_j| < k$. *Obviously* $\eta = \bigcup\limits_{k \geq 1} \eta_k$ *is a reflexive and symmetric relation on* G.

Theorem 6. [**10**] *Let* $(G, \{E_x\}_{x \in G})$ *be a hypergraph and* $\eta = \eta^*$. *Then for any* $i \in \mathbb{N}^*$ *there exists an operation* $*_i$ *on* G/η *such that* $(G/\eta, *_i)$ *is a graph.*

Theorem 7. [**9**] *Let* G *be a set and* $\mathcal{H}g(G)$ *be the set of all hypergraphs are constructed on* G. *If* $|G| = n$, *then* $|\mathcal{H}g(G)| = \sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} 2^{\mathbf{2^{(n-k)}-1}}$.

3. EIGENVALUES OF DISCRETE COMPLETE HYPERGRAPHS

In this section, we introduce the notation of weak adjacency matrix and compute eigenvalues of weak adjacency matrix.

Let $(H, \{E_i\}_{i=1}^{m})$ be a discrete complete hypergraph and $k \in \mathbb{N}$. For any $1 \leq j, j' \leq m$, will denote the set of all discrete complete hypergraph with $|E_j| = |E_{j'}| = k$ on H, by $\mathcal{D}_c^{(k)}(H)$.

Theorem 8. *Let* $|H| = n$, $(H, \{E_i\}_{i=1}^{m})$ *be a hypergraph and* $x \in H$.

(i) If $\mathcal{A}_{i,x} = \{E_j \mid x \in E_j$ *and* $|E_j| = i\}$, *then* $|\mathcal{A}_{i,x}| \leq \binom{n-1}{i-1}$.

(ii) $deg(x) \leq 2^{n-1}$.

(iii) $m \leq 2^n - 1$.

Proof. (i) Let $G = \{x_1, x_2, \ldots, x_n\}$ and $x_i \in H$. Because $|H \setminus \{x_i\}| = n - 1$, if $x \in E_j$ and $|E_j| = i$ then $(i - 1)$ elements from $(n - 1)$ elements can be selected in $\binom{n-1}{i-1}$ ways.

(ii) By (i), $deg(x) = \sum\limits_{i=1}^{n} \binom{n-1}{i-1} = 2^{n-1}$. □

Definition 9. *Let $(G, \{E_x\}_{x\in G})$ be a hypergraph. We define a weak adjacency matrix $A^w_n(G) = (a^w_{ij})_{n\times n}$ of G by square matrix which rows and columns are indexed by the vertices of G, where $a^w_{ij} = |\{E_k \mid a_i, a_j \in E_k\}|$.*

The following example describes the notion of weak adjacency matrix and adjacency matrix.

Example 10. *Let $G = \{a, b, c\}$ be a hypergraph in Figure 1 (K^*_3).*

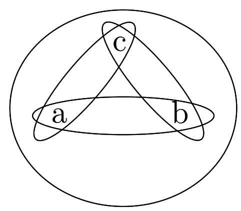


Figure 1: Complete hypergraph(K^*_3).

Then we have the following matrices:

$$A_3(K^*_3) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \text{ and } \quad A^w_3(K^*_3) = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$

where $A_3(K^*_3)$ is adjacency matrix respect to hypergraph 1 (K^*_3) and $A^w_3(K^*_3)$ is adjacency matrix respect to hypergraph 1 (K^*_3).

Proposition 11. *Let $|H| = n$ and $(H, \{E_i\}_{i=1}^m)$ be a joint complete hypergraph.*

(i) $\sum_{j=1}^{n}\sum_{i=1}^{n} a^w_{ij} = \sum_{i=1}^{m} |E_i|$,

(ii) $Tr(A^w_n(G)) = \frac{n^2+n}{2}$,

(iii)

$$\sum_{j=1}^{n}\sum_{i\neq j=1}^{n} a^w_{ij} = \begin{cases} \frac{-2[n/2]}{3}(2[n/2]^2 - (6+3n)[n/2] + (3n+1)) & n \text{ is even}, \\ \frac{-4[n/2]}{3}(2[n/2]^2 - (6+3n)[n/2] + (3n+1)) & n \text{ is odd}, \end{cases}$$

(iv) $Det(A^w_n) = 1$.

Proof. Let $H = \{b_1, b_2, \ldots, b_n\}$. Since $(G, \{E_i\}_{i=1}^m)$ is a joint complete hypergraph, we get $m = n$ and for any $1 \le i \le n$, $|E_i| = i, E_i \subseteq E_{i+1}$. For any $b_i, b_j \in H$, $\{b_i, b_j\} \subseteq E_{\max\{i,j\}}$, then $a^w_{ij} = t_{ij}$ where $t_{ij} = n + 1 - \max\{i, j\}$.

(ii) Since for any $1 \le i \le n, t_{ii} = n+1-i$, we get that

$$Tr(A_n^w) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^w = \sum_{i=1}^{n} t_{ii} = \frac{n^2+n}{2}.$$

(iii) Let n be an even number. Since for any $1 \le i,j \le n, a_{ij}^w = a_{ji}^w$ and $a_{1j} = a_{2j} = a_{3j} = \ldots = a_{j-1j}$, then

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i\neq j=1}^{n} a_{ij}^w = \sum_{j=1}^{n}\sum_{i\neq j=1}^{n} t_{ij} &= \sum_{i=1}^{n-1}(n-i)i = 2\Big(\sum_{i=1}^{[n/2]-1}(n-i)i\Big) + [n/2]^2 \\ &= \frac{-2[n/2]}{3}(2[n/2]^2 - (6+3n)[n/2] + (3n+1)).\end{aligned}$$

In a similar way, for even number n, is proved. □

Proposition 12. *Let* $|H| = n$ *and* $(G, \{E_i\}_{i=1}^{m+1}) \in \mathcal{D}_c^{(k)}(H)$.

(i) $Tr(A_n^w(G)) = 2n$,

(ii) $\sum_{j=1}^{n}\sum_{i\neq j=1}^{n} a_{ij}^w = n(n+k-2)$,

(iii) $\sum_{j=1}^{n}\sum_{i,j=1}^{n} a_{ij}^w = n(n+k)$,

(iv) $Det(A_n^w) = 0$.

Proof. Let $H = \{b_1, b_2, \ldots, b_n\}$. Since $(G, \{E_i\}_{i=1}^{m+1}) \in \mathcal{D}_c^{(k)}(H)$, we get $m = n/k$. Let $E_i = \{a_{(i-1)(n/k)+1}, a_{(i-1)(n/k)+2}, \ldots, a_{(i-1)(n/k)+(n/k)}\}$, where $1 \le i \le n/k$ and $E_{m+1} = H$. Let $1 \le i \le n/k$. Then $a_{rs}^w = \begin{cases} 2 & \text{if}(i-1)(n/k)+1 \le r, s \le i(n/k), \\ 1 & \text{otherwise.}\end{cases}$

(i) $\sum_{i=1}^{n} a_{ii}^w = \sum_{r,r} a_{rr}^w = 2n.$

(ii) $\sum_{j=1}^{n}\sum_{i\neq j=1}^{n} a_{ij}^w = \sum_{r}\sum_{s} a_{rs}^w = 2(n/k(k(k-1))) + (n(n-1) - n/k(k(k-1))) = n(n+k-2).$ □

Theorem 13. *Let* $|H| = n$ *and* $(H, \{E_i\}_{i=1}^{m})$ *be a hypergraph.*

(i) $1 \le a_{ii}^w \le \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$;

(*ii*) *if* $i \neq j$, *then* $1 \leq a^w_{ij} \leq -1 + \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$;

(*iii*) $\sum_{i=1}^{n} a^w_{ii} \leq n \sum_{k=1}^{n} \binom{n-1}{k-1}$;

(*iv*) $\sum_{j=1}^{n} \sum_{j\neq i=1}^{n} a^w_{ij} \leq (n^2-n) \sum_{k=2}^{n} \binom{n-1}{k-1}$;

(*v*) $\sum_{j=1}^{n} \sum_{i=1}^{n} a^w_{ij} \leq n + n^2 \sum_{k=2}^{n} \binom{n-1}{k-1}$.

Proof. Let $H = \{b_1, b_2, \ldots, b_n\}$ and $b_i \in H$. By Theorem 8, for any $1 \leq k \leq n$, $|\mathcal{A}_{i,x}| = \binom{n}{k} - \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!}$ and so $a^w_{ii} \leq \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$.

(*ii*) By (*i*), since $i \neq j$ we get that $a^w_{ij} \leq \sum_{k=2}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$. (*iii*), (*iv*) and (*v*) are obtained by (*i*) and (*ii*), immediately. □

In the following example, we show that the converse of Theorem 13, is not true necessarily.

Example 14. *Consider* $A_3 = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$. *It is satisfied in Theorem 13, but one can not find a hypergraph with corresponding weak adjacency matrix* A_3.

Let $n \in \mathbb{N}, \lambda, a \in \mathbb{R}$ and $W_n(a, -\lambda) = (w_{ij})_{n\times n} = \begin{cases} w_{ij} = w_{11} = a & i \neq j, \\ w_{ii} = -\lambda & i \neq 1. \end{cases}$
Then $Det(W_n(a, -\lambda)) = a(-1)^{n-1}(a+\lambda)^{n-1}$. From now on, let $P_n(\lambda)$ be the characteristic polynomial of matrix $A_n(G)$, $P^w_n(\lambda)$ be the characteristic polynomial of matrix $A^w_n(G)$ and for any matrix $B, Ev(B) = \{\lambda \mid \lambda \text{ is an eigenvalue of B}\}$.

Theorem 15. *Let* $(G, \{E_x\}_{x\in G})$ *be an ordinary complete hypergraph. Then* $Ev(A_n(G)) = \{-1, n-1\}$.

Proof. If $n = 2$, then $P_2(\lambda) = Det(A_2(G) - \lambda I_2) = Det(\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix})$. Thus $P_2(\lambda) = (\lambda+1)(\lambda-1)$. Now, let $P_{n-1}(\lambda) = (-1)^{n-1}(\lambda+1)^{n-2}(\lambda-n+2)$. Then $P_n(\lambda) = Det(A_n(G) - \lambda I_n)$, so for $a = 1$, we have:

$$\begin{aligned} Det(A_n(G) - \lambda I_n) &= (-\lambda)P_{n-1}(\lambda) - (n-1)Det(W_{n-1}(1, -\lambda)) \\ &= (-\lambda)(-1)^{n-1}(\lambda+1)^{n-2}(\lambda-n+2) \\ &- (n-1)(-1)^{n-2}(1+\lambda)^{n-2} \\ &= (-1)^n(\lambda+1)^{n-1}(\lambda-n+1). \end{aligned}$$

Thus $P_n(\lambda)=(-1)^n((\lambda+1)^{n-1}(\lambda-n+1))$ and $Ev(A_n(G))=\{-1,n-1\}$. □

Let $n\in\mathbb{N},\lambda,b\in\mathbb{R}$ and $W_n(b,b{-}\lambda)=(w_{ij})_{n\times n}=\begin{cases} w_{ij}=w_{11}=b-1 & i\neq j,\\ w_{ii}=b-\lambda & i\neq 1.\end{cases}$
Then $Det(W_n(b,b-\lambda))=(-1)^{n-1}(b-1)(\lambda-1)^{n-1}$.

Theorem 16. *Let $(G,\{E_x\}_{x\in G})$ be a hypergraph. Then $(G,\{E_x\}_{x\in G})$ is an ordinary complete hypergraph if and only if $Ev(A_n^w(G))=\{0,n\}$.*

Proof. Let $(G,\{E_x\}_{x\in G})$ be an ordinary complete hypergraph. If $n=2$, then $P_2^w(\lambda)=Det(A_2^w(G)-\lambda I_2)=P_2^w(\lambda)=\lambda^2-2\lambda$. Now, let

$$P_{n-1}^w(\lambda)=(-1)^{n-1}(\lambda^{n-1}-(n-1)\lambda^{n-2}).$$

Then $P_n^w(\lambda)=Det(A_n^w(G)-\lambda I_n)$, so for $b=1$, we have:

$$\begin{aligned} Det(A_n^w(G)-\lambda I_n) &= (1-\lambda)P_{n-1}^w(\lambda)-(n-1)Det(W_{n-1}(1,1-\lambda))\\ &= (1-\lambda)(-1)^{n-1}(\lambda^{n-1}-(n-1)\lambda^{n-2})-(n-1)(-1)^{n-2}\lambda^{n-2}\\ &= (-1)^n(\lambda^n-n\lambda^{n-1}).\end{aligned}$$

Thus $P_n^w(\lambda)=(-1)^n(\lambda^n-n\lambda^{n-1})$ and so $Ev(A_n^w(G))=\{0,n\}$.

Conversely, let $A_n^w=(a_{ij})_{n\times n}$ and $Ev(A_n^w(G))=\{0,n\}$. It is clear to see that

$$\begin{aligned} P_n^w(\lambda) &= (-1)^n\lambda^n+\big((-1)^{n-1}tr(A_n^w)\big)\lambda^{n-1}\\ &+ \big((-1)^n\sum_{j=1}^{n}\sum_{j\neq i=1}^{n}a_{ii}^w a_{jj}^w+(-1)^{n-1}\sum_{j=1}^{n}\sum_{j\neq i=1}^{n}(a_{ij}^w)^2)\big)\lambda^{n-2}+g(\lambda),\end{aligned}$$

where $deg(g(\lambda))\leq n-3$. Since $Ev(A_n^w(G))=\{0,n\}$ and $Det(A_n^w(G)-\lambda I_n)=(-1)^n(\lambda^n-n\lambda^{n-1})$, we get that $tr(A_n^w)=n$, $\sum_{j\neq i=1}^{n}a_{ii}^w a_{jj}^w=\sum_{j=1}^{n}\sum_{j\neq i=1}^{n}(a_{ij}^w)^2$ and $g(\lambda)=0$. By Theorem 13, $tr(A_n^w)=n$ implies that $a_{11}=a_{22}=\ldots=a_{nn}=1$ and thus $\sum_{j\neq i=1}^{n}a_{ii}^w a_{jj}^w=\sum_{j=1}^{n}\sum_{j\neq i=1}^{n}(a_{ij}^w)^2$ implies that $\sum_{j\neq i=1}^{n}(a_{ij}^w)^2=\frac{n(n-1)}{2}$ and so for any $1\leq i,j\leq n, a_{ij}^w=1$. Hence $(G,\{E_x\}_{x\in G})$ is an ordinary complete hypergraph. □

Theorem 17. *Let $(G,\{E_x\}_{x\in G})\in\mathcal{D}_c^{(1)}(H)$. Then $Ev(A_n(G))=\{-1,n-1\}$.*

Proof. If $n=2$, then $P_2(\lambda)=Det(A_2(G)-\lambda I_2)=(\lambda+1)(\lambda-1)$. Now, let $P_{n-1}(\lambda)=(-1)^{n-1}(\lambda+1)^{n-2}(\lambda-n+2)$, then $P_n(\lambda)=Det(A_n(G)-\lambda I_n)$, so for $a=1$, we have:

$$\begin{aligned} Det(A_n(G){-}\lambda I_n) &= (-\lambda)P_{n-1}(\lambda){-}(n{-}1)Det(W_{n-1}(1,{-}\lambda))\\ &= (-\lambda)(-1)^{n-1}(\lambda{+}1)^{n-2}(\lambda{-}n{+}2)(-1)^{n-2}(n{-}1)(1{+}\lambda)^{n-2}\\ &= (-1)^n(\lambda{+}1)^{n-1}(\lambda{-}n{+}1).\end{aligned}$$

Thus $P_n(\lambda) = (–1)^n(\lambda+1)^{n-1}(\lambda - n + 1)$ and so $Ev(A_n(G)) = \{-1, n-1\}$. □

Remark 18. *By Theorems 15 and 17, the set of eigenvalues of adjacency matrix of an ordinary complete hypergraph and discrete complete hypergraph with hyperedges of order 1 are equal, but ordinary complete hypergraphs and discrete complete hypergraphs are not isomorphic necessarily. It follows that the converse of Theorems 15 and 17 is not true necessarily.*

Theorem 19. *Let $(G, \{E_x\}_{x\in G})$ be a hypergraph. If $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(1)}(H)$, then $Ev(A_n^w(G)) = \{1, n+1\}$.*

Proof. If $n = 2$, then $P_2^w(\lambda) = Det(A_2^w(G) - \lambda I_2) = (1-\lambda)(3-\lambda)$. Now, let $P_{n-1}^w(\lambda) = (-1)^{n-1}(\lambda-1)^{n-2}(\lambda - n)$. Then $P_n^w(\lambda) = Det(A_n^w(G) - \lambda I_n)$, so for $b = 2$, we have:

$$\begin{aligned} Det(A_n^w(G)-\lambda I_n) &= (2-\lambda)P_{n-1}^w(\lambda)-(n-1)Det(W_{n-1}(2, 2-\lambda)) \\ &= (2-\lambda)(-1)^{n-1}(\lambda-1)^{n-2}(\lambda-n)-(-1)^{n-2}(n-1)(\lambda-1)^{n-2} \\ &= (-1)^n(\lambda-1)^{n-1}(\lambda-n-1). \end{aligned}$$

Thus $P_n^w(\lambda) = (-1)^n(\lambda-1)^{n-1}(\lambda - n - 1)$ and $Ev(A_n^w(G)) = \{1, n+1\}$. □

Example 20. *Consider the hypergraph $H = (G, \{E_i\}_{i=1}^3)$ in Figure 2. One can see that $Ev(A_n^w(G)) = \{1, 4\}$, while $H = (G, \{E_i\}_{i=1}^4) \notin \mathcal{D}_c^{(2)}(H)$. It follows that the converse of Theorem 19, is not necessarily true.*

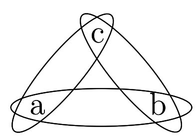


Figure 2: Hypergraph $H = (G, \{E_i\}_{i=1}^3)$.

Lemma 21. *Let* $T_n(r, s, -\lambda) = (t_{ij})_{n\times n} = \begin{cases} t_{ii} = -\lambda, & \\ t_{2i2i-1} = t_{2i-12i} = r & 1 \le 2i \le n, \text{ where} \\ s & \text{otherwise,} \end{cases}$
$\lambda, r, s \in \mathbb{R}$ and $4 \le n \in \mathbb{N}$. If $T_n(r, s, -\lambda) \sim U$, then

$$u_{kk} = \begin{cases} -(\lambda^2 - \lambda(r + (k-3)s) - (k-1)s^2)/(\lambda - (r + (k-3)s)) & k \text{ is odd,} \\ -a/(\lambda^2 - \lambda(r + (k-4)s) - (k-2)s^2) & k \text{ is even.} \end{cases}$$

where $a = \lambda^3 - \lambda^2(r+(k-4)s) - \lambda(2(k-2)s^2 + r^2) + r^3 + (k-4)r^2s - 2(k-2)s^2r$.

In the following example, we describe Lemma 21.

Example 22. *For $n = 4$ and $W_4(r,s,-\lambda) = \begin{pmatrix} -\lambda & r & s & s \\ r & -\lambda & s & s \\ s & s & -\lambda & r \\ s & s & r & -\lambda \end{pmatrix}$, we have $u_{11} = -\lambda$, $u_{22} = -(\lambda^2 - r^2)/\lambda$, $u_{33} = -(\lambda^2 - r\lambda - 2s^2)/(\lambda - r)$ and $u_{44} = -(\lambda^3 - \lambda^2 r - 4s^2\lambda - r^2\lambda + r^3 - 4s^2 r)/(\lambda^2 - r\lambda - 2s^2)$.*

Theorem 23. *Let $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(2)}(H)$. Then $Ev(A_n(G)) = \{-2, 0, n\}$, where $n \geq 4$.*

Proof. Since for any $x \in G, |E_x| = 2$, we get that $n = 2r$, whence $r \geq 2$, so for $r = 1$ and $s = 2$ by Lemma 21, we get that

$$u_{kk} = \begin{cases} -(\lambda^2 - \lambda(k-1) - (k-1))/(\lambda - (k-1)) & k \text{ is odd,} \\ -\lambda(\lambda^2 - \lambda(k-2) - 2k)/(\lambda^2 - \lambda(k-2) - (k-2)) & k \text{ is even.} \end{cases}$$

Thus

$$\begin{aligned} P_n(\lambda) &= \prod_{k=1}^{n} u_{kk} = (u_{11}u_{22})(u_{33}u_{44})\dots(u_{(n-1)(n-1)}u_{nn}) \\ &= \left(\left(\frac{\lambda^2}{\lambda}\right)\left(\frac{\lambda(\lambda^2-4)}{\lambda^2}\right)\right)\left(\left(\frac{\lambda^2-2\lambda-2}{\lambda-2}\right)\left(\frac{\lambda(\lambda^2-2\lambda-4)}{\lambda^2-2\lambda-2}\right)\right)\dots \\ \dots & \left(\left(\frac{(\lambda^2-(\lambda+1)(n-1))}{\lambda-(n-1)}\right)\left(\frac{\lambda(\lambda^2-\lambda(n-2)-2n)}{\lambda^2-(\lambda+1)(n-2)}\right)\right). \end{aligned}$$

If $n = 4$, $P_4(\lambda) = (\lambda-2)(\lambda+2)\left(\frac{\lambda(\lambda+2)(\lambda-4)}{\lambda-2}\right) = \lambda(\lambda+2)^2(\lambda-4)$. Now, let $P_{n-2}(\lambda) = \lambda^{[(n-2)/2]-1}(\lambda+2)^{[(n-2)/2]}(\lambda - n + 2)$. Then

$$\begin{aligned} P_n(\lambda) &= \left(\prod_{k=1}^{n-2} u_{kk}\right)\left(u_{(n-1)(n-1)}\right)\left(u_{nn}\right) \\ &= \lambda^{\frac{n}{2}-2}(\lambda+2)^{\frac{n}{2}-1}(\lambda-n+2)\left(\frac{(\lambda^2-(\lambda+1)(n-2))}{\lambda-(n-2)}\right)\left(\frac{\lambda(\lambda+2)(\lambda-n)}{\lambda^2-(\lambda+1)(n-2)}\right) \\ &= \lambda^{[n/2]-1}(\lambda+2)^{[n/2]}(\lambda-n). \end{aligned}$$

It follows that $P_n(\lambda) = \lambda^{[n/2]-1}(\lambda+2)^{[n/2]}(\lambda - n)$ and so $Ev(A_n(G)) = \{0, -2, n\}$. □

Example 24. *Consider the hypergraph $H = (G, \{E_i\}_{i=1}^4)$ in Figure 3. One can see that $Ev(A_n(G)) = \{0, -2, 4\}$, while $H = (G, \{E_i\}_{i=1}^4) \notin \mathcal{D}_c^{(2)}(H)$. It follows that the converse of Theorem 23, is not necessarily true.*

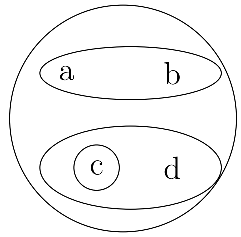


Figure 3: Hypergraph $H = (G, \{E_i\}_{i=1}^{4})$.

Lemma 25. *Let* $T_n(r, s, 2-\lambda) = (t_{ij})_{n\times n} = \begin{cases} t_{ii} = 2-\lambda, & \\ t_{2i2i-1} = t_{2i-12i} = r & 1 \le 2i \le n, \\ s & \text{otherwise}, \end{cases}$
where $\lambda, r, s \in \mathbb{R}$ *and* $4 \le n \in \mathbb{N}$. *If* $T_n(r, s, 2-\lambda) \sim U$, *then*

$$u_{kk} = \begin{cases} -b/((\lambda-2) - (r + (k-3)s)) & k \text{ is odd}, \\ -a/((\lambda-2)^2 - (\lambda-2)(r+(k-4)s) - (k-2)s^2) & k \text{ is even}. \end{cases}$$

where $a = (\lambda-2)^3 - (\lambda-2)^2(r+(k-4)s) - (\lambda-2)(2(k-2)s^2 + r^2) + r^3 + (k-4)r^2s - 2(k-2)s^2r$ *and* $b = (\lambda-2)^2 - (\lambda-2)(r+(k-3)s) - (k-1)s^2$.

In the following example, we describe Lemma 25.

Example 26. *For* $n = 4$ *and* $W_4(r, s, 2-\lambda) = \begin{pmatrix} 2-\lambda & r & s & s \\ r & 2-\lambda & s & s \\ s & s & 2-\lambda & r \\ s & s & r & 2-\lambda \end{pmatrix}$,
we have $u_{11} = -(\lambda-2)$, $u_{22} = -((\lambda-2)^2 - r^2)/(\lambda-2)$,

$$u_{33} = -((\lambda-2)^2 - r(\lambda-2) - 2s^2)/((\lambda-2) - r) \text{ and}$$

$$u_{44} = -((\lambda-2)^3 - (\lambda-2)^2 r - 4s^2(\lambda-2) - r^2(\lambda-2) + r^3 - 4s^2 r)/((\lambda-2)^2 - r(\lambda-2) - 2s^2).$$

Theorem 27. *Let* $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(2)}(H)$. *Then* $Ev(A_n^w(G)) = \{0, 2, n+2\}$.

Proof. Since for any $x \in G, |E_x| = 2$, we get that $n = 2r$, whence $r \ge 2$, so for $r = 1$ and $s = 2$ by Lemma 25, we get that

$$u_{kk} = \begin{cases} -((\lambda-2)^2 - (\lambda-2)(k-1) - (k-1))/((\lambda-2) - (k-1)) & k \text{ is odd}, \\ -(\lambda-2)((\lambda-2)(\lambda-k) - 2k)/((\lambda-2)(\lambda-k) - (k-2)) & k \text{ is even}. \end{cases}$$

In the similar way of proof of Theorem 23, it follows that

$$P_n^w(\lambda) = \lambda^{[n/2]}(\lambda-2)^{[n/2]-1}(\lambda-n-2)$$

and $Ev(A_n^w(G)) = \{0, 2, n+2\}$. □

Example 28. *Consider the hypergraph* $H = (G, \{E_i\}_{i=1}^4)$ *in Figure 4. One can see that* $Ev(A_n^w(G)) = \{0, 2, 8\}$, *while* $H = (G, \{E_i\}_{i=1}^4) \notin \mathcal{D}_c^{(2)}(H)$. *It follows that the converse of Theorem 27, is not necessarily true.*

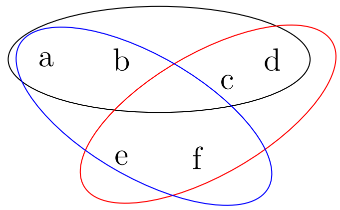


Figure 4: Hypergraph $H = (G, \{E_i\}_{i=1}^4)$.

Corollary 29. *Let* $k \geq 2$ *and* $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(k)}(H)$. *Then*

(*i*) $P_n(\lambda) = (\lambda+2)^{[(k-1)n/k]}(\lambda-k+2)^{[n/k]-1}(\lambda-n-k+2)$ *and so* $Ev(A_n(G)) = \{-2, k-2, n+k-2\}$.

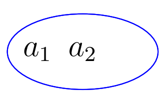
(*ii*) $P_n^w(\lambda) = \lambda^{[(k-1)n/k]}(\lambda-k)^{[n/k]-1}(\lambda-n-k)$ *and so* $Ev(A_n^w(G)) = \{0, k, n+k\}$.

Corollary 30. *Let* $k \geq 2$ *and* $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(k)}(H)$. *Then*

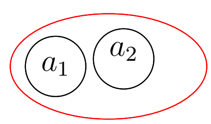
$$Spec(A_n^w(G)) - Spec(A_n(G)) = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 31. *Examples 20 and 24, disprove the converse of above corollaries.*

Example 32. *Consider the hypergraphs in Figure 5. Obviously,* $Ev(A_n(G)) =$



(a) G_1



(b) G_2

Figure 5: Hypergraphs G_1 and G_2 of order 2.

$Ev(A_n^w(G)) = \{-1, 1\}$, *while* $G_1 \ncong G_2$.

4. EIGENVALUES OF PARTITIONED HYPERGRAPHS

In this section, we compute eigenvalues of weak adjacency matrix and eigenvalues of adjacency matrix of all types of partitioned hypergraphs.

Definition 33. *Let $(H, \{E_i\}_{i=1}^n)$ be a hypergraph, $1 \leq i, j \leq n$ and $k \in \mathbb{N}$. Then H is called a partitioned hypergraph, if $\mathcal{P} = \{E_1, E_2, \ldots, E_n\}$ is a partition set of H. We will denote the set of partitioned hypergraphs with $|\mathcal{P}| = k$ on H that $|E_i| = |E_j|$, by $\mathcal{P}_h^{(k)}(H)$ and the set of all partitioned hypergraphs on H, by $\mathcal{P}_h(H)$.*

Theorem 34. *Let $|H| = n$, $k \mid n$ and $(H, \{E_i\}_{i=1}^k) \in \mathcal{P}_h^{(k)}(H)$. Then*

(i) $|\mathcal{P}_h^{(k)}(H)| = \dfrac{1}{k!} \displaystyle\prod_{r=0}^{n(k-1)/k} \binom{n-r}{n/k} = \dfrac{n!}{k!((n/k)!)^k}$;

(ii) $|\mathcal{P}_h(H)| = \displaystyle\sum_{k|n} \dfrac{1}{k!} \prod_{r=0}^{n(k-1)/k} \binom{n-r}{n/k} = \sum_{k|n} \dfrac{n!}{k!((n/k)!)^k}$.

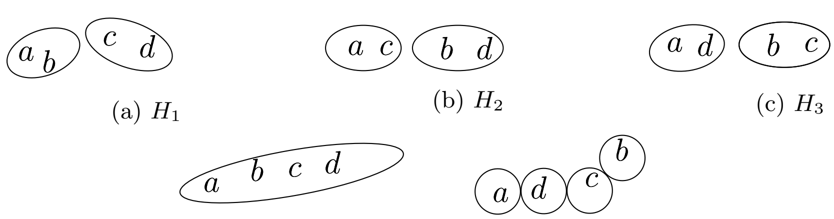
Proof. (i) Let $H = \{a_1, a_2, \ldots, a_n\}$. Because $H = E_1 \cup E_2 \cup \ldots E_k$ and $(H, \{E_i\}_{i=1}^k)$ is a discrete complete hypergraph, so $|E_1| + |E_2| + \ldots + |E_k| = n$. Thus for any $1 \leq j \leq k, |E_j| = n/k$(because of $k \mid n$). n/k elements from n elements can be selected for E_1, n/k elements from $n - n/k$ elements can be selected for E_2 and so on n/k elements from $n - (j-1)n/k$ elements can be selected for E_j, where $1 \leq j \leq k$. Moreover for $E_1, E_2, \ldots, E_k$ there exist $k!$ permutations, hence $|\mathcal{P}_h^{(k)}(H)| = \dfrac{1}{k!} \displaystyle\prod_{r=0}^{n(k-1)/k} \binom{n-r}{n/k}$.

(ii) By (i), for any $1 \leq j \leq k, |E_j| \in D(n)$, so

$$|\mathcal{P}_h(H)| = \sum_{k|n} \frac{1}{k!} \prod_{r=0}^{n(k-1)/k} \binom{n-r}{n/k} = \sum_{k|n} \frac{n!}{k!((n/k)!)^k}.$$

□

Example 35. *Let $H = \{a, b, c, d\}$, $k = 2, k = 1$ and $k = 4$. Then $\mathcal{P}_h^{(2)}(H) = \{H_1, H_2, H_3\}$, $\mathcal{P}_h^{(1)}(H) = \{H_4\}$ and $\mathcal{P}_h^{(4)}(H) = \{H_5\}$ in Figure 6. It is easy to*



(a) H_1 (b) H_2 (c) H_3 (d) H_4 (e) H_5

Figure 6: Partitioned hypergraphs with $k = 2$, $k = 1$ and $k = 4$.

check that $|\mathcal{P}_h^{(2)}(H)| = 4!/(2!)(4) = 3$ *and* $|\mathcal{P}_h^{(4)}(H)| = 4!/(2!)(4) = 1$

Proposition 36. *Let $|H| = n$ and $(G, \{E_i\}_{i=1}^m) \in \mathcal{P}_h^{(k)}(H)$.*

(i) $Tr(A_n^w(G)) = n$,

(ii) $\sum_{i \neq j=1}^{n} a_{ij}^w = n(k-1)$,

(iii) $\sum_{i,j=1}^{n} a_{ij}^w = nk$,

(iv) $Det(A_n^w) = 0$.

Proof. Let $H = \{b_1, b_2, \ldots, b_n\}$. Since $(G, \{E_i\}_{i=1}^m) \in \mathcal{P}_h^{(k)}(H)$, we get $m = n/k$. Let $E_i = \{a_{(i-1)(n/k)+1}, a_{(i-1)(n/k)+2}, \ldots, a_{(i-1)(n/k)+(n/k)}\}$, where $1 \leq i \leq n/k$. Suppose that $1 \leq i \leq n/k$. Then $a_{rs}^w = \begin{cases} 1 & \text{if } (i-1)(n/k)+1 \leq r, s \leq i(n/k), \\ 0 & \text{otherwise.} \end{cases}$

(i) $\sum_{i=1}^{n} a_{ii}^w = \sum_{r,r} a_{rr}^w = n$.

(ii) $\sum_{i \neq j=1}^{n} a_{ij}^w = \sum_{r,s} a_{rs}^w = 2(n/k(k(k-1))) + 0(n(n-1) - n/k(k(k-1))) = n(k-1)$. □

Theorem 37. *Let $k \mid |H|$ and $G \in \mathcal{P}_h^{(|H|/k)}(H)$. Then there exists $C_{k\times k}$ such that*

(i) $A_n(G) = [C, C, \ldots, C]$ *is a diagonal matrix,*

(ii) $Ev(C) = \{-1, k-1\}$,

(iii) $Ev(A_n(G)) = Ev(C)$.

Proof. (i) Let $G = \{a_1, a_2, \ldots, a_n\}$. Since $k \mid |G|$, there is $m \in \mathbb{N}$ such that $G = (H, \{E_j\}_{j=1}^m)$ and for any $1 \leq j \leq m$, $|E_j| = k$. Suppose for any $1 \leq j \leq m$, $E_j = \{a_{(j-1)k+1}, a_{(j-1)k+2}, \ldots, a_{jk}\}$, then by definition we obtain the matrix $C_{k\times k} = (c_{ij})$, where for any $1 \leq i \neq j \leq k, c_{ij} = 1$ and $c_{ii} = 0$. Since for any $1 \leq j \neq j' \leq m, E_j \cap E_{j'} = \emptyset$, we get that $\{a_{(j-1)k+p}, a_{(j'-1)k+p'}\} \nsubseteq E_j \cup E_{j'}$, where $1 \leq p, p' \leq k$. Hence for any $a_{rs} \neq b_{ij}$ we have $a_{rs} = 0$ and so we obtain $A_n(G)$ as above.

(ii) A simple computation shows that $P_k(\lambda) = (\lambda+1)^{k-1}(k-1-\lambda)$ and so $Spec(C) = \begin{pmatrix} -1 & k-1 \\ k-1 & 1 \end{pmatrix}$.

(iii) By (i), $P_k^w(\lambda) = ((\lambda+1)^{k-1}(k-1-\lambda))^m = (\lambda+1)^{mk-m}(k-1-\lambda)^m$ and so $Spec(A_n(G)) = \begin{pmatrix} -1 & k-1 \\ mk-m & m \end{pmatrix}$. □

Theorem 38. *Let H be a non–empty set and $(G, \{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. Then there exist $C^1_{k_1\times k_1}, C^2_{k_2\times k_2}, \ldots C^{m-1}_{k_{m-1}\times k_{m-1}}$ and $C^m_{k_m\times k_m}$ such that*

(*i*) *for any $1 \leq j \leq m, k_j = |E_j|$ and $A_n(G) = [C^1, C^2, \ldots, C^{m-1}, C^m]$,*

(*ii*) *for any $1 \leq j \leq m$, have $Ev(C^j_{k_j\times k_j}) = \{-1, j-1\}$,*

(*iii*) $Ev(A_n(G)) = \bigcup_{j=1}^{m} Ev(C^j_{k_j\times k_j})$.

Proof. (*i*) Let $G = \{a_1, a_2, \ldots, a_n\}$, then there exist $m_1, m_2, \ldots, m_t$ such that $n = \sum_{j=1}^{t} m_j$ and we rearrange $E_1 = \{a_1, a_2, \ldots, a_{m_j}\}$ and $E_j = \{a_{m_{(j-1)}+1}, a_{m_{(j-1)}+2}, \ldots, a_{m_j}\}$. So for any $1 \leq r \leq m$, by definition we obtain the matrix $C^r_{m_r\times m_r} = (c^r_{ij})$, where for any $1 \leq i \neq j \leq m_r, c^r_{ij} = 1$ and $c^r_{jj} = 0$. Since for any $1 \leq j \neq j' \leq m, E_j \cap E_{j'} = \emptyset$, we get that $\{a_{m_{j-1}+p}, a_{m_{j'-1}+p'}\} \nsubseteq E_j \cup E_{j'}$, where $m_{j-1} \leq p \leq m_j$ and $m_{j'-1} \leq p' \leq m_{j'}$. Hence for any $a_{st} \neq c^r_{ij}$ we have $a_{st} = 0$ and so $A_n(G)$ is obtained.

(*ii*) A simple computation shows that for any $1 \leq j \leq m$,

$$P_{m_j}(\lambda) = (-\lambda - 1)^{m_j - 1}(\lambda - m_j + 1)$$

and so

$$Spec(C^j_{m_j\times m_j}) = \begin{pmatrix} -1 & m_j - 1 \\ m_j - 1 & 1 \end{pmatrix}.$$

(*iii*) By (*i*),

$$\begin{aligned} P_n(\lambda) &= \prod_{j=1}^{m} P_{m_j}(\lambda) = ((-\lambda - 1)^{m_j - 1}(\lambda - m_j + 1))^m \\ &= (-\lambda - 1)^{mm_j - m}(\lambda - m_j + 1)^m \text{ and so } Ev(A_n(G)) = \bigcup_{j=1}^{m} Ev(C^j_{m_j\times m_j}). \end{aligned}$$

□

In what follows, we explain Theorem 38.

Example 39. *Let $H = \{1, 2, 3, \ldots, 6\}$ and $G \in \mathcal{P}^{(3)}_h(H)$. Consider $m_1 = 2, m_2 = 2, m_3 = 2 (6 = 2 + 2 + 2)$ and so $G = (H, E_1 = \{1, 2\}, E_2 = \{3, 4\}, E_3 = \{5, 6\})$. It follows that an adjacency matrix* $A_6(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} =$

$\begin{pmatrix} C_{2\times 2} & B_{2\times 2} & B_{2\times 2} \\ B_{2\times 2} & C_{2\times 2} & B_{2\times 2} \\ B_{2\times 2} & B_{2\times 2} & C_{2\times 2} \end{pmatrix}$, *where* $B_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $C_{2\times 2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *It is easy to check that* $P_6(\lambda) = Det(\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix})^3 = (\lambda - 1)^3(\lambda + 1)^3$.

Corollary 40. *Let H be a non–empty set and $(G, \{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. Then* $Spec(A_n(G)) = \begin{pmatrix} |E_1| - 1 & |E_2| - 1 & \dots & |E_{n-1}| - 1 & |E_n| - 1 & -1 \\ 1 & 1 & \dots & 1 & 1 & n - m \end{pmatrix}$.

Theorem 41. *Let H be a set and $|H| = p$ be a prime. If $G \in \mathcal{P}_h^{(k)}(H)$, then $A_p^w(G) = (a_{ij})$, where for any $1 \leq j \neq j \leq p, a_{jj'} = 0$, $a_{jj} = 1$ and* $Spec(A_p^w(G)) = \begin{pmatrix} 1 \\ p \end{pmatrix}$.

Proof. Let $H = \{a_1, a_2, \dots, a_p\}$. Since p is prime, we get for any $1 \leq j \leq p, E_j = \{a_j\}$, then by definition we obtain the matrix $A_p^w(G) = I_p$. □

Example 42. *Let $H = \{1, 2, 3, 4, 5\}$ and $G \in \mathcal{P}_h^{(5)}(H)$ be in Figure 7.*

Figure 7: Partitioned hypergraph G.

Then it is easy to see that $G = (H, \{E_j\}_{j=1}^5)$ is a partitioned hypergraph, where for any $1 \leq j \leq 5$, $E_j = \{a_j\}$. Hence $A_5(G) = [1, 1, 1, \dots, 1, 1]$. It is easy to check that $P_5(\lambda) = (\lambda - 1)^5$.

Theorem 43. *Let $k \mid |H|$ and $G \in \mathcal{P}_h^{(|H|/k)}(H)$. Then there exists $B_{k\times k}$ such that*

(i) $A_n^w(G) = [B, B, \dots, B]$ *is a diagonal matrix,*

(ii) $Ev(B) = \{0, k\}$,

(iii) $Ev(A_n^w(G)) = Ev(B)$.

Proof. *(i)* Let $G = \{a_1, a_2, \dots, a_n\}$. Since $k \,\big|\, |G|$, there is $m \in \mathbb{N}$ such that $G = (G, \{E_j\}_{j=1}^m)$ and for any $1 \leq j \leq m$, $|E_j| = k$. Suppose that for any $1 \leq j \leq m$, $E_j = \{a_{(j-1)k+1}, a_{(j-1)k+2}, \dots, a_{jk}\}$, then by definition we obtain the matrix $B_{k\times k} = (b_{ij})$, where for any $1 \leq i, j \leq k$, $b_{ij} = 1$. Since for any $1 \leq j \neq j' \leq m$, $E_j \cap E_{j'} = \emptyset$, we get that $\{a_{(j-1)k+p}, a_{(j'-1)k+p'}\} \nsubseteq E_j \cup E_{j'}$, where $1 \leq p, p' \leq k$. Hence for any $a_{rs} \neq b_{ij}$ we have $a_{rs} = 0$ and so we obtain $A_n^w(G)$ as above.

(ii) A simple computation shows that $P_k^w(\lambda) = (-\lambda)^{k-1}(k - \lambda)$ and so $Spec(B) = \begin{pmatrix} 0 & k \\ k-1 & 1 \end{pmatrix}$.

(iii) By *(i)*, $P_n^w(\lambda) = ((-\lambda)^{k-1}(k - \lambda))^m = (-\lambda)^{mk-m}(k - \lambda)^m$ and so $Spec(A_n^w(G)) = \begin{pmatrix} 0 & k \\ mk - m & m \end{pmatrix}$. □

Example 44. *Let $H = \{1,2,3,\ldots,9\}$ and $G \in \mathcal{P}_h^{(3)}(H)$. Hence $G = (H, E_1 = \{1,2,3\}, E_2 = \{4,5,6\}, E_3 = \{7,8,9\})$ and so* $A_9(G) = \begin{pmatrix} C_{3\times 3} & A_{3\times 3} & A_{3\times 3} \\ A_{3\times 3} & C_{3\times 3} & A_{3\times 3} \\ A_{3\times 3} & A_{3\times 3} & C_{3\times 3} \end{pmatrix}$, *where* $C_{3\times 3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, *and* $A_{4\times 4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *It is easy to check that* $P_9^w(\lambda) = Det(\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix})^3 = (-\lambda)^6(3-\lambda)^3$.

Theorem 45. *Let H be a non–empty set and $(G, \{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. If for any $1 \leq j \leq n$, $|E_j| \mid |E_{j+1}|$, then there exist $B^1_{k_1\times k_1}, B^2_{k_2\times k_2}, \ldots B^{m-1}_{k_{m-1}\times k_{m-1}}$ and $B^m_{k_m\times k_m}$ such that*

(i) for any $1 \leq j \leq m, k_j = |E_j|$ and $A_n^w(G) = [B^1, B^2, \ldots, B^{m-1}, B^m]$,

(ii) for any $1 \leq j \leq m$, have $Ev(B^j_{k_j\times k_j}) = \{0, j\}$,

(iii) $Ev(A_n^w(G)) = \bigcup_{j=1}^{m} Ev(B^j_{k_j\times k_j})$.

Proof. (i) Let $G = \{a_1, a_2, \ldots, a_n\}$. Since for any $1 \leq j \leq n, |E_j| \mid |E_{j+1}|$, then there exist $1 \leq j \leq m$ such that $|E_j| = k_j$ and $k_j \mid k_{j+1}$. Hence for any $1 < j \leq m$ we can rearrange $E_1 = \{a_1, a_2, \ldots, a_{k_j}\}$ and $E_j = \{a_{k_{j-1}+1}, a_{k_{j-1}+2}, \ldots, a_{k_j}\}$. So for any $1 \leq r \leq m$, by definition we obtain the matrix $B^r_{k_r\times k_r} = (b^r_{ij})$, where for any $1 \leq i, j \leq k_r, b^r_{ij} = 1$. Since for any $1 \leq j \neq j' \leq m, E_j \cap E_{j'} = \emptyset$, we get that $\{a_{k_{j-1}+p}, a_{k_{j'-1}+p'}\} \nsubseteq E_j \cup E_{j'}$, where $k_{j-1} \leq p \leq k_j$ and $k_{j'-1} \leq p' \leq k_{j'}$. Hence for any $a_{st} \neq b^r_{ij}$ we have $a_{st} = 0$ and so $A_n^w(G)$ is obtained.

(ii) A simple computation shows that for any $1 \leq j \leq m$,

$$P_{k_j}(\lambda) = (-\lambda)^{k_j-1}(k_j - \lambda)$$

and so $Spec(B^j_{k_j\times k_j}) = \begin{pmatrix} 0 & k_j \\ k_j - 1 & 1 \end{pmatrix}$.

(iii) By (i),

$$P_n^w(\lambda) = \prod_{j=1}^{m} P_{k_j}^w(\lambda) = ((-\lambda)^{k_j-1}(k_j-\lambda))^m = (-\lambda)^{mk_j-m}(k_j-\lambda)^m$$

and so $Ev(A_n^w(G)) = \bigcup_{j=1}^{m} Ev(B^j_{k_j\times k_j})$. □

Example 46. *Let $H = \{a_1, a_2, a_3, \ldots, a_{14}\}$, $G \in \mathcal{P}_h(H)$ which in $G = (H, E_1 = \{a_1, a_2\}, E_2 = \{a_3, a_4, a_5, a_6\}$ and $E_3 = \{a_7, a_8, a_9, \ldots, a_{14}\})$. Clearly $A_{14}^w(G) = [B^2, B^4, B^8]$ and so* $Spec(A_n^w(G)) = \begin{pmatrix} 0 & 2 & 4 & 8 \\ 11 & 1 & 1 & 1 \end{pmatrix}$.

Corollary 47. *Let H be a non–empty set and $(G,\{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. Then* $Spec(A_n^w(G)) = \begin{pmatrix} |E_1| & |E_2| & \dots & |E_{n-1}| & |E_n| & 0 \\ 1 & 1 & \dots & 1 & 1 & n-m \end{pmatrix}$.

Corollary 48. *Let H be a non–empty set and $(G,\{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. If for any $1 \leq j \leq n, |E_j| \leq |E_{j+1}|$, then there exist $B^1_{k_1\times k_1}, B^2_{k_2\times k_2}, \dots B^{m-1}_{k_{m-1}\times k_{m-1}}$ and $B^m_{k_m\times k_m}$ such that*

(i) *for any $1 \leq j \leq m, k_j = |E_j|$ and $A_n^w(G) = [B^1, B^2, \dots, B^{m-1}, B^m]$,*

(ii) *for any $1 \leq j \leq m$, have $Ev(B^j_{k_j\times k_j}) = \{0, j\}$,*

(iii) $Ev(A_n^w(G)) = \bigcup_{j=1}^{m} Ev(B^j_{k_j\times k_j})$.

Example 49. *Let $H = \{1,2,3,\dots,9\}$, $G \in \mathcal{P}_h(H)$ which in $G = (H, E_1 = \{1,2\}, E_2 = \{3,4,5\}$ and $E_3 = \{6,7,8,9\})$. Then* $A_9(G) = \begin{pmatrix} C_{2\times2} & A_{2\times3} & E_{2\times4} \\ A^t_{2\times3} & C_{3\times3} & D_{3\times4} \\ E^t_{2\times4} & D^t_{3\times4} & C_{4\times4} \end{pmatrix}$, *where* $C_{2\times2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_{2\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_{2\times4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C_{3\times3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $D_{3\times4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $C_{4\times4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. *It is easy to check that* $P_9(\lambda) = Det(C_1C_2C_3) = (-\lambda-1)^6(\lambda-1)(\lambda-2)(\lambda-3)$ *whence* $C_{2\times2} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$, $C_{3\times3} = \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$ *and* $C_{4\times4} = \begin{pmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{pmatrix}$.
Also obviously, $Spec(A_n^w(G)) = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 6 & 1 & 1 & 1 \end{pmatrix}$.

Corollary 50. *Let H be a non–empty set and $(G,\{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. Then*

$$Spec(A_n^w(G)) - Spec(A_n(G)) = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

5. CONNECTION BETWEEN EIGENVALUES OF GRAPHS AND EIGENVALUES OF HYPERGRAPHS

In this section, we compute eigenvalues of quotient of partitioned hypergraphs and quotient of discrete complete hypergraphs via positive equivalence relation η^*. Also eigenvalues of hypergraphs and eigenvalues of quotient of corresponding hypergraphs is compared.

Theorem 51. *Let $(G, \{E_x\}_{x\in G})$ be a hypergraph. Then $(G, \{E_x\}_{x\in G})$ is an ordinary complete hypergraph if and only if $Ev(A_n(G/\eta^*)) = \{0\}$.*

Proof. Let $G = \{a_1, a_2, \ldots, a_n\}$. Since G is an ordinary complete hypergraph, there exists a unique hyperedge $E = G$ so that for any $1 \le j \ne j' \le n, \eta^*(a_j) = \eta^*(a_{j'}) = E$. Thus $G/\eta^* = \{\eta^*(a_1)\}$ and so $Ev(A_n(G/\eta^*)) = \{0\}$. If $Ev(A_n(G/\eta^*)) = \{0\}$, then $|G/\eta^*| = 1$. Thus for all $x, y \in G$, there exists a unique hyperedge E such that $\{x, y\} \subseteq E$. It concludes that $(G, \{E_x\}_{x\in G})$ is an ordinary complete hypergraph. □

Corollary 52. *Let $(G, \{E_x\}_{x\in G})$ be a hypergraph. Then $(G, \{E_x\}_{x\in G})$ is an ordinary complete hypergraph if and only if $Ev(A^w_n(G/\eta^*)) = \{0\}$.*

Example 53. *Let $G = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a hypergraph in Figure 8.*

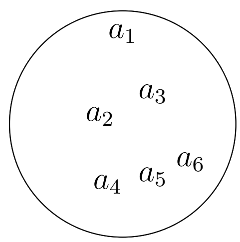


Figure 8: Ordinary complete hypergraphs G of order 6.

We can see that $\eta^*(a_j) = G$, where $1 \le j \le 6$. Hence $G/\eta^* = \{\eta^*(a_1)\}$, so $Ev(A_n(G/\eta^*)) = Ev(A^w_n(G/\eta^*)) = \{0\}$.

Theorem 54. *Let $(G, \{E_x\}_{x\in G}) \in \mathcal{D}^{(1)}_c(H)$ and $*_i = *_0$. Then*

(*i*) $Ev(A_n(G/\eta^*)) = \{-1, n-1\}$;

(*ii*) $Ev(A^w_n(G/\eta^*)) = \{n-2, 2n-2\}$.

Proof. (*i*) Let $G = \{a_1, a_2, \ldots, a_n\}$. For any $1 \le j \le n$, set $E_j = \{a_j\}$ and $E_{j+1} = G$. A simple computation shows that $(G, \overline{E} = \{E_j\}_{j=1}^{n+1})$ is a discrete complete hypergraph and for any $1 \le j \le n$, $\eta^*(a_j) = E_j$. Thus $G/\eta^* = \{\eta^*(a_1), \eta^*(a_2), \ldots, \eta^*(a_n)\}$, since for $1 \le j,\ j' \le n$, $|E_j| = |E_{j'}|$ and $*_i = *_0$, we get that for any $a_j \in E_j$, $a_{j'} \in E'_j$, $\eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\eta^*(a_j), \eta^*(a_{j'})}$ and $|\overline{E}/\eta^*| = (n^2 - n)/2$. It follows that $G/\eta^* \cong K_n$ and so $Ev(A_n(G/\eta^*)) = \{-1, n-1\}$.

(*ii*) Since any complete graph is a hypergraph by (*i*), we get $Spec(A^w_n(G/\eta^*)) = \begin{pmatrix} n-2 & 2n-2 \\ n-1 & 1 \end{pmatrix}$. □

Example 55. *Consider the hypergraph $H_1 = (\{a, b, c, d\}, E_1 = \{a, b\}, E_2 = \{c, d\})$ in Example 35. It is easy to see that $Ev(A_n(G/\eta^*, *_0)) = \{-1, 1\}$, while $H_1 \notin \mathcal{D}^{(1)}_c(H)$ and it disproves the converse of Theorem 54.*

Corollary 56. *Let* $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(1)}(H)$ *and* $*_i = *_0$. *Then* $Ev(M_n(G/\eta^*)) = \{0, n\}$ *if and only if* $n = 3$.

Example 57. *Let* $G = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ *be a discrete complete hypergraph in Figure 9.*

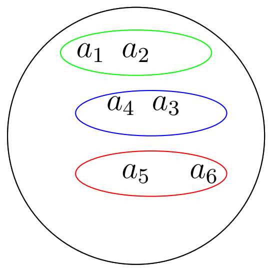


Figure 9: Discrete complete hypergraph G of order 6.

Computation shows that $\eta^*(a_1) = \{a_1, a_2\}, \eta^*(a_2) = \{a_3, a_4\}$ *and* $\eta^*(a_5) = \{a_5, a_6\}$. *Since for* $1 \leq j \neq j' \leq 6$, $|E_j| = |E_{j'}|$ *and* $*_i = *_0$, *we get that for any* $a_j \in E_j$, $a_{j'} \in E'_j$, $\eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\eta^*(a_j), \eta^*(a_{j'})}$. *Hence* $G/\eta^* \cong K_3$ *as Figure 10.*

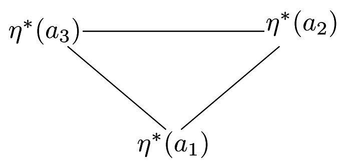


Figure 10: Graph G/η^* obtained from hypergraph Figure 9 for $i = 0$.

So $Ev(A_n(G/\eta^*)) = Ev(K_3)) = \{-1, 2\}$ *and* $Ev(M_n(G/\eta^*)) = \{0, 3\}$.

Theorem 58. *Let* $2 \geq k \in \mathbb{N}, k \bigm| |H|$ *and* $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(k)}(H)$. *Then*

(*i*) $Ev(A_n(G/\eta^*)) = \{-1, (n-k)/k\}$.

(*ii*) $Ev(A_n^w(G/\eta^*)) = \{(n-2k)/k, (2n-2k)/k\}$.

(*iii*) $Ev(M_n(G/\eta^*)) = \{0, n/k\}$ *if and only if* $n = 3k$.

Proof. (*i*) Let $G = \{a_1, a_2, \ldots, a_n\}$. Then for any $1 \leq j \leq n/k$, rearrange $E_j = \{a_{(j-1)k+1}, a_{(j-1)k+2}, \ldots, a_{jk}\}$. It is clear that $(G, \overline{E} = \{E_j\}_{j=1}^{n/k} \cup G)$ is a discrete complete hypergraph and computations show that for any $1 \leq j \leq n/k$, $\eta^*(a_{(j-1)k+1}) = E_j$. Thus $G/\eta^* = \{E_1, E_2, \ldots, E_{n/k}\}$, since for $1 \leq j, j' \leq n/k$, $|E_j| = |E_{j'}|$ and $*_i = *_0$, concludes that for any $a_j \in E_j$, $a_{j'} \in E'_j$, $\eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\eta^*(a_j), \eta^*(a_{j'})}$ and $|\overline{E}/\eta^*| = (n^2 - kn)/k^2$. It follows that $G/\eta^* \cong K_{n/k}$. Hence $Ev(A_n(G/\eta^*)) = \{-1, (n-k)/k\}$.

(*ii*) and (*iii*) are clearly obtained from (*i*), Theorem 54 and Corollary 56. □

Corollary 59. *Let* $(G, \{E_x\}_{x\in G}) \in \mathcal{D}_c^{(2)}(H)$. *Then for* $n \geq 4$

(*i*) $Ev(A_n(G/\eta^*)) = \{-1, (n-2)/2\}$;

(*ii*) $Ev(A_n^w(G)) = \{(n-4)/2, n-2\}$.

Example 60. (*i*) *Let* $G = \{a_1, a_2, a_3, \dots, a_{15}, a_{16}\}$ *be a discrete complete hypergraph in Figure 11.*

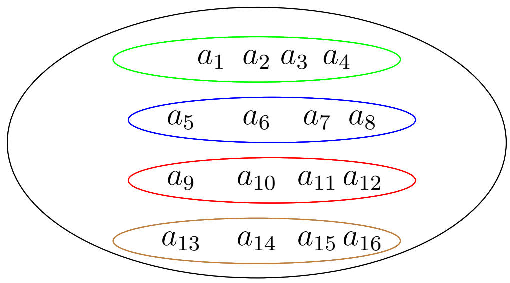


Figure 11: Discrete complete hypergraphs G of order 16.

Computations show that $\eta^*(a_1) = \{a_1, a_2, a_3, a_4\}$, $\eta^*(a_5) = \{a_5, a_6, a_7, a_8\}$, $\eta^*(a_9) = \{a_9, a_{10}, a_{11}, a_{12}\}$ *and* $\eta^*(a_{13}) = \{a_{13}, a_{14}, a_{15}, a_{16}\}$. *Since for* $1 \leq j \neq j' \leq 16$, $|E_j| = |E_{j'}|$ *and* $*_i = *_0$, *we get that for any* $a_j \in E_j$, $a_{j'} \in E'_j$, $\eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\eta^*(a_j), \eta^*(a_{j'})}$. *Hence* $G/\eta^* \cong K_4$ *in Figure 12.*

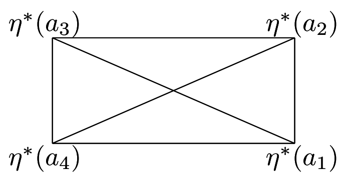


Figure 12: Graph G/η^* obtained from hypergraph Figure 11 for $i = 0$.

So $Ev(A_n(G/\eta^*)) = Ev(K_4)) = \{-1, 3\}$ *and* $Ev(A_n^w(G/\eta^*)) = \{2, 6\}$.

(*ii*) *Inputting* $n = 2$ *in Theorem 54 and* $n = 4, k = 2$ *in Theorem 58, shows that the converse of Theorem 58, is not true necessarily.*

Theorem 61. *Let* H *be a set and* $|H| = p$ *be a prime. If* $G \in \mathcal{P}_h^{(k)}(H)$, *then*

(*i*) $Ev(A_n(G/\eta^*)) = \{-1, 0, p-1\}$.

(*ii*) $Ev(A_n^w(G/\eta^*)) = \{0, p-2, 2p-2\}$.

Proof. Let $H = \{a_1, a_2, \dots, a_p\}$, $k = 1$ and $i = 0$. Consider the following hypergraphs: (Figure 13):

Then it is easy to see that $G = (H, \{E_j\}_{j=1}^p)$ is a partitioned hypergraph, where for any $1 \leq j \leq n$, $E_j = \{a_j\}$. Clearly for any $1 \leq j \leq p$, $E_{a_j}^s = \{a_j\}$.

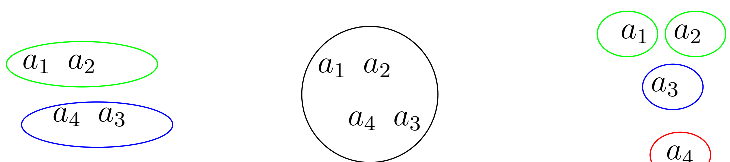
Figure 13: Partitioned hypergraph G.

Hence $G/\eta^* = \{\eta(a_j) \mid 1 \leq j \leq p\}$ and so $G/\eta^* \cong K_p$. Hence $Spec(A_n(G/\eta^*)) = \begin{pmatrix} -1 & p-1 \\ p-1 & 1 \end{pmatrix}$. Also for $k = p$, by Theorem 51 and Corollary 52, we get

$$Spec(A_n(G/\eta^*)) = Spec(A_n^w(G/\eta^*)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

□

Example 62. *Let $H = \{a_1, a_2, a_3, a_4\}$, then consider the hypergraphs in Figure 14.*



(a) G_1 (b) G_2 (c) G_3

Figure 14: Partitioned hypergraphs H of order 4.

One can see that $Ev(A_n(G_1/\eta^*)) = \{-1, 1\}, Ev(A_n(G_2/\eta^*)) = \{0\}, Ev(A_n(G_3/\eta^*)) = \{-1, 3\}, Ev(A_n^w(G_1/\eta^*)) = \{0, 2\}, Ev(A_n^w(G_2/\eta^*)) = \{0\}$ *and* $Ev(A_n^w(G_3/\eta^*)) = \{4, 6\}$.

Example 62, shows that in Theorem 61, p must be a prime.

Theorem 63. *Let* $2 \geq k \in \mathbb{N}, k \mid |H| = n$ *and* $G \in \mathcal{P}_h^{(n/k)}(H)$. *Then*

(*i*) $Ev(A_n(G/\eta^*)) = \{-1, k-1\}$.

(*ii*) $Ev(A_n^w(G/\eta^*)) = \{k-2, 2k-2\}$.

Proof. (*i*) Let $G = \{a_1, a_2, \ldots, a_n\}$ and $k \in \mathbb{N}$. For any $1 \leq j \leq k$, rearrange $E_j = \{a_{(j-1)n/k+1}, a_{(j-1)n/k+2}, \ldots, a_{jn/k}\}$. A simple computation shows that $(G, \overline{E} = \{E_j\}_{j=1}^k)$ is a partitioned hypergraph and for any $1 \leq j \leq k$, $\eta^*(a_{(j-1)n/k+1}) = E_j$. Thus $G/\eta^* = \{E_1, E_2, \ldots, E_k\}$, since for $1 \leq j, j' \leq k$, $|E_j| = |E_{j'}|$ and $*_i = *_0$, implies that for any $a_j \in E_j, a_{j'} \in E'_j, \eta^*(a_j) *_i \eta^*(a_{j'}) = \widehat{\eta^*(a_j), \eta^*}(a_{j'})$ and $|\overline{E}/\eta^*| = k^2 - k/2$. It follows that $G/\eta^* \cong K_k$ and $Spec(A_n(G/\eta^*)) = \begin{pmatrix} -1 & k-1 \\ k-1 & 1 \end{pmatrix}$.

(*ii*) It is clearly obtained from (*i*). □

Theorem 64. *Let H be a non–empty set, $k\in\mathbb{N}$ and $(G,\{E_j\}_{j=1}^m)\in\mathcal{P}_h(H)\backslash\mathcal{P}_h^k(H)$. If $|E_1| \mid |E_2| \mid |E_3| \dots \mid |E_{m-1}| \mid |E_m|$, then $Ev(G/\eta^*) = \{0,1,-1\}$.*

Proof. Let $H = \{a_1, a_2, \dots, a_n\}$ and $(G,\{E_j\}_{j=1}^m) \in \mathcal{P}_h(H) \setminus \mathcal{P}_h^k(H)$. Since for any $1 \leq j \leq m, |E_j| \mid |E_{j+1}|$, we get that $|E_j| \mid \Big||E_{j+1}| - |E_j|\Big|$. Clearly $G/\eta^* = \{\eta^*(a_1), \eta^*(a_2), \dots, \eta^*(a_m)\}$, so for any $*_i$, such that $i = \big||E_{j+1}| - |E_j|\big|$ we have

$$\eta^*(a_r) *_i \eta^*(a_s) = \begin{cases} \widehat{\eta^*(a_r), \eta^*}(a_s) & \textit{if } i = \big||\eta^*(a_r)| - |\eta^*(a_s)|\big| \\ \widehat{\emptyset} & \textit{otherwise.} \end{cases}$$

Hence G/η^* is obtained as follows: (Figure 15):

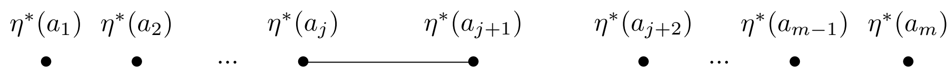


Figure 15: Graph G/η^* with m vertices and 1 edge.

In this case it is to see that $P_n(\lambda) = (-\lambda)^{m-2}(\lambda^2 - 1)$ and so $Spec(A_n(G)) = \begin{pmatrix} 0 & 1 & -1 \\ m-2 & 1 & 1 \end{pmatrix}$. If for any $i \in \mathbb{N}^*, i \nmid \big||E_{j+1}| - |E_j|\big|$, then $G/\eta^* \cong \overline{K}_m$ and so $Spec(A_n(G)) = \begin{pmatrix} 0 \\ m \end{pmatrix}$. □

Let $V = \{a_1, a_2, \dots, a_n\}$. Then we denote the path tree on V by Figure 16 and denote it by T_m^l.

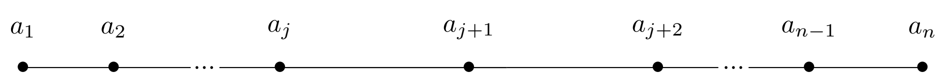


Figure 16: Path graph tree T_m^l.

Theorem 65. *Let H be a non–empty set, $k\in\mathbb{N}$ and $(G,\{E_j\}_{j=1}^m)\in\mathcal{P}_h(H)\backslash\mathcal{P}_h^k(H)$. If there exists $1 \leq t \leq m$ such that for any $1 \leq j \leq m, |E_t| \mid |E_j|$, then*

(*i*) $G/\eta^* \cong T_m^l$

(*ii*) $\sum Ev(G/\eta^*) = \{0\}$.

Proof. (*i*) Let $H = \{a_1, a_2, \dots, a_n\}$ and $(G,\{E_j\}_{j=1}^m) \in \mathcal{P}_h(H)$. We rearrange the hypergraph $(G,\{E_j\}_{j=1}^m)$ such that for any $1 \leq j \leq m$, $|E_1|\big||E_j|$, so we get that $|E_1|\Big|\Big||E_{j+1}|-|E_j|\Big|$. Hence for $*_i = *_{|E_1|}$ we have $G/\eta^* = \{\eta^*(a_1),\eta^*(a_2),\dots,\eta^*(a_m)\}$, where for any $1 \leq j \leq m, \eta^*(a_j) = E_j$. In this case we have

$$\eta^*(a_r) *_i \eta^*(a_s) = \begin{cases} \widehat{\eta^*(a_r), \eta^*}(a_s) & \textit{if } |E_1| = \big||\eta^*(a_r)| - |\eta^*(a_s)|\big| \\ \widehat{\emptyset} & \textit{otherwise.} \end{cases}$$

Hence G/η^* is obtained in Figure 17 and clearly $G/\eta^* \cong T^l_m$.
(ii) A simple computation shows that for any $k \in \mathbb{N}$,

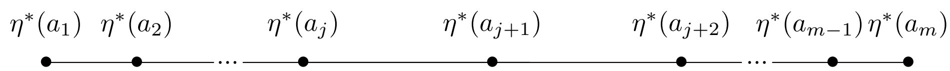


Figure 17: Path graph tree T^l_m.

$$\begin{aligned} P_{2k}(\lambda) &= \lambda^{2k} - (2k-1)\lambda^{2k-2} + a_1\lambda^{2k-4} - a_2\lambda^{2k-6} + \\ \ldots &\quad +(-1)^{[n/2-1]}a_{[n/2]}\lambda^2 + (-1)^{[n/2]}a_{[n/2]-1} \end{aligned}$$

and

$$\begin{aligned} P_{2k+1}(\lambda) &= -\lambda(\lambda^{2k} - (2k)\lambda^{2k-2} + a_1\lambda^{2k-4} - a_2\lambda^{2k-6} + \\ \ldots &\quad +(-1)^{[n/2-1]}a_{[n/2]}\lambda^2 + (-1)^{[n/2]}a_{[n/2]-1}). \end{aligned}$$

Hence we obtain that $Spec(A_{2k}(G)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{2k} \\ 1 & 1 & \ldots & 1 \end{pmatrix}$ and $Spec(A_{2k+1}(G)) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \ldots & \lambda_{2k} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$, where for any $1 \leq j \leq 2k, \lambda_{j+1} = -\lambda_j$. □

We give an Example for describing of Theorem 65.

Example 66. *Let $G = \{a_1, a_2, a_3, a_4, \ldots, a_{12}\}$ be hypergraph H in Figure 18.*

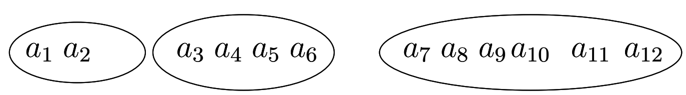


Figure 18: Hypergraph H.

Clearly $|E_1| = 2$, $|E_2| = 4$ and $|E_2| = 6$. It follows that $\eta^* = \eta$, $\eta(a_1) = E_1, \eta(a_3) = E_2$ and $\eta(a_7) = E_3$. Now, for $i = 2$, we obtain the graph in Figure 19.

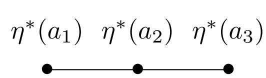


Figure 19: Path graph tree T^l_3.

and so $P_3(\lambda) = -\lambda(\lambda^2 - 2)$.

6. CONCLUSION

The current paper has considered the notion of partitioned hypergraphs, discrete complete hypergraphs and introduced weak adjacency matrix related these hypergraphs. Eigenvalues of (weak) adjacency matrix of these class of hypergraphs is computed and formulated. We investigated some properties of weak adjacency matrix, spectrum of weak adjacency matrix of discrete complete hypergraphs (partitioned hypergraphs) and spectrum of adjacency matrix of discrete complete hypergraphs (partitioned hypergraphs). With respect to the properties of weak adjacency matrix, it is presented that there exists a two sided connection between ordinary complete hypergraphs and their eigenvalues, spectrum of any complete graph is a accessible spectrum graph. For making connection between spectrum of graphs and hypergraphs, a positive equivalence relation η^* is introduced, moreover is proved that spectrum of path graph trees are accessible from hypergraphs.

In our future studies, we want to obtain more results regarding spectrum of graphs, hypergraphs and their applications.

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