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MATRIX REPRESENTATIONS FOR A CERTAIN CLASS OF COMBINATORIAL NUMBERS ASSOCIATED WITH BERNSTEIN BASIS FUNCTIONS AND CYCLIC DERANGEMENTS AND THEIR PROBABILISTIC AND ASYMPTOTIC ANALYSES

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In this paper, we mainly concerned with an alternate form of the generating functions for a certain class of combinatorial numbers and polynomials. We give matrix representations for these numbers and polynomials with their applications. We also derive various identities such as Rodrigues-type formula, recurrence relation and derivative formula for the aforementioned combinatorial numbers. Besides, we present some plots of the generating functions for these numbers. Furthermore, we give relationships of these combinatorial numbers and polynomials with not only Bernstein basis functions, but the two-variable Hermite polynomials and the number of cyclic derangements. We also present some applications of these relationships. By applying Laplace transform and Mellin transform respectively to the aforementioned functions, we give not only an infinite series representation, but also an interpolation function of these combinatorial numbers. We also provide a contour integral representation of these combinatorial numbers. In addition, we construct exponential generating functions for a new family of numbers arising from the linear combination of the numbers of cyclic derangements in the wreath product of the finite cyclic group and the symmetric group of permutations of a set. Finally, we analyse the aforementioned functions in probabilistic and asymptotic manners, and we give some of their relationships with not only the Laplace distribution, but also the standard normal distribution. Then, we provide an asymptotic power series representation of the aforementioned exponential generating functions.

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1. INTRODUCTION

Undoubtedly, generating functions are one of the most significant tools to be able to used to find out identities such as recurrence relations, derivative formulas and other fundamental properties for some classes of special numbers and polynomials. Among others, the cited references herein originally leads us to think that way. Particularly, motivated by these papers and inspired by [**19**], Kucukoglu et al. [**9**] has recently introduced the combinatorial numbers $Y_n^{(k)}(\lambda)$ by the following exponential generating functions:

$$\mathcal{F}(t,k;\lambda) := \left(\frac{2}{\lambda(1+\lambda t)-1}\right)^k = \sum_{n=0}^{\infty} Y_n^{(k)}(\lambda)\frac{t^n}{n!}, \tag{1}$$

in which, $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and λ is a real or complex number (*cf.* [**9**]). Notice that the special case of (1) when $k=1$ gives us the generating functions for the numbers $Y_n(\lambda) = Y_n^{(1)}(\lambda)$ introduced by the second author [**19**].

Afterwards, at the continuation, by considering the reciprocal of the function $\mathcal{F}(t,k;\lambda)$ with respect to multiplication, Kucukoglu et al. [**10**] introduced the another families of the combinatorial numbers $Y_n^{(-k)}(\lambda)$, unifying the Bernstein basis functions, by the following generating functions:

$$\mathcal{G}(t,k;\lambda) := 2^{-k}(\lambda(1+\lambda t)-1)^k = \sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda)\frac{t^n}{n!},$$

in which, $k \in \mathbb{N} = \{1,2,\dots\}$ and λ is a real or complex number (*cf.* [**10**]).

The explicit formula for the numbers $Y_n^{(k)}(\lambda)$ and the numbers $Y_n^{(-k)}(\lambda)$ is respectively given as follows:

$$Y_n^{(k)}(\lambda) = (-1)^n \binom{n+k-1}{n}\frac{2^k n! \lambda^{2n}}{(\lambda-1)^{k+n}} \qquad (\textit{cf. } [\mathbf{9}]),$$

and

$$Y_n^{(-k)}(\lambda) = 2^{-k} n! \binom{k}{n} \lambda^{2n}(\lambda-1)^{k-n} \qquad (\textit{cf. } [\mathbf{10}]), \tag{2}$$

where $n \leq k$ and the notation $\binom{k}{n}$ denotes as usual the binomial coefficients given by the formula:

$$\binom{k}{n} = \begin{cases} \frac{k!}{n!(k-n)!} & \text{for } 0 \leq n \leq k \\ 0 & \text{for other cases.} \end{cases}$$

In conjunction with the above statement of the binomial coefficients notice that the numbers $Y_n^{(-k)}(\lambda)$ becomes zero when $n > k$. Namely,

$$Y_n^{(-k)}(\lambda) = 0 \quad \text{if} \quad n > k.$$

Noting that when the parameter λ is considered as a variable, $Y_n^{(-k)}(\lambda)$ yields us a polynomial of degree $k+n$.

Let $(x)_j$ stand for the falling factorial defined as in the following formula:

$$(x)_j = \prod_{m=1}^{j} (x - m + 1).$$

By making calculation with the equation (2), we compute a few values of $Y_n^{(-k)}(\lambda)$ in the following manner:

$$\begin{aligned}
Y_0^{(-k)}(\lambda) &= \frac{(\lambda-1)^k}{2^k}, \\
Y_1^{(-k)}(\lambda) &= \frac{k\lambda^2(\lambda-1)^{k-1}}{2^k}, \\
Y_2^{(-k)}(\lambda) &= \frac{k(k-1)\lambda^4(\lambda-1)^{k-2}}{2^k}, \\
&\vdots \\
Y_j^{(-k)}(\lambda) &= \frac{(k)_j\lambda^{2j}(\lambda-1)^{k-j}}{2^k} \quad \textit{if} \quad j \le k, \\
&\vdots \\
Y_k^{(-k)}(\lambda) &= \frac{k!\lambda^{2k}}{2^k}.
\end{aligned} \tag{3}$$

On the other hand, by changing the studied index, we have

$$\begin{aligned}
Y_n^{(-1)}(\lambda) &= \frac{(1)_n\lambda^{2n}(\lambda-1)^{1-n}}{2} \quad \textit{if} \quad n \le 1, \\
Y_n^{(-2)}(\lambda) &= \frac{(2)_n\lambda^{2n}(\lambda-1)^{2-n}}{4} \quad \textit{if} \quad n \le 2,
\end{aligned}$$

and so on.

In more specific cases depending on the above, we have

$$\begin{aligned}
Y_0^{(-1)}(\lambda) &= \frac{\lambda-1}{2}, \\
Y_1^{(-1)}(\lambda) &= \frac{\lambda^2}{2}, \\
Y_0^{(-2)}(\lambda) &= \frac{(\lambda-1)^2}{4}, \\
Y_1^{(-2)}(\lambda) &= \frac{\lambda^2(\lambda-1)}{2}, \\
Y_2^{(-2)}(\lambda) &= \frac{\lambda^4}{2}.
\end{aligned}$$

In order to give some interesting relations among some special numbers, special functions and $Y_n^{(-k)}(\lambda)$, we need to recall the following definitions and formulas:

Let $x \in [0,1]$ and $n \in \mathbb{N}_0$. Then, the Bernstein basis functions, $B_k^n(x)$, are defined by

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad (k = 0, 1, \ldots, n) \tag{4}$$

of which the generating function is given below:

$$F_B(t, x; k) := \frac{(xt)^k e^{(1-x)t}}{k!} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!},$$

where $t \in \mathbb{C}$ (*cf.* [**1**], [**12**], [**15**], [**17**], [**18**], [**23**]).

The numbers $Y_n^{(-k)}(\lambda)$ have some interesting relationships with some special functions and special numbers. One of the most important of these is the relationship with the Bernstein basis functions given as follows (*cf.* [**10**, p. 6, Eq. (16)]):

$$Y_n^{(-k)}(\lambda) = (-1)^{k-n} \frac{n!\lambda^n}{2^k} B_n^k(\lambda)$$

in the case of $n \le k$ and $\lambda \in [0,1]$.

With the aid of (4), we obtain another interesting relation between $Y_n^{(-k)}(\lambda)$ and the Bernoulli numbers of higher-order given as follows:

$$Y_k^{(-k)}(\lambda) = \frac{(-1)^k \lambda^{2k}}{2^k} B_k^{(k+1)}$$

where $B_k^{(n)}$ denotes the n-th order Bernoulli numbers defined by the following generating functions:

$$\left(\frac{t}{e^t - 1} \right)^n = \sum_{k=0}^{\infty} B_k^{(n)} \frac{t^k}{k!},$$

(see, for details, [**4**], [**7**]).

Two-variable Hermite polynomials $H_k^{(j)}(x, y)$ are defined by means of the following generating functions (*cf.* [**5**], [**14**]):

$$F_H(t, x, y; j) := e^{xt + yt^j} = \sum_{k=0}^{\infty} H_k^{(j)}(x, y) \frac{t^k}{k!} \tag{5}$$

and the explicit formula for these numbers is given by

$$H_k^{(j)}(x, y) = k! \sum_{r=0}^{\left[\frac{k}{j}\right]} \frac{x^{k-jr} y^r}{r!\,(k - jr)!}, \tag{6}$$

where $j \geq 2$ is an integer and $[x]$ denotes the greatest integer (floor) function (see, for details, [**5**], [**14**] and the cited references therein).

To state briefly, the rest of this paper is organized as below:

In Section 1, we give some matrix representations for $Y_n^{(-k)}(\lambda)$ and their applications and observations with some open questions which may lead to new research topics.

In Section 2, by considering an alternate form of the generating functions for $Y_n^{(-k)}(\lambda)$, we obtain multifarious formulae such as Rodrigues-type formula, recurrence relation and derivative formula concerning $Y_n^{(-k)}(\lambda)$. We also provide some plots to examine the behaviour of the aforementioned alternate form of the generating functions for $Y_n^{(-k)}(\lambda)$ in some special cases. Besides, it is shown that there exist a relationship between $Y_n^{(-k)}(\lambda)$ and the two-variable Hermite polynomials $H_k^{(j)}(x,y)$. We present some applications of this relationship.

In Section 3, we present some applications of integral transforms to the aforementioned alternate form of the generating functions for $Y_n^{(-k)}(\lambda)$. With the application of the Laplace transform, we give an infinite series representation involving $Y_n^{(-k)}(\lambda)$. With the application of the Mellin transform, we present an interpolation function of $Y_n^{(-k)}(\lambda)$ at negative integers. In addition, we present a contour integral representation of $Y_n^{(-k)}(\lambda)$.

In Section 4, it is shown that there exist a relationship between $Y_n^{(-k)}(\lambda)$ and the number of cyclic derangements. We present some applications of this relationship. We also construct exponential generating functions for a new family of numbers arising from the linear combination of the numbers of cyclic derangements in the wreath product of the finite cyclic group and the symmetric group of permutations of a set.

In Section 5, we analyse the aforementioned alternate form of the generating functions for $Y_n^{(-k)}(\lambda)$ in probabilistic and asymptotic manners, and we give some of their relations with not only the Laplace distribution, but also the standard normal distribution. In addition, we provide an asymptotic power series representation of the aforementioned exponential generating functions.

1. MATRIX REPRESENTATIONS FOR $Y_n^{(-k)}(\lambda)$

In this section, we give some matrix representations for $Y_n^{(-k)}(\lambda)$ and their applications and observations with some open questions which may lead to new research topics. This section is inspired by the papers [**15**]-[**22**] of the second author who presented matrix representation formulas for the Bernstein basis functions and Beta polynomials with their generating functions.

1.1. A matrix representation obtained by means of the base consisting of $Y_n^{(-k)}(\lambda)$

It is well-known that the polynomials of degree up to $2k$ can be written as a linear combination of the standard basis vectors $\{1,\lambda,\lambda^2,\ldots,\lambda^{2k}\}$ as in the following form:

$$P_{2k}(\lambda)=\sum_{n=0}^{2k}a_n\lambda^n.$$

At this stage, since $Y_n^{(-k)}(\lambda)$ are polynomials of degree $k+n$ in the variable λ, the following question is came up with:

Can the polynomials $P_{2k}(\lambda)$ be written as a linear combination of the polynomials $Y_n^{(-k)}(\lambda)$ as in the following form:

$$P_{2k}(\lambda)=\sum_{n=0}^{k}c_nY_n^{(-k)}(\lambda).$$

Before answering the above question, it is first required show whether the set $\{Y_0^{(-k)}(\lambda),Y_1^{(-k)}(\lambda),\ldots,Y_k^{(-k)}(\lambda)\}$ is linearly independent.

It is well-known that the set $\{Y_0^{(-k)}(\lambda),Y_1^{(-k)}(\lambda),\ldots,Y_k^{(-k)}(\lambda)\}$ becomes a linearly independent set if the following equality:

$$0=\sum_{n=0}^{k}c_nY_n^{(-k)}(\lambda)$$

possess only the trivial solution $c_n=0$ for $n=1,2,\ldots,k$.

Let us assume that the above proposition is true. Then, by (2), we write

$$\begin{aligned}0 &= c_0(\lambda-1)^k+c_1k\lambda^2(\lambda-1)^{k-1}+c_2(k)_2\lambda^4(\lambda-1)^{k-2}\\ &\quad+\cdots+c_{k-2}(k)_{k-2}\lambda^{2(k-2)}(\lambda-1)^2+c_{k-1}(k)_{k-1}\lambda^{2(k-1)}(\lambda-1)+c_kk!\lambda^{2k}.\end{aligned}$$

From the above equation, we get

$$\begin{aligned}0 &= c_0\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\lambda^j+c_1k\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\lambda^{2+j}\\ &\quad+c_2(k)_2\sum_{j=0}^{k-2}(-1)^{k-2-j}\binom{k-2}{j}\lambda^{4+j}\\ &\quad+\cdots+c_{k-2}(k)_{k-2}\sum_{j=0}^{2}(-1)^{2-j}\binom{2}{j}\lambda^{2(k-2)+j}\\ &\quad+c_{k-1}(k)_{k-1}\sum_{j=0}^{1}(-1)^{1-j}\binom{1}{j}\lambda^{2(k-1)+j}+c_kk!\lambda^{2k}.\end{aligned}$$

Thus, we have

$$\begin{aligned}
0 &= c_0(-1)^k + c_0(-1)^{k-1}\binom{k}{1}\lambda \\
&+ \left(c_1(-1)^{k-1}(k)_1\binom{k-1}{0} + c_0(-1)^{k-2}\binom{k}{2}\right)\lambda^2 \\
&+ \left(c_1(-1)^{k-2}(k)_1\binom{k-1}{2} + c_0(-1)^{k-3}\binom{k}{3}\right)\lambda^3 \\
&+ \cdots + \left(c_{k-2}(k)_{k-2} - c_{k-3}(k)_{k-3}\binom{3}{2} + c_{k-4}(k)_{k-4}\right)\lambda^{2k-4} \\
&+ \left(-c_{k-2}(k)_{k-2}\binom{2}{1} + c_{k-3}(k)_{k-3}\right)\lambda^{2k-3} \\
&+ \left(-c_{k-1}(k)_{k-1} + c_{k-2}(k)_{k-2}\right)\lambda^{2k-2} + c_{k-1}(k)_{k-1}\lambda^{2k-1} + c_k k!\lambda^{2k}.
\end{aligned}$$

As a result, we get the following equations:

$$\begin{aligned}
(-1)^k c_0 &= 0 \\
(-1)^{k-1}\binom{k}{1}c_0 &= 0 \\
(-1)^{k-1}(k)_1\binom{k-1}{0}c_1 + (-1)^{k-2}\binom{k}{2}c_0 &= 0 \\
(-1)^{k-2}(k)_1\binom{k-1}{2}c_1 + (-1)^{k-3}\binom{k}{3}c_0 &= 0 \\
&\vdots \\
(k)_{k-2}c_{k-2} - (k)_{k-3}\binom{3}{2}c_{k-3} + (k)_{k-4}c_{k-4} &= 0 \\
-(k)_{k-2}\binom{2}{1}c_{k-2} + (k)_{k-3}c_{k-3} &= 0 \\
-(k)_{k-1}c_{k-1} + (k)_{k-2}c_{k-2} &= 0 \\
(k)_{k-1}c_{k-1} &= 0 \\
k!c_k &= 0.
\end{aligned}$$

It follows from the above equations that

$$c_0 = c_1 = \cdots = c_k = 0$$

which means that the set $\{Y_0^{(-k)}(\lambda), Y_1^{(-k)}(\lambda), \ldots, Y_k^{(-k)}(\lambda)\}$ is linearly independent. By the above investigation, we also conclude that any polynomial of degree less than or equal to $2k$ can be expressed as a linear combination of $Y_n^{(-k)}(\lambda)$. In other words, the set $\{Y_0^{(-k)}(\lambda), Y_1^{(-k)}(\lambda), \ldots, Y_k^{(-k)}(\lambda)\}$ forms a basis for the vector space of the polynomials of degree less than or equal to $2k$.

Remark 1. *It can be easily seen that the parameter λ parameter goes up to the degree $2k$ in all expansions above. On the other hand, by (2), we have*

$$P_{2k}(\lambda) = 2^{-k} \sum_{n=0}^{k} c_n \sum_{j=0}^{k-n} (-1)^{k-n-j}\, n! \binom{k}{n} \binom{k-n}{j} \lambda^{2n+j}. \tag{7}$$

Thus, due to both all the above investigations and the right-hand side of the equation (7), the polynomials $P_{2k}(\lambda)$ are considered as the polynomials of degree less than or equal to $2k$. Note that the polynomials of odd degree less than $2k$ disappear, as can be also seen in the rest of the paper.

Now, we are ready to give matrix form of the polynomials $P_{2k}(\lambda)$:

It is clear that with the aid of the dot product of two vectors, we have

$$P_{2k}(\lambda) = \begin{bmatrix} Y_0^{(-k)}(\lambda) & Y_1^{(-k)}(\lambda) & Y_2^{(-k)}(\lambda) & \dots & Y_k^{(-k)}(\lambda) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}.$$

With the application of the mathematical induction method to the above equation and with the aid of (7), we get a matrix representation for the polynomials $P_{2k}(\lambda)$ as in the following formula:

$$P_{2k}(\lambda) = 2^{-k} XAC \tag{8}$$

where

$$X = \begin{bmatrix} 1 & \lambda & \lambda^2 & \lambda^3 & \dots & \lambda^{2k} \end{bmatrix}_{1\times(2k+1)}, \quad C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}_{(k+1)\times 1}$$

and the matrix A is given as a block matrix below:

$$A = \left[\begin{array}{c|c} V_{(k+1)\times k} & \mathbf{0}_{(k+1)\times 1} \\ \hline \mathbf{0}_{k\times 1} & W_{k\times k} \end{array}\right] \tag{9}$$

in which

$$W = \begin{bmatrix} (-1)^0 (k)_1 \binom{k-1}{k-1} & (-1)^1 (k)_2 \binom{k-2}{k-3} & (-1)^1 (k)_3 \binom{k-3}{k-5} & \dots & 0 \\ 0 & (-1)^0 (k)_2 \binom{k-2}{k-2} & (-1)^1 (k)_3 \binom{k-3}{k-4} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & (-1)^0 (k)_k \binom{0}{0} \end{bmatrix}_{k\times k}$$

and we conjecture that

$$V=\begin{bmatrix} (-1)^{k}(k)_0\binom{k}{0} & 0 & \cdots & 0 \\ (-1)^{k-1}(k)_0\binom{k}{1} & 0 & \cdots & 0 \\ (-1)^{k-2}(k)_0\binom{k}{2} & (-1)^{k-1}(k)_1\binom{k-1}{0} & \cdots & 0 \\ (-1)^{k-3}(k)_0\binom{k}{3} & (-1)^{k-2}(k)_1\binom{k-1}{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{0}(k)_0\binom{k}{k} & (-1)^{1}(k)_1\binom{k-1}{k-2} & \cdots & (k \bmod 2)\left((-1)^{k-\gamma(k)}(k)_{\gamma(k)}\binom{\gamma(k)}{(k \bmod 2)}\right) \end{bmatrix}_{(k+1)\times k}$$

so that

$$\gamma(k)=\left[\frac{k}{2}\right]+(k \bmod 2).$$

Remark 2. *Notice that the matrix W is a upper triangular matrix. On the other hand, the matrix V is closely related to the family of the lower triangular ladder matrix which is the concept of the Ladder matrix algebras defined in [**3**], and this type matrices forms a set which corresponds to the Lie algebra with respect to the standard Lie bracket, see for details [**6**].*

In order to explain the matrix representation of the polynomials $P_{2k}(\lambda)$, it is time to give some examples below:

Example 1. *In the case of (8) when $k=1$, the following matrix representation of the polynomials $P_2(\lambda)$ is obtained:*

$$P_2(\lambda)=2^{-1}\begin{bmatrix} 1 & \lambda & \lambda^2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

Example 2. *In the case of (8) when $k=2$, the following matrix representation of the polynomials $P_4(\lambda)$ is obtained:*

$$P_4(\lambda)=2^{-2}\begin{bmatrix} 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

By using the decomposition given in (9), we have

$$A=\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \\ 1 & -2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{bmatrix}.$$

Example 3. *In the case of* (8) *when* $k=3$, *the following matrix representation of the polynomials* $P_6(\lambda)$ *is obtained:*

$$P_6(\lambda)=2^{-3}\begin{bmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^6 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Example 4. *In the case of* (8) *when* $k=4$, *the following matrix representation of the polynomials* $P_8(\lambda)$ *is obtained:*

$$P_8(\lambda)=2^{-4}\begin{bmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 \\ 6 & -4 & 0 & 0 & 0 \\ -4 & 12 & 0 & 0 & 0 \\ 1 & -12 & 12 & 0 & 0 \\ 0 & 4 & -24 & 0 & 0 \\ 0 & 0 & 12 & -24 & 0 \\ 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

1.2. Other matrix representations with entries $Y_n^{(-k)}(\lambda)$

Here, we study on other kinds of matrix representations with entries $Y_n^{(-k)}(\lambda)$.

By considering that $Y_0^{(0)}(\lambda)=1$ and $Y_n^{(0)}(\lambda)=0$ when $n\neq 0$, we can give all values of $Y_n^{(-k)}(\lambda)$ in the form of a lower triangular matrix as follows:

$$Y=\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{\lambda-1}{2} & \frac{\lambda^2}{2} & 0 & 0 & \ldots & 0 \\ \frac{(\lambda-1)^2}{4} & \frac{\binom{2}{1}\lambda^2(\lambda-1)}{4} & \frac{2!\binom{2}{2}\lambda^4}{4} & 0 & \ldots & 0 \\ \frac{(\lambda-1)^3}{8} & \frac{\binom{3}{1}\lambda^2(\lambda-1)^2}{8} & \frac{2!\binom{3}{2}\lambda^4(\lambda-1)}{4} & \frac{3!\binom{3}{3}\lambda^6}{8} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(\lambda-1)^k}{2^k} & \frac{\binom{k}{1}\lambda^2(\lambda-1)^{k-1}}{2^k} & \frac{2!\binom{k}{2}\lambda^4(\lambda-1)^{k-2}}{2^k} & \frac{3!\binom{k}{3}\lambda^6(\lambda-1)^{k-3}}{2^k} & \ldots & \frac{k!\lambda^{2k}}{2^k} \end{bmatrix}_{(k+1)\times(k+1)}$$

namely, $Y=[Y_{k,n}]$ so that

$$Y_{k,n}=\begin{cases} Y_n^{(-k)}(\lambda) & \text{for } k\geq n \\ 0 & \text{for } k<n. \end{cases}$$

Some properties of the matrix Y are given below:

It is known that the eigenvalues of a lower triangular matrix are completely its diagonal inputs. Thus, the eigenvalues of the matrix Y are equal to

$$Y_j^{(-j)}(\lambda); \quad j = 0, 1, 2, \ldots, k.$$

In addition, the characteristic polynomial $p_Y(\lambda)$ of the matrix Y is in the following form:

$$p_Y(\lambda) = \det(\lambda I - Y)$$

where I denotes the identity matrix. Therefore, we have

$$p_Y(\lambda) = \prod_{j=0}^{k} \left(\lambda - Y_j^{(-j)}(\lambda)\right).$$

The determinant of the matrix Y is given as follows:

$$\det(Y) = \prod_{j=0}^{k} \frac{(-1)^j \lambda^{2j}}{2^j} B_j^{(j+1)}.$$

Remark 3. *In his recent paper, Simsek* [**21**] *defined a class of special numbers* $\beta_n(\overrightarrow{x_v}, \overrightarrow{y_v})$ *with degree* n *and order* $\overrightarrow{y_v}$ *by the following generating functions:*

$$\begin{aligned} h(w; \overrightarrow{x_v}, \overrightarrow{y_v}) &= \sum_{n=0}^{\infty} \beta_n(\overrightarrow{x_v}, \overrightarrow{y_v}) w^n \\ &= \prod_{j=0}^{v-1} (f(w) - x_j)^{y_j}. \end{aligned} \tag{10}$$

Observe that if we substitute $v = k+1$, $w = \lambda$, $f(\lambda) = \lambda$, $y_j = 1$ *and* $x_j = Y_j^{(-j)}(\lambda)$ *for* $j = 0, 1, \ldots, k$ *into* (10), *we have*

$$p_Y(\lambda) = \sum_{m=0}^{\infty} \beta_m(\overrightarrow{x_j}, \overrightarrow{y_j}) \lambda^m. \tag{11}$$

As a result of (11), *we conclude that the characteristic polynomial* $p_Y(\lambda)$ *of the matrix* Y *is the ordinary generating function of the numbers* $\beta_m(\overrightarrow{x_j}, \overrightarrow{y_j})$.

It should be noted here that the functions $h(w; \overrightarrow{x_v}, \overrightarrow{y_v})$ *has interesting connections with some concepts such as the characteristic polynomial of a complex matrix, the minimal polynomial over the set of all polynomials on vector space over a finite field, the chordal graph regarding the* w*-colouring of a graph (colored with no more than* w *colors), the chromatic functions, the chromatic polynomials and so on. The interested reader may refer to* [**21**] *and cited references therein to find further details about these connections.*

1.3. Some open problems concerning the matrix representations of $Y_n^{(-k)}(\lambda)$

Here, by taking into account the above observations on the matrix representations of $Y_n^{(-k)}(\lambda)$, we consider it to be necessary to motivate the interested reader by raising the following open questions:

Are there other matrix representations that accept $Y_n^{(-k)}(\lambda)$ as inputs?

What is the Hankel determinants of $Y_n^{(-k)}(\lambda)$?

Are there any family of orthogonal polynomials and corresponding three-term recurrence relations concerning the Hankel determinants of $Y_n^{(-k)}(\lambda)$? If there exist, which family of orthogonal polynomials?

Notice that any study will be conducted for achieving an answer to the above questions can serve as an introduction to a subject for the researchers working in the relevant areas and bring one to the frontiers of research conducted on the finite dimensional linear algebra in combinatorial aspects.

2. FORMULAE ARISING FROM AN ALTERNATE FORM OF THE GENERATING FUNCTIONS FOR $Y_n^{(-k)}(\lambda)$

In this section, we consider an alternate form of the generating functions for $Y_n^{(-k)}(\lambda)$. With the aid of these alternate generating functions, we derive multifarious formulae such as Rodrigues-type formula, recurrence relation and derivative formula concerning $Y_n^{(-k)}(\lambda)$. Furthermore, we provide some plots to examine the behaviour of the aforementioned alternate form of the generating functions for $Y_n^{(-k)}(\lambda)$ in some special cases. At the end of this section, we show that there exist a relationship between $Y_n^{(-k)}(\lambda)$ and the two-variable Hermite polynomials $H_k^{(j)}(x,y)$. We also present some applications of this relationship.

It is clear from the right-hand side of the equation (1) that the summation runs over the index n of $Y_n^{(k)}(\lambda)$. At this stage, the following question comes to our mind:

What is the explicit formula for the function f_Y given as the following formal power series:

$$f_Y(t,n;\lambda) = \sum_{k=0}^{\infty} Y_n^{(-k)}(\lambda)\frac{t^k}{k!} \tag{12}$$

whose coefficients give the sequence $\{Y_n^{(-k)}(\lambda)\}_{k=0}^{\infty}$, namely what happens when the summation runs over the index k of $Y_n^{(-k)}(\lambda)$ instead of its index n?

The above question allows us to glance at the generating functions from a different perspective. With this motivation, in this section, we first present a formula for the function $f_Y(t,n;\lambda)$. Note that the functions to be obtained in this way gives one generating function corresponding to each values of n.

The function $f_Y(t,n;\lambda)$ is given explicitly by the following theorem:

Theorem 1. *Let $n \in \mathbb{N}_0$ and $\lambda, t \in \mathbb{C}$. Then we have*

$$f_Y(t,n;\lambda) = \left(\frac{t\lambda^2}{2}\right)^n e^{t\left(\frac{\lambda-1}{2}\right)}. \tag{13}$$

Proof. Substituting (2) into the right-hand side of (12), we obtain

$$f_Y(t,n;\lambda) = \sum_{k=0}^{\infty}\left(2^{-k}n!\binom{k}{n}\lambda^{2n}(\lambda-1)^{k-n}\right)\frac{t^k}{k!}$$

which yields

$$f_Y(t,n;\lambda) = \left(\frac{\lambda^2}{2}\right)^n t^n \sum_{k=n}^{\infty}\frac{\left(\frac{\lambda-1}{2}\right)^{k-n} t^{k-n}}{(k-n)!}.$$

By using the Taylor series expansion of the function $e^{t\left(\frac{\lambda-1}{2}\right)}$, we thus arrive at the desired result. □

Observe that putting $n=0$ in (13), we get

$$f_Y(t,0;\lambda) = e^{t\left(\frac{\lambda-1}{2}\right)}.$$

2.1. Rodrigues-type formula, Derivative formula and recurrence relation for $Y_n^{(-k)}(\lambda)$

Here, we derive multifarious formulae such as Rodrigues-type formula, recurrence relation and derivative formula concerning $Y_n^{(-k)}(\lambda)$.

A Rodrigues-type formula for $Y_n^{(-k)}(\lambda)$ is given by the following theorem:

Theorem 2. *Let $k,n \in \mathbb{N}$. Then we have*

$$Y_n^{(-k)}(\lambda) = \lambda^{2n}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\frac{\lambda-1}{2}\right)^k.$$

Proof. By using (13), we derive the following partial differential equation:

$$\frac{\partial^n}{\partial\lambda^n}\left\{e^{t\left(\frac{\lambda-1}{2}\right)}\right\} = \lambda^{-2n} f_Y(t,n;\lambda).$$

By combining the Taylor series expansion of the function $e^{t\left(\frac{\lambda-1}{2}\right)}$ and (12) with the above equation, we have

$$\sum_{k=0}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\frac{\lambda-1}{2}\right)^k\frac{t^k}{k!} = \lambda^{-2n}\sum_{k=0}^{\infty}Y_n^{(-k)}(\lambda)\frac{t^k}{k!}.$$

With the comparison of the coefficients $\frac{t^k}{k!}$ on both sides of the above equation, we thus arrive at the desired result. □

Remark 4. *The numbers $Y_n^{(-k)}(\lambda)$ also satisfy the following differential equation:*

$$Y_n^{(-k)}(\lambda) = \frac{\partial^n}{\partial t^n}\left\{\mathcal{G}(t,k;\lambda)\right\}\Big|_{t=0}$$

which tells us that the following identity holds true:

$$\lambda^{2n}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\frac{\lambda-1}{2}\right)^k = \frac{\partial^n}{\partial t^n}\left\{\mathcal{G}(t,k;\lambda)\right\}\Big|_{t=0}.$$

Differentiating both sides of (13) with respect to λ, we get the following partial derivative equation:

$$\frac{\partial}{\partial\lambda}\{f_Y(t,n;\lambda)\} = \left(\frac{t}{2}+\frac{2n}{\lambda}\right)f_Y(t,n;\lambda). \tag{14}$$

On the other hand, if we differentiate both sides of the equation (13) with respect to t, we also get the following partial derivative equation:

$$\frac{\partial}{\partial t}\{f_Y(t,n;\lambda)\} = \left(\frac{n}{t}+\frac{\lambda-1}{2}\right)f_Y(t,n;\lambda). \tag{15}$$

The above partial derivative equations allow us to obtain not only a derivative formula, but also a recurrence relation for $Y_n^{(-k)}(\lambda)$.

Next, by the following theorem, we first give a derivative formula for $Y_n^{(-k)}(\lambda)$:

Theorem 3. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\{Y_n^{(-k)}(\lambda)\} = \frac{k}{2}Y_n^{(-k+1)}(\lambda) + \frac{2n}{\lambda}Y_n^{(-k)}(\lambda).$$

Proof. By substituting the right-hand side of (13) into (14), we have

$$\sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\lambda}\{Y_n^{(-k)}(\lambda)\}\frac{t^k}{k!} = \frac{t}{2}\sum_{k=0}^{\infty}Y_n^{(-k)}(\lambda)\frac{t^k}{k!} + \frac{2n}{\lambda}\sum_{k=0}^{\infty}Y_n^{(-k)}(\lambda)\frac{t^k}{k!}$$

which yields

$$\sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\lambda}\{Y_n^{(-k)}(\lambda)\}\frac{t^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{k}{2}Y_n^{(-k+1)}(\lambda) + \frac{2n}{\lambda}Y_n^{(-k)}(\lambda)\right)\frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the aforementioned equation yields the desired result. □

By the following theorem, we second give a recurrence relation for $Y_n^{(-k)}(\lambda)$:

Theorem 4. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\left(1-\frac{n}{k}\right)Y_n^{(-k)}(\lambda) = \frac{\lambda-1}{2}Y_n^{(-k+1)}(\lambda). \tag{16}$$

Proof. Substituting the right-hand side of (13) into (15), we have

$$\sum_{k=1}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^{k-1}}{(k-1)!} = \frac{n}{t} \sum_{k=0}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^k}{k!} + \frac{\lambda-1}{2} \sum_{k=0}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^k}{k!}.$$

which yields

$$\sum_{k=1}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^{k-1}}{(k-1)!} = \sum_{k=1}^{\infty} \left(\frac{n}{k} Y_n^{(-k)}(\lambda) + \frac{\lambda-1}{2} Y_n^{(-k+1)}(\lambda) \right) \frac{t^{k-1}}{(k-1)!}.$$

Comparing the coefficients of $\frac{t^{k-1}}{(k-1)!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

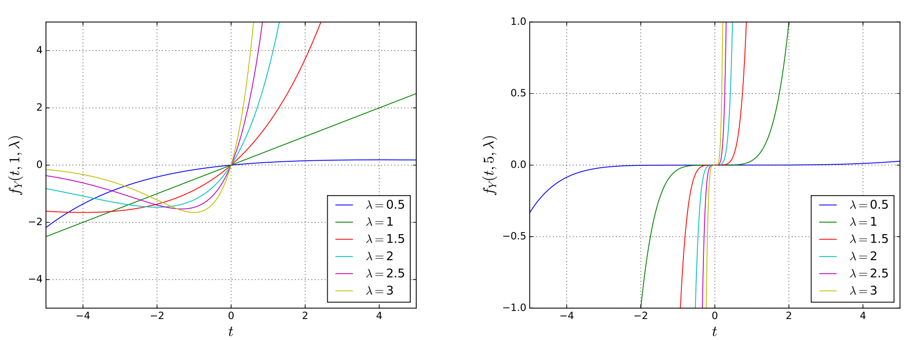
Remark 5. *As an application of* (16)*, by replacing n by k in* (16)*, we get*

$$Y_k^{(1-k)}(\lambda) = 0$$

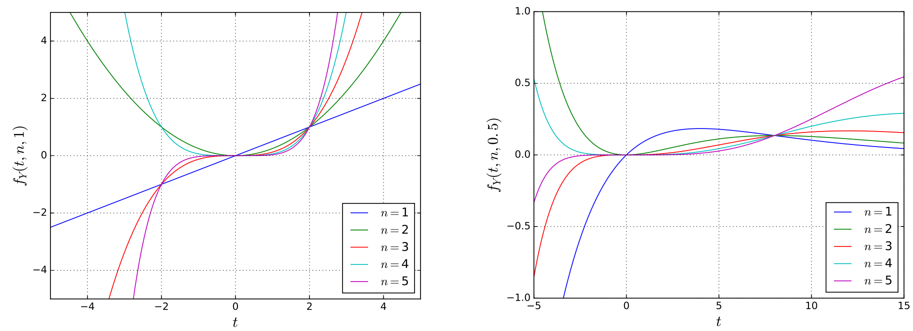
where $k \in \mathbb{N}$.

2.2. Evaluations on the graphs of the functions $f_Y(t, n; \lambda)$

Here, in order to illustrate some special cases of the functions $f_Y(t, n; \lambda)$, Figure 1 is provided. The graphs in Figure 1 allow us to examine how the functions $f_Y(t, n; \lambda)$ behaves in some specific cases.



(a) $n = 1$**,** $t \in [-5, 5]$ **and** $\lambda \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$**. (b)** $n = 5$**,** $t \in [-5, 5]$ **and** $\lambda \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$**.**



(c) $\lambda = 1$**,** $t \in [-5, 5]$ **and** $n \in \{1, 2, 3, 4, 5\}$**.** **(d)** $\lambda = \frac{1}{2}$**,** $t \in [-5, 5]$ **and** $n \in \{1, 2, 3, 4, 5\}$**.**

Figure 1. Graphs of the functions $f_Y(t, n; \lambda)$ **for some special cases.**

2.3. Relations between $Y_n^{(-k)}(\lambda)$ and the two-variable Hermite polynomials

Here, we give relations between $Y_n^{(-k)}(\lambda)$ and the two-variable Hermite polynomials $H_k^{(j)}(x, y)$ by the following theorem:

Theorem 5. *Let* $n, k \in \mathbb{N}_0$ *and* $j \in \mathbb{N}$*. Then we have*

$$H_k^{(j)}\left(\frac{\lambda - 1}{2}, y\right) = \frac{2^n k!}{\lambda^{2n}} \sum_{v=0}^{\left[\frac{k+n}{j}\right]} \frac{Y_n^{(-k-n+vj)}(\lambda)}{(k+n-vj)!} \frac{y^v}{v!}. \tag{17}$$

Proof. By (5) and (13), we set the following functional equation:

$$\left(\frac{2}{\lambda^2 t}\right)^n f_Y(t, n; \lambda) e^{yt^j} = F_H\left(t, \frac{\lambda - 1}{2}, y; j\right). \tag{18}$$

By combining (5) and (12) with the above equation, we get

$$\frac{2^n}{\lambda^{2n}} \sum_{k=0}^{\infty} \sum_{v=0}^{\left[\frac{k}{j}\right]} y^v Y_n^{(-k+vj)}(\lambda) \frac{t^k}{(k-vj)! v!} = \sum_{k=0}^{\infty} H_k^{(j)}\left(\frac{\lambda - 1}{2}, y\right) \frac{t^{n+k}}{k!}.$$

Thus we have

$$\frac{2^n}{\lambda^{2n}} \sum_{k=0}^{\infty} \sum_{v=0}^{\left[\frac{k}{j}\right]} \frac{k! y^v Y_n^{(-k+jv)}(\lambda)}{(k-vj)! v!} \frac{t^k}{k!} = \sum_{k=0}^{\infty} (k)_n H_{k-n}^{(j)}\left(\frac{\lambda - 1}{2}, y\right) \frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation yields the desired result. □

By the combination of (17) with (6), we achieve a combinatorial sum given by the following corollary:

Corollary 1. *Let* $n,k \in \mathbb{N}_0$ *and* $j \in \mathbb{N}$. *Then we have*

$$\frac{2^n}{\lambda^{2n}} \sum_{v=0}^{\left[\frac{k+n}{j}\right]} \frac{Y_n^{(-k-n+vj)}(\lambda)}{(k+n-vj)!} \frac{y^v}{v!} = \sum_{r=0}^{\left[\frac{k}{j}\right]} \frac{y^r}{r!\,(k-jr)!} \left(\frac{\lambda-1}{2}\right)^{k-jr}.$$

By the combination of (17) with (2), we also achieve the following corollary:

Corollary 2. *Let* $k \in \mathbb{N}_0$. *Then we have*

$$H_k^{(2)}\left(\frac{\lambda-1}{2}, y\right) = k! \left(\frac{\lambda-1}{2}\right)^k \sum_{v=0}^{\left[\frac{k}{2}\right]} \frac{1}{(k-2v)!v!} \left(\frac{4y}{\lambda^2-2\lambda+1}\right)^v.$$

As an application of the Corollary 2, we present the following formula for a combinatorial sum which gives the value of $H_k^{(2)}(-1,-1)$:

$$H_k^{(2)}(-1,-1) = (-1)^k\, k! \sum_{v=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^v}{(k-2v)!v!},$$

where $k \in \mathbb{N}_0$.

3. INTEGRAL REPRESENTATIONS FOR THE FUNCTIONS $f_Y(t,n;\lambda)$

In this section, we present some applications of integral transforms to the functions $f_Y(t,n;\lambda)$. With the application of the Laplace transform, we give an infinite series representation involving $Y_n^{(-k)}(\lambda)$. With the application of the Mellin transform, we present an interpolation function of $Y_n^{(-k)}(\lambda)$ at negative integers. Moreover, we give a contour integral representation of $Y_n^{(-k)}(\lambda)$.

3.1. Applications of the Laplace transform and Mellin transform to the functions $f_Y(t,n;\lambda)$

Considering an alternative form of (13), it can be rewritten in the following form:

$$e^{-\frac{t\lambda}{2}} f_Y(t,n;\lambda) = \left(\frac{\lambda^2}{2}\right)^n t^n e^{-\frac{t}{2}}.$$

Therefore, by (12), we have

$$e^{-\frac{t\lambda}{2}}\sum_{k=0}^{\infty}Y_n^{(-k)}(\lambda)\frac{t^k}{k!}=\left(\frac{\lambda^2}{2}\right)^n t^n e^{-\frac{t}{2}}.$$

By integrating both sides of the above equation, with respect to t, from zero to ∞, we have

$$\sum_{k=0}^{\infty}\frac{Y_n^{(-k)}(\lambda)}{k!}\int_0^{\infty}t^k e^{-\frac{t\lambda}{2}}\mathrm{d}t=\left(\frac{\lambda^2}{2}\right)^n\int_0^{\infty}t^n e^{-\frac{t}{2}}\mathrm{d}t. \tag{19}$$

By using the following Laplace transform of the function $f(t)=t^k;\ (t\geq 0)$:

$$\mathcal{L}\left[t^k\right]\left(\frac{\lambda}{2}\right)=\left(\frac{2}{\lambda}\right)^{k+1}k!$$

and

$$\mathcal{L}\left[t^n\right]\left(\frac{1}{2}\right)=2^{n+1}n!$$

in the equation (19), we obtain an infinite series representation, involving $Y_n^{(-k)}(\lambda)$, given by the following theorem:

Theorem 6. *Let $n\in\mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{\infty}\left(\frac{2}{\lambda}\right)^{k+1}Y_n^{(-k)}(\lambda)=2\lambda^{2n}n!$$

where $\left|\frac{\lambda-1}{\lambda}\right|<1$.

Next, we provide a new function which interpolates $Y_n^{(-k)}(\lambda)$ at negative integers.

Let $s\in\mathbb{C}$, and $\lambda\in\mathbb{R}$ with $\lambda\neq 1$. By applying the Mellin transform to (13), we give integral representation of the interpolation function $I_Y(s,n;\lambda)$ as follows:

$$\begin{aligned} I_Y(s,n;\lambda) &= \frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}f_Y(-t,n;\lambda)\,\mathrm{d}t \\ &= \frac{1}{\Gamma(s)}\left(-\frac{\lambda^2}{2}\right)^n\int_0^{\infty}t^{s+n-1}e^{-t\left(\frac{\lambda-1}{2}\right)}\mathrm{d}t \end{aligned} \tag{20}$$

where $\Gamma(s)$ denotes the Euler gamma function, $\min\{\mathrm{Re}(s+n),\mathrm{Re}\left(\frac{\lambda-1}{2}\right)\}>0$ and the additional constraint $\mathrm{Re}\left(\frac{\lambda-1}{2}\right)>0$ is required for the convergence of the infinite integral, which is given in the aforementioned integral at its upper terminal.

By using the integral representation given by (20), we define interpolation function of $Y_n^{(-k)}(\lambda)$ by the following definition:

Definition 1. *Let $n \in \mathbb{N}_0$, $s \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $\lambda \neq 1$. The interpolation function $I_Y(s,n;\lambda)$ is defined by*

$$I_Y(s,n;\lambda) = \frac{\Gamma(s+n)}{\Gamma(s)} \left(-\frac{\lambda^2}{\lambda-1}\right)^n \left(\frac{2}{\lambda-1}\right)^s .$$

Theorem 7. *Let $m \in \mathbb{N}$. Then we have*

$$I_Y(-m,n;\lambda) = Y_n^{(-m)}(\lambda).$$

3.2. Contour integral representation of $Y_n^{(-k)}(\lambda)$

Here, we present a contour integral representation of $Y_n^{(-k)}(\lambda)$.

With the aid of the Cauchy residue theorem and by using the same method of Lopez and Temme [**11**], the Contour integral representation of $Y_n^{(-k)}(\lambda)$ is given by the following theorem:

Theorem 8. *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_n^{(-k)}(\lambda) = \frac{n!}{2\pi i} \int_{\mathcal{C}} f_Y(z,n;\lambda) \frac{\mathrm{d}z}{z^{k+1}},$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction.

4. RELATIONS OF THE FUNCTIONS $f_Y(t,n;\lambda)$ WITH THE NUMBER OF CYCLIC DERANGEMENTS

In this section, we show that there exist a relationship between $Y_n^{(-k)}(\lambda)$ and the number of cyclic derangements. We present some applications of this relationship. We also construct exponential generating functions for a family of numbers arising from the linear combination of the numbers of cyclic derangements in the wreath product of the finite cyclic group and the symmetric group of permutations of a set.

In [**2**, Theorem 2.1, p. 3], Assaf introduced and investigated the number of cyclic derangements in the wreath product of the finite cyclic group of order r and the symmetric group of permutations of a set of n objects by the following formula:

$$d_n^{(r)} = r^n n! \sum_{j=0}^{n} \frac{(-1)^j}{r^j j!},$$

and the exponential generating function for the number $d_n^{(r)}$ of cyclic derangements is given by

$$\frac{e^{-x}}{1-rx} = \sum_{n=0}^{\infty} d_n^{(r)} \frac{x^n}{n!}, \tag{21}$$

(*cf.* [**2**, p. 5]).

Remark 6. *For interesting combinatorial interpretations regarding the number of derangements of a finite set, the interested reader may consult* [**2**], [**8**] *and* [**13**]*; and also cited references therein.*

By using (13), we get the following functional equation:

$$\mathcal{F}_K(t;\lambda) := \frac{e^{t\left(\frac{\lambda-1}{2}\right)}}{1-\frac{\lambda^2}{2}t} = \sum_{n=0}^{\infty} f_Y(t,n;\lambda), \tag{22}$$

where $|\frac{\lambda^2 t}{2}| < 1$.

By (21) and (22), we have

$$\begin{aligned}\sum_{n=0}^{\infty} f_Y(t,n;\lambda) &= \frac{e^{-\frac{t}{2}}}{1-\frac{\lambda^2}{2}t} e^{\frac{t\lambda}{2}} \\ &= \left(\sum_{n=0}^{\infty} \frac{d_n^{(\lambda^2)}}{2^n}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\right)^n \frac{t^n}{n!}\right)\end{aligned}$$

such that $\lambda^2 \in \mathbb{N}$.

By using the Cauchy product rule in the above equation, we obtain

$$\sum_{n=0}^{\infty} f_Y(t,n;\lambda) = \sum_{n=0}^{\infty}\left(\left(\frac{1}{2}\right)^n \sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j} d_j^{(\lambda^2)}\right)\frac{t^n}{n!}.$$

Next, we set

$$K_n(\lambda) := \left(\frac{1}{2}\right)^n \sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j} d_j^{(\lambda^2)} \tag{23}$$

such that $\lambda^2 \in \mathbb{N}$.

Observe that the numbers $K_n(\lambda)$, defined in (23), is the linear combination of the number of cyclic derangements in the wreath product of the finite cyclic group of order λ^2 and the symmetric group of permutations of a set of j objects for $j = 0, \ldots, n$.

It follows from (23) that

$$\sum_{n=0}^{\infty} K_n(\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} f_Y(t,n;\lambda). \tag{24}$$

Therefore, by (22) and (24), we arrive at the following theorem which gives exponential generating function for the numbers $K_n(\lambda)$:

Theorem 9. *Let $\lambda^2 \in \mathbb{N}$. Then we have*

$$\mathcal{F}_K(t;\lambda) = \sum_{n=0}^{\infty} K_n(\lambda)\frac{t^n}{n!}.$$

5. PROBABILISTIC AND ASYMPTOTIC ANALYSES OF THE FUNCTIONS $f_Y(t, n; \lambda)$ AND $\mathcal{F}_K(t; \lambda)$

In this section, we analyse the functions $f_Y(t, n; \lambda)$ in probabilistic and asymptotic manners. As a result of this analysis, we give some relations of the functions $f_Y(t, n; \lambda)$ with not only the Laplace distribution, but also the standard normal distribution. In addition, we provide an asymptotic power series representation for the function $\mathcal{F}_K(t; \lambda)$ which is the exponential generating function for the numbers $K_n(\lambda)$.

By using (13), we have the following functional equation:

$$\sum_{n=0}^{\infty} (-1)^n f_Y(t, n; \lambda) = \frac{e^{t\left(\frac{\lambda-1}{2}\right)}}{1 + \frac{\lambda^2}{2} t}, \tag{25}$$

where $|\frac{\lambda^2 t}{2}| < 1$.

Multiplying (22) and (25) yields

$$\sum_{n=0}^{\infty} f_Y(t, n; \lambda) \sum_{n=0}^{\infty} (-1)^n f_Y(t, n; \lambda) = \frac{e^{t(\lambda-1)}}{1 - \frac{\lambda^4}{4} t}.$$

By applying the Cauchy product rule to the above equation, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j f_Y(t, j; \lambda) f_Y(t, n-j; \lambda) &= \frac{e^{t(\lambda-1)}}{1 - \frac{\lambda^4}{4} t} \\ &= \mathrm{MGF}\left(\mathrm{L}\left(\lambda - 1, \frac{\lambda^2}{2}\right)\right) \end{aligned}$$

where $|t| < \frac{2}{\lambda^2}$ and $\mathrm{MGF}\left(\mathrm{L}\left(\lambda - 1, \frac{\lambda^2}{2}\right)\right)$ denotes the moment generating function of the Laplace distribution with $\lambda - 1$ (location parameter) and $\frac{\lambda^2}{2}$ (scale parameter) with real λ parameter.

On the other hand, substituting $j = 2$, $y = -\frac{1}{2}$ and $\lambda = \frac{1}{\sqrt[4n]{2\pi}}$, the equation (18) yields the standard normal distribution with mean 0 and variance 1 as follows:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = \left(\frac{2}{t}\right)^n e^{-\frac{t}{2}\left(\frac{1}{\sqrt[4n]{2\pi}} - 1\right)} f_Y\left(t, n; \frac{1}{\sqrt[4n]{2\pi}}\right),$$

where $t \in (-\infty, \infty)$. The right-hand side of the above equation gives us a modification of the standard normal distribution.

As for asymptotic analyse of the function $\mathcal{F}_K(t; \lambda)$, it is well-known that asymptotic power series representation of a function $f(z)$ is given as follows:

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n \tag{26}$$

as $z \to 0$ in some region R if and only if

(27) $$\lim_{z\to 0 \text{ in } R} \frac{|f(z) - s_n(z)|}{|z|^n} = 0,$$

for each fixed n, with

$$s_n(z) = \sum_{k=0}^{n} a_k z^k$$

(*cf.* [**14**, p. 33, Eq.(3)-(5)]).

By using (26), we have the following asymptotic representation:

(28) $$\int_0^\infty \frac{e^{-t}\mathrm{d}t}{1-xt} \sim \sum_{n=0}^{\infty} n! x^n$$

as $x \to 0$ in $\mathrm{Re}(x) \leq 0$ (*cf.* [**14**, p. 34]).

Let $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. By substituting $t = \frac{1-\lambda}{2}u$ into (28), we have

$$\frac{1-\lambda}{2} \int_0^\infty \frac{e^{-u\left(\frac{1-\lambda}{2}\right)}}{1-\frac{\lambda^2}{2}u} \mathrm{d}u \sim \sum_{n=0}^{\infty} n! \left(\frac{\lambda^2}{1-\lambda}\right)^n$$

as $\lambda \to 0$ in $\mathrm{Re}\left(\frac{\lambda^2}{1-\lambda}\right) \leq 0$.

Thus, we get the following theorem:

Theorem 10. *Asymptotic power series representation of the function* $\mathcal{F}_K(u;\lambda)$ *is given by*

(29) $$\int_0^\infty \mathcal{F}_K(u;\lambda)\,\mathrm{d}u \sim \frac{2}{1-\lambda} \sum_{n=0}^{\infty} n! \left(\frac{\lambda^2}{1-\lambda}\right)^n$$

as $\lambda \to 0$ *in* $\mathrm{Re}\left(\frac{\lambda^2}{1-\lambda}\right) \leq 0$.

Let $o(x)$ denote the Little-o notation (*cf.* [**14**, p. 13]). By using (27) with $o(x)$, we have another expression of (29) by the following corollary:

Corollary 3.

$$\left| \int_0^\infty \mathcal{F}_K(u;\lambda)\,\mathrm{d}u - \frac{2}{1-\lambda} \sum_{n=0}^{\infty} n! \left(\frac{\lambda^2}{1-\lambda}\right)^n \right| \leq (n+1)! \left| \frac{\lambda^2}{1-\lambda} \right|^{n+1}$$

which means for fixed n *the following holds true:*

$$\int_0^\infty \mathcal{F}_K(u;\lambda)\,\mathrm{d}u - \frac{2}{1-\lambda} \sum_{n=0}^{\infty} n! \left(\frac{\lambda^2}{1-\lambda}\right)^n = o\left(\left(\frac{\lambda^2}{1-\lambda}\right)^n\right).$$

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