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COUNTING PATTERN AVOIDING PERMUTATIONS BY NUMBER OF MOVABLE LETTERS

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By a *movable* letter within a pattern avoiding permutation, we mean one that may be transposed with its predecessor while still avoiding the pattern. In this paper, we enumerate permutations avoiding a single pattern of length three according to the number of movable letters, thereby obtaining new q-analogues of the Catalan number sequence. Indeed, we consider the joint distribution with the statistics recording the number of descents and occurrences of certain vincular patterns. To establish several of our results, we make use of the kernel method to solve the functional equations that arise.

1. INTRODUCTION

A permutation $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$ *contains* the pattern $\tau = \tau_1 \cdots \tau_m \in \mathcal{S}_m$ where $m \leq n$ in the classical sense if π contains a subsequence that is isomorphic to τ. Otherwise, π is said to *avoid* τ. A *vincular pattern* [1] is one containing dashes separating some of the elements and is of the form $\tau = \alpha_1\text{-}\alpha_2\text{-}\cdots\text{-}\alpha_r$, where the α_i represent disjoint non-empty subsets of distinct letters whose union is $[m] = \{1, 2, \ldots, m\}$. Then π is said to contain the vincular pattern τ if it contains some subsequence that is isomorphic to τ in which the entries of π corresponding to each α_i are required to be consecutive, and avoids τ otherwise. See [12, Chapter 7] for a complete list of results concerning avoidance of vincular patterns.

Note that a classical pattern then corresponds to a vincular pattern in which each α_i is of length one, while a consecutive pattern (i.e., subword) corresponds to

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a vincular where $r = 1$. Here, we will denote classical patterns without dashes as we will not be discussing avoidance of subwords. Given any set T of patterns, let $\mathcal{S}_n(T)$ denote the subset of $\mathcal{S}_n$ whose members avoid each pattern in T. In the case where T is a singleton, one often indicates the set by just writing the pattern in question. Recall from [16] that $|\mathcal{S}_n(\tau)| = C_n$ for all $\tau \in \mathcal{S}_3$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the n-th Catalan number (see [17, A000108]). Let $\mathcal{S}_{n,i}(\tau)$ for $1 \leq i \leq n$ denote the subset of $\mathcal{S}_n(\tau)$ whose members have final letter i.

The problem of enumerating the members of $\mathcal{S}_n(\tau)$ for some pattern τ according to a combinatorial statistic has been a frequent object of research. See, e.g., [2,3,8,11–15] and references contained therein. Note that such an enumeration leads to a polynomial generalization of the underlying counting sequence $|\mathcal{S}_n(\tau)|$. Here, we undertake an enumeration of $\mathcal{S}_n(\tau)$, where $\tau \in \mathcal{S}_3$, according to a new parameter which tracks certain kinds of adjacencies. In fact, we consider a joint distribution of this parameter with statistics recording the number of descents and occurrences of certain vincular patterns. This leads to new (multivariate) polynomial generalizations of the Catalan number sequence.

Our work also extends prior results of Burstein and Elizalde [6] concerning the total number of occurrences of a vincular pattern of length three over all members of $\mathcal{S}_n(\tau)$, where $\tau \in \mathcal{S}_3$. Indeed, we find recurrences and generating function formulas for the entire distribution of the statistic on $\mathcal{S}_n(\tau)$ that records the number of occurrences of a vincular pattern in several instances from which the totals of these statistics over $\mathcal{S}_n(\tau)$ may be deduced as a corollary. For other results in this direction, we refer the reader to the paper of Vajnovszki [18], where some bijective results involving the equidistribution of vincular patterns are given, and of Homberger [9], where totals over $\mathcal{S}_n(\tau)$ for the number of occurrences of classical patterns of length three are found. See also [4, 5] for further related results.

We now define the new statistic on $\mathcal{S}_n(\tau)$. By a *movable* letter within $\pi \in \mathcal{S}_n(\tau)$, we mean one that may be transposed with its predecessor while still avoiding τ. Letters that cannot be moved in this way will be referred to as *immovable*. The first letter of any permutation will always be taken to be immovable. Let $\mu(\pi)$ denote the number of movable letters in π. For example, if $\pi = 912786453 \in \mathcal{S}_9(213)$, then we have $\mu(\pi) = 4$ since each of 1, 8, 4 and 5 may be transposed with its predecessor without introducing an occurrence of 213. Note that 2 and 7, which are immovable, both correspond to the second entry within an occurrence of 12-3, whereas the immovable letters 6 and 3 both correspond to the third entry within a 2-31. More generally, immovable letters in $\pi \in \mathcal{S}_n(213)$ that do not start a permutation correspond to the second entry of a 12-3 or the third entry of a 2-31, with these options seen to be mutually exclusive. Similar remarks apply when avoiding other patterns τ.

This paper is divided as follows. In the next section, we consider the movable letter statistic on $\mathcal{S}_n(213)$ and compute a recurrence for its joint distribution with the statistics recording the number of descents, occurrences of 2-31 and adjacencies xy such that x and y are the last two letters in an occurrence of 2-31. An explicit formula is computed for the generating function of the joint distribution on $\mathcal{S}_n(213)$

as well as for the one corresponding to each statistic taken separately. From these, simple closed form expressions for the totals of the various statistics may be derived. In the third section, a similar treatment is provided for the μ statistic on $\mathcal{S}_n(123)$ and the parameters recording the number of descents and occurrences of 1-32. In the fourth section, a comparable distribution is considered on $\mathcal{S}_n(132)$ involving μ and the pattern 1-23. Combinatorial proofs are provided in the final section for the totals of some of the statistics, including μ, on the avoidance classes in question and for a related equidistribution result.

Note that μ, when taken alone, is equally distributed on $\mathcal{S}_n(\tau)$ for τ =132, 213, 231 or 312 (as seen upon applying the reversal and/or the complementation operations). However, its joint distribution with the statistics on $\mathcal{S}_n(213)$ considered in the second section will be seen to differ from the one obtained from the statistics on $\mathcal{S}_n(132)$ considered in the fourth. Moreover, μ itself has a different distribution on $\mathcal{S}_n(123)$ than it does on $\mathcal{S}_n(132)$. To establish several of our results, we will make use of various refinements and restrictions of the avoidance class in question. This leads to functional equations that are satisfied by the related generating functions, which can often be solved explicitly using the *kernel method* [10]. In our combinatorial proofs, we will make use of the Simion-Schmidt bijection from [16] as well as a well-known bijection (see [7]) between $\mathcal{S}_n(132)$ and Dyck paths.

2. JOINT DISTRIBUTION OVER $\mathcal{S}_N(213)$

In this section, we consider the joint distribution of μ on $\mathcal{S}_n(213)$ with some closely related statistics. Let $a_n = a_n(p,q,s,t)$ denote the joint distribution on $\mathcal{S}_n(213)$ for the following four statistics marked by p, q, s and t, respectively: (i) number of adjacencies xy such that the x and y serve as the 3 and 1 in an occurrence of 2-31, (ii) movable letters, (iii) descents, and (iv) occurrences of 2-31. Let $a_{n,i} = a_{n,i}(p,q,s,t)$ denote the restriction of a_n to $\mathcal{S}_{n,i}(213)$ for $1 \le i \le n$, with $a_{n,0} = 0$. Note that $a_n = \sum_{i=1}^{n} a_{n,i}$, by the definitions.

The $a_{n,i}$ are determined recursively as follows.

Lemma 1. *If $n \ge 2$ and $1 \le i \le n-1$, then*

$$a_{n,i} = qsa_{n-1,i} + ps\sum_{\ell=i+1}^{n-1} a_{n-1,\ell}t^{\ell-i} + q^2s\sum_{j=1}^{i-1} a_{n-j-1,i-j} + pqs\sum_{j=1}^{i-1}\sum_{\ell=i+1}^{n-1} a_{n-j-1,\ell-j}t^{\ell-i}, \tag{1}$$

with $a_{n,n} = q$ for $n \ge 2$ and $a_{1,1} = 1$.

Proof. If $i = n$, then there is a single member of $\mathcal{S}_{n,i}(213)$, namely $12\cdots n$, which contains no occurrences of 2-31 with only the last letter movable for $n > 1$. So assume $1 \le i \le n-1$ and let $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_{n,i}(213)$. Note that if the last two letters of π form a descent, then the penultimate letter of π must be $i-1$, for otherwise π would contain 213. Let $\ell+1$ denote the rightmost letter of π that is greater than i where $i \leq \ell \leq n-1$. Then we may write $\pi = \pi'(\ell+1)(i-j)(i-j+1)\cdots(i-1)i$ for some $0 \leq j \leq i-1$. We consider now cases based on j and ℓ. If $j=0$ and $\ell=i$, then $\pi = \pi'(i+1)i$ and the i which is movable and responsible for a descent may be deleted, yielding $qsa_{n-1,i}$ possibilities. If $j=0$ and $\ell \geq i+1$, then each element of $[i+1,\ell]$ is the first letter in an occurrence of 2-31 in which i is the last. Since i is extraneous concerning avoidance of 213 due to a larger predecessor, it may once again be deleted, which yields $ps\sum_{\ell=i+1}^{n-1} a_{n-1,\ell}t^{\ell-i}$ possibilities.

If $j \geq 1$ and $\ell = i$, then both $i-j$ and i are seen to be movable (note that $i-j$ is movable since no element strictly between $i-j$ and $i+1$ occurs to the left of $i+1$). Furthermore, no element of $[i-j+1, i-1]$ is movable, for otherwise a 213 is introduced with i corresponding to the 3. As there is a single descent between $i+1$ and $i-j$, deleting all elements of $[i-j, i]$ in this case gives $q^2s\sum_{j=1}^{i-1} a_{n-j-1,i-j}$ additional possibilities. If $j \geq 1$ and $\ell \geq i+1$, then i is movable but $i-j$ is not, with the adjacency $(\ell+1)(i-j)$ corresponding to the last two letters in exactly $\ell - i$ occurrences of 2-31. Deleting the final $j+1$ letters of π and considering all j and ℓ in this case gives the last sum on the right side of (1) and completes the proof. □

Let $a_n(w) = a_n(w; p, q, s, t)$ be given by $a_n(w) = \sum_{i=1}^{n} a_{n,i}w^i$ for $n \geq 1$. We have the following recurrence formula for $a_n(w)$.

Lemma 2. *If $n \geq 3$, then*

$$a_n(w) = (qs+w)a_{n-1}(w) + q(q-1)swa_{n-2}(w) + \frac{ps}{t-w}(wa_{n-1}(t) - ta_{n-1}(w))$$
$$+ \frac{p(q-1)sw}{t-w}(wa_{n-2}(t) - ta_{n-2}(w)), \qquad (2)$$

with $a_1(w) = w$ and $a_2(w) = qsw + qw^2$.

Proof. The initial conditions may be verified using the definitions, so assume $n \geq 3$. Replacing n by $n-1$ and i by $i-1$ in (1), and subtracting, yields

$$a_{n,i} - a_{n-1,i-1} = qs(a_{n-1,i} - a_{n-2,i-1}) + ps\sum_{\ell=i+1}^{n-1} a_{n-1,\ell}t^{\ell-i} - ps\sum_{\ell=i}^{n-2} a_{n-2,\ell}t^{\ell-i+1}$$
$$+ q^2s\sum_{j=1}^{i-1} a_{n-j-1,i-j} - q^2s\sum_{j=1}^{i-2} a_{n-j-2,i-j-1}$$
$$+ pqs\sum_{j=1}^{i-1}\sum_{\ell=i+1}^{n-1} a_{n-j-1,\ell-j}t^{\ell-i} - pqs\sum_{j=1}^{i-2}\sum_{\ell=i}^{n-2} a_{n-j-2,\ell-j}t^{\ell-i+1}.$$

Replacing j by $j-1$ in the second subtracted sum and replacing both ℓ by $\ell-1$ and j by $j-1$ in the final subtracted sum, and observing the cancellations that result, one gets

$$a_{n,i} = qsa_{n-1,i} + a_{n-1,i-1} + q(q-1)sa_{n-2,i-1} + ps\sum_{\ell=i+1}^{n-1} a_{n-1,\ell}t^{\ell-i} - ps\sum_{\ell=i}^{n-2} a_{n-2,\ell}t^{\ell-i+1} + pqs\sum_{\ell=i+1}^{n-1} a_{n-2,\ell-1}t^{\ell-i}, \tag{3}$$

for $2 \leq i \leq n-1$, with

$$a_{n,1} = qsa_{n-1,1} + ps\sum_{\ell=2}^{n-1} a_{n-1,\ell}t^{\ell-1}, \qquad n \geq 2. \tag{4}$$

Multiplying both sides of (3) by w^i, summing over $2 \leq i \leq n-1$ and adding w times equation (4) implies

$$\begin{aligned}
a_n(w) &= qw^n + qsa_{n-1}(w) + \sum_{i=2}^{n-1} a_{n-1,i-1}w^i + q(q-1)swa_{n-2}(w) \\
&\quad + ps\sum_{\ell=2}^{n-1} a_{n-1,\ell}t^\ell \sum_{i=1}^{\ell-1}\left(\frac{w}{t}\right)^i - ps\sum_{\ell=2}^{n-2} a_{n-2,\ell}t^{\ell+1}\sum_{i=2}^{\ell}\left(\frac{w}{t}\right)^i \\
&\quad + pqs\sum_{\ell=3}^{n-1} a_{n-2,\ell-1}t^\ell \sum_{i=2}^{\ell-1}\left(\frac{w}{t}\right)^i \\
&= qw^n + qsa_{n-1}(w) + w(a_{n-1}(w) - qw^{n-1}) + q(q-1)swa_{n-2}(w) \\
&\quad + \frac{ps}{t-w}\sum_{\ell=1}^{n-1} a_{n-1,\ell}t^{\ell+1}\left(\frac{w}{t} - \left(\frac{w}{t}\right)^\ell\right) \\
&\quad - \frac{ps}{t-w}\sum_{\ell=1}^{n-2} a_{n-2,\ell}t^{\ell+2}\left(\left(\frac{w}{t}\right)^2 - \left(\frac{w}{t}\right)^{\ell+1}\right) \\
&\quad + \frac{pqs}{t-w}\sum_{\ell=2}^{n-1} a_{n-2,\ell-1}t^{\ell+1}\left(\left(\frac{w}{t}\right)^2 - \left(\frac{w}{t}\right)^{\ell}\right) \\
&= (qs+w)a_{n-1}(w) + q(q-1)swa_{n-2}(w) + \frac{ps}{t-w}(wa_{n-1}(t) - ta_{n-1}(w)) \\
&\quad + \frac{p(q-1)s}{t-w}\sum_{\ell=1}^{n-2} a_{n-2,\ell}t^{\ell+2}\left(\left(\frac{w}{t}\right)^2 - \left(\frac{w}{t}\right)^{\ell+1}\right) \\
&= (qs+w)a_{n-1}(w) + q(q-1)swa_{n-2}(w) + \frac{ps}{t-w}(wa_{n-1}(t) - ta_{n-1}(w)) \\
&\quad + \frac{p(q-1)sw}{t-w}(wa_{n-2}(t) - ta_{n-2}(w)),
\end{aligned}$$

as desired. □

Let $f(x;w) = f(x;w;p,q,s,t)$ be given by

$$f(x;w) = \sum_{n\geq 1} a_n(w)x^n.$$

Then multiplying both sides of (2) by x^n, and summing over $n \geq 3$, gives

$$f(x;w) - wx - (qsw + qw^2)x^2 = (qs+w)x(f(x;w) - wx) + q(q-1)swx^2 f(x;w) + \frac{psx + p(q-1)swx^2}{t-w}(wf(x;t) - tf(x;w)),$$

which may be rewritten as

$$\left(1 - x(w + qs + q(q-1)swx) + \frac{pstx(1+(q-1)wx)}{t-w}\right) f(x;w) = wx + (q-1)w^2x^2 + \frac{pswx(1+(q-1)wx)}{t-w} f(x;t). \tag{5}$$

Let $C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$ denote the generating function for the Catalan numbers. To solve the functional equation (5), we apply the kernel method and let $w = w_0$ be given by

$$\frac{t(1+(p-q)sx)}{1+(t-qs)x+(1-q)(p-q)stx^2} C\left(\frac{tx(1+(p-q)sx)(1-q(1-q)sx)}{(1+(t-qs)x+(1-q)(p-q)stx^2)^2}\right),$$

which is seen to cancel out the left-hand side. Solving for $f(x;t)$ then yields

$$f(x;t) = \frac{w_0 - t}{ps}.$$

Hence, by (5), we obtain the following result.

Theorem 1. *The generating function $f(x;w) = f(x;w;p,q,s,t)$ is given by*

$$f(x;w) = \frac{wx(1+(q-1)wx)(w-w_0)}{(1-x(w+qs+q(q-1)swx))(w-t) - pstx(1+(q-1)wx)},$$

where $w_0 = \frac{t(1+(p-q)sx)}{1+(t-qs)x+(1-q)(p-q)stx^2} C\left(\frac{tx(1+(p-q)sx)(1-q(1-q)sx)}{(1+(t-qs)x+(1-q)(p-q)stx^2)^2}\right)$.

Considering p, q and t separately in Theorem 1, setting all other parameters equal to one and simplifying, gives the following generating function formulas for the distributions of the statistics marked by the respective variables.

Corollary 1. *We have*

$$\begin{aligned}
&f(x;1;p,1,1,1) = \frac{1-(1-p)x}{p} C\left(x(1-(1-p)x)\right) - \frac{1}{p}\\
&= x + 2x^2 + (p+4)x^3 + (6p+8)x^4 + (2p^2+24p+16)x^5 + \cdots,\\
&f(x;1;1,q,1,1)\\
&= \frac{1+(1-q)x}{1+(1-q)x+(1-q)^2x^2} C\left(\frac{x(1+(1-q)x)(1-q(1-q)x)}{(1+(1-q)x+(1-q)^2x^2)^2}\right) - 1\\
&= x + 2qx^2 + q(3q+2)x^3 + q(5q^2+6q+3)x^4 + q(8q^3+17q^2+12q\\
&\quad + 5)x^5 + \cdots,\\
&f(x;1;1,1,1,t) = \frac{x - \frac{tx}{1-(1-t)x} C\left(\frac{tx}{(1-(1-t)x)^2}\right)}{1-t-(2-t)x}\\
&= x + 2x^2 + (t+4)x^3 + (t^2+5t+8)x^4 + (t^3+8t^2+17t+16)x^5 + \cdots.
\end{aligned}$$

Remarks: Taking the limit as $p \to 0$ in the first formula in Corollary 1 and noting the Catalan transform $C(x(1-x)) = \frac{1}{1-x}$, or taking $t = 0$ in the third, gives in either case $\frac{x}{1-2x}$. This is in accord with the fact that there are 2^{n-1} members of $\mathcal{S}_n(213, 2\text{-}31)$ for $n \geq 1$, which can be shown directly. Taking $q = 0$ in the second formula of the prior corollary, and noting $C\left(\frac{x(1+x)}{(1+x+x^2)^2}\right) = 1 + x + x^2$, yields $f(x;1;1,0,1,1) = x$, which implies that every member of $\mathcal{S}_n(213)$ contains a movable letter for $n \geq 2$. This may be realized directly by noting that the second letter is always movable if n starts a permutation, with n itself movable if it does not start a permutation. Finally, note that the generating function $f(x;1;p,1,1,1)$ may also be written as $\frac{x}{1-2x} C\left(\frac{px^2}{(1-2x)^2}\right)$. Extracting the coefficient of $x^n p^m$ in the last formula, and equating with the equivalent expression obtained from Corollary 1, yields the following identity for $m \geq 0$ and $n \geq 2m+1$:

$$(-1)^{n-m-1} 2^{n-2m-1} \binom{n-1}{2m} C_m = \sum_{i=0}^{n-m-1} (-1)^i \binom{i+1}{n-i} \binom{n-i}{m+1} C_i,$$

where the absolute value of both sides gives the number of members of $\mathcal{S}_n(213)$ that have exactly m adjacencies serving as the last two letters in an occurrence of 2-31.

From the prior formulas, we obtain the following further result.

Corollary 2. *The total number of (a) adjacencies xy such that the x and y serve as the last two letters in an occurrence of* 2-31*, (b) movable letters, and (c) occurrences* *of* 2-31 *over all members of* $\mathcal{S}_n(213)$ *for* $n \geq 1$ *is given respectively by*

$$(a) \quad \binom{2n-2}{n-1} - \frac{1}{n+1}\binom{2n}{n},$$
$$(b) \quad \binom{2n}{n} - 2\binom{2n-2}{n-1},$$
$$(c) \quad \frac{5}{2}\binom{2n}{n} - \frac{5}{2}\binom{2n+2}{n+1} + \frac{1}{2}\binom{2n+4}{n+2}.$$

Proof. Partially differentiating the formulas in Corollary 1 with respect to the variable marking the statistic in question in each case gives

$$\frac{\partial}{\partial p} f(x;1;p,1,1,1)\mid_{p=1} = 1 + \frac{x}{\sqrt{1-4x}} - C(x),$$
$$\frac{\partial}{\partial q} f(x;1;1,q,1,1)\mid_{q=1} = -1 + \frac{1-2x}{\sqrt{1-4x}},$$
$$\frac{\partial}{\partial t} f(x;1;1,1,1,t)\mid_{t=1} = -\frac{1}{2} + \frac{3}{2x} - \frac{1}{2x^2} + \frac{1-5x+5x^2}{2x^2\sqrt{1-4x}}.$$

Extracting the coefficient of x^n in each case, using the fact $\sum_{n\geq 0}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$, yields the stated formulas. □

Remark: An equivalent expression to the third of the prior corollary in terms of the number of occurrences of 2-13 over $\mathcal{S}_n(231)$ was found in [6, Corollary 3.10], using a different method, though no recurrence or generating function is given there for the distribution of the statistic on $\mathcal{S}_n(231)$ recording the number of such occurrences.

We conclude this section with some further observations concerning the distribution $a_n(w)$. Given $\pi \in \mathcal{S}_n(213)$, let $\mu_1(\pi)$ and $\mu_2(\pi)$ denote the number of adjacencies xy such that x and y are the last two letters in an occurrence of 2-31 within π or the first two letters in an occurrence of 12-3, respectively. Then we have $\mu(\pi) + \mu_1(\pi) + \mu_2(\pi) = n - 1$. To see this, note that within every adjacency xy, either y is movable or is immovable because $x > y$ and transposing them would create a 213 of the form ayx or is immovable because $x < y$ and an occurrence of 213 of the form yxb would be created. Note that in the first case of y being immovable, the adjacency in question is of the form enumerated by μ_1, whereas in the latter, it is enumerated by μ_2. Since the three options above are seen to be mutually exclusive and exhaustive, the stated formula follows. Thus, the u-variable in the joint distribution polynomial $u^{n-1}a_n\left(w; \frac{p}{u}, \frac{q}{u}, s, t\right)$ for $n \geq 1$ marks the number of adjacencies serving as the first two letters in an occurrence of the pattern 12-3.

Upon applying the reversal and complementation operations, one sees that the movable letter statistic is equally distributed on $\mathcal{S}_n(\tau)$ for $\tau = 132, 213, 231, 312$. In particular, taking reverse complements, one has that $a_n(w)$ is the joint distribution on $\mathcal{S}_n(132)$ for the same statistics as before except that the p and t variables now pertain to occurrences of the pattern 31-2. A similar remark concerning $a_n(w)$

applies to the patterns 231 and 312, where now p and t pertain to 2-13 and 13-2, respectively, and s tracks ascents instead of descents.

3. DISTRIBUTION ON $\mathcal{S}_N(123)$

Here, we consider a comparable multivariate distribution on $\mathcal{S}_n(123)$. Let $b_n = b_n(p,q,s,t)$ denote the joint distribution on $\mathcal{S}_n(123)$ for the following four statistics marked by p, q, s and t, respectively: (i) number of adjacencies xy such that the x and y serve as the 3 and 2 in an occurrence of 1-32, (ii) movable letters, (iii) descents, and (iv) occurrences of 1-32. Let $b_{n,i} = b_{n,i}(p,q,s,t)$ denote the restriction to $\mathcal{S}_{n,i}(123)$ for $1 \le i \le n$.

The $b_{n,i}$ are determined by the following recursion.

Lemma 3. *If $n \geq 3$, then*

$$(6)\quad b_{n,i} = pst^{i-1}\sum_{\ell=i+1}^{n} b_{n-1,\ell-1} + q^2\sum_{j=1}^{i-1}\sum_{\ell=i+1}^{n} b_{n-j-1\ell-j-1}s^j, \qquad 2 \le i \le n-1,$$

with $b_{n,1} = qsb_{n-1}$ and $b_{n,n} = qs^{n-2}$ for $n \geq 2$ and $b_{1,1} = 1$.

Proof. Deletion of the 1 within a member of $\mathcal{S}_{n,1}(123)$, which is seen to be movable, implies $b_{n,1} = qsb_{n-1}$ for $n > 1$. Note that $\mathcal{S}_{n,n}(123)$ consists of only the permutation $(n-1)(n-2)\cdots 1n$, which has no occurrences of 1-32 with only the n being movable, whence $b_{n,n} = qs^{n-2}$ for $n > 1$. So assume $2 \le i \le n-1$ and let ℓ be the rightmost letter of $\pi = \pi_1\cdots\pi_n \in \mathcal{S}_{n,i}(123)$ that is greater than i. If ℓ is the penultimate letter of π, then each element of $[i-1]$ corresponds to a 1 within an occurrence of 1-32 in which i plays the role of the 2. Deleting the i then results in an arbitrary member of $\mathcal{S}_{n-1,\ell-1}(123)$ on the set $[n]-\{i\}$. Considering all ℓ thus gives $pst^{i-1}\sum_{\ell=i+1}^{n} b_{n-1,\ell-1}$ possibilities in this case. On the other hand, if ℓ is not the penultimate letter and if $j \in [i-1]$ is the largest letter between ℓ and i, then all elements of $[j-1]$ must occur between ℓ and j in decreasing order, for otherwise π would contain a 123 with i corresponding to the 3. Thus, π is of the form $\pi = \pi'\ell j(j-1)\cdots 1i$, where $j < i < \ell$. No element of $[j]\cup\{i\}$ within π can be the second letter of an adjacency enumerated by the p variable, with only the j and i being movable out of this set (note that transposing two adjacent elements of $[j]$ results in an occurrence of 123 since $i > j$). Deleting the final $j+1$ letters of π, which are seen to account for j descents, then yields $b_{n-j-1,\ell-j-1}\cdot q^2s^j$ possibilities. Considering all possible j and ℓ accounts for the remaining possibilities and implies (6). □

Let $b_n(w) = b_n(w;p,q,s,t)$ be given by $b_n(w) = \sum_{i=1}^{n} b_{n,i}w^i$ for $n \geq 1$.

Lemma 4. *If $n \geq 2$, then*

$$b_n(w) = qs^{n-2}w^n + \left(qsw + \frac{pstw^2}{1-tw}\right) b_{n-1}(1) - \frac{psw}{1-tw} b_{n-1}(tw)$$
$$+ \frac{q^2 w}{1-w} \sum_{j=1}^{n-2} (sw)^{n-j-1} (b_j(1) - b_j(w)), \tag{7}$$

with $b_1(w) = w$.

Proof. Note first that (7) is seen to hold when $n = 2$ since $b_2(w) = qsw + qw^2$ and $b_1(w) = w$, so assume $n \geq 3$. Multiplying both sides of (6) by w^i, summing over $2 \leq i \leq n-1$ and noting $b_{n,1} = qsb_{n-1} = qsb_{n-1}(1)$ yields

$$\begin{aligned}
&b_n(w) - qs^{n-2}w^n - qswb_{n-1}(1) \\
&\quad = ps \sum_{i=2}^{n-1} t^{i-1} w^i \sum_{\ell=i+1}^{n} b_{n-1,\ell-1} + q^2 \sum_{i=2}^{n-1} w^i \sum_{j=1}^{i-1} \sum_{\ell=i+1}^{n} b_{n-j-1,\ell-j-1} s^j \\
&\quad = ps \sum_{\ell=3}^{n} b_{n-1,\ell-1} \sum_{i=2}^{\ell-1} t^{i-1} w^i + q^2 \sum_{j=1}^{n-2} \sum_{\ell=j+2}^{n} b_{n-j-1,\ell-j-1} s^j \sum_{i=j+1}^{\ell-1} w^i \\
&\quad = \frac{psw}{1-tw} \sum_{\ell=3}^{n} b_{n-1,\ell-1} (tw - (tw)^{\ell-1}) \\
&\qquad + \frac{q^2}{1-w} \sum_{j=1}^{n-2} \sum_{\ell=j+2}^{n} b_{n-j-1,\ell-j-1} s^j (w^{j+1} - w^{\ell}) \\
&\quad = \frac{psw}{1-tw} (twb_{n-1}(1) - b_{n-1}(tw)) \\
&\qquad + \frac{q^2}{1-w} \sum_{j=1}^{n-2} \sum_{\ell=1}^{n-j-1} b_{n-j-1,\ell} s^j (w^{j+1} - w^{\ell+j+1}) \\
&\quad = \frac{psw}{1-tw} (twb_{n-1}(1) - b_{n-1}(tw)) \\
&\qquad + \frac{q^2}{1-w} \sum_{j=1}^{n-2} s^j w^{j+1} (b_{n-j-1}(1) - b_{n-j-1}(w)).
\end{aligned}$$

Replacing j by $n-1-j$ in the last sum gives (7). □

Let $g(x;w) = g(x;w;p,q,s,t)$ be given by $g(x;w) = \sum_{n \geq 1} b_n(w) x^n$. Multiplying both sides of (7) by x^n, and summing over $n \geq 2$, then gives

$$\begin{aligned}
g(x;w) - wx &= \frac{qw^2x^2}{1-swx} + swx\left(q + \frac{ptw}{1-tw}\right) g(x;1) - \frac{pswx}{1-tw} g(x;tw) \\
&\quad + \frac{q^2 w}{1-w} \sum_{j \geq 1} (b_j(1) - b_j(w)) \sum_{n \geq j+2} (sw)^{n-j-1} x^n
\end{aligned}$$

$$= \frac{qw^2x^2}{1-swx} + swx\left(q + \frac{ptw}{1-tw}\right)g(x;1) - \frac{pswx}{1-tw}g(x;tw) + \frac{s(qwx)^2}{(1-w)(1-swx)}(g(x;1)-g(x;w)),$$

which may be rewritten as

$$\left(1 + \frac{s(qwx)^2}{(1-w)(1-swx)}\right)g(x;w) = wx + \frac{qw^2x^2}{1-swx} - \frac{pswx}{1-tw}g(x;tw) + swx\Bigg(q + \frac{ptw}{1-tw} + \frac{q^2wx}{(1-w)(1-swx)}\Bigg)g(x;1). \tag{8}$$

We may solve (8) explicitly when $t = 1$ using the kernel method. Let $w = w_0$ be defined by

$$w_0 = \frac{1+(1-p)sx - \sqrt{(1+(1-p)sx)^2 - 4sx(1+(q^2-ps)x)}}{2sx(1+(q^2-ps)x)}.$$

Taking $w = w_0$ in (8) gives

$$g(x;1;p,q,s,1) = -\frac{1+\frac{qw_0x}{1-sw_0x}}{s(q+\frac{pw_0}{1-w_0}+\frac{q^2w_0x}{(1-w_0)(1-sw_0x)})}.$$

Then (8) with $t = 1$ leads to the following result.

Theorem 2. *The generating function $g(x;w;p,q,s,1)$ is given by*

$$g(x;w;p,q,s,1) = \frac{wx\left(1+\frac{qwx}{1-swx}\right)}{1+\frac{s(qwx)^2}{(1-w)(1-swx)}+\frac{pswx}{1-w}} - \frac{wx\left(q+\frac{pw}{1-w}+\frac{q^2wx}{(1-w)(1-swx)}\right)}{1+\frac{s(qwx)^2}{(1-w)(1-swx)}+\frac{pswx}{1-w}} \frac{1+\frac{qw_0x}{1-sw_0x}}{q+\frac{pw_0}{1-w_0}+\frac{q^2w_0x}{(1-w_0)(1-sw_0x)}}, \tag{9}$$

where $w_0 = \frac{1+(1-p)sx-\sqrt{(1+(1-p)sx)^2-4sx(1+(q^2-ps)x)}}{2sx(1+(q^2-ps)x)}$.

We are able to obtain the following formulas for the totals over $\mathcal{S}_n(123)$.

Corollary 3. *The total number of (a) adjacencies xy such that the x and y are the last two letters in an occurrence of* 1-32 *and (b) movable letters over all members of $\mathcal{S}_n(123)$ for $n \geq 1$ is given respectively by*

$$(a)\ \frac{1}{2}\binom{2n+2}{n+1} - 2\binom{2n}{n} + \binom{2n-2}{n-1}, \qquad (b)\ \binom{2n}{n} - 2\binom{2n-2}{n-1}.$$

Proof. Taking $w = q = s = 1$, differentiating with respect to p and setting $p = 1$ in (9) yields

$$\frac{\partial}{\partial p} g(x; 1; p, 1, 1, 1) \mid_{p=1} = 1 - \frac{1}{2x} + \frac{1 - 4x + 2x^2}{2x\sqrt{1-4x}},$$

and extracting the coefficient of x^n leads to the first formula. Similarly, differentiating with respect to q implies

$$\frac{\partial}{\partial q} g(x; 1; 1, q, 1, 1) \mid_{q=1} = -1 + \frac{1-2x}{\sqrt{1-4x}},$$

which gives the second. □

A comparable formula may be obtained for the total in the t-variable as follows. By Theorem 2, we have

$$g(x; w; 1, 1, 1, 1) = \frac{wx(C(x) - w)}{1 - w + w^2 x}. \tag{10}$$

Define $g'(x; w) = \frac{\partial}{\partial t} g(x; w; 1, 1, 1, t) \mid_{t=1}$. Differentiating (8) with respect to t, and setting $t = 1$ (where here we assume $s = p = q = 1$), yields

$$\begin{aligned} \frac{1 - w + w^2 x}{1 - wx} g'(x; w) = &-\frac{w^2 x}{1 - w} g(x; w) - w^2 x \frac{\partial}{\partial w} g(x; w) \\ &+ \frac{w^2 x}{1 - w} g(x; 1) + \frac{wx}{1 - wx} g'(x; 1), \end{aligned}$$

which, by (10), implies

$$\begin{aligned} (1 - w + w^2 x)^3 g'(x; w) = &\frac{1}{2} w^2 (1 - w)(1 - wx)(w^2 x^2 + 4wx^2 - 2wx - 3x + 1) \\ &- \frac{1}{2} w^2 (1 - w)(1 - wx)(w^2 x^2 - 2wx - x + 1)\sqrt{1-4x} \\ &+ wx(1 - w + w^2 x)^2 g'(x; 1). \end{aligned}$$

By differentiating this last equation twice with respect to w and then taking $w = C(x)$, we obtain

$$g'(x; 1) = \frac{x^3 C^3(x)}{1 - 4x} = \frac{x - 1}{2\sqrt{1-4x}} + \frac{1 - 3x}{2(1 - 4x)}.$$

Extracting the coefficient of x^n leads to the following result.

Corollary 4. *If $n \geq 1$, then the total number of occurrences of* 1-32 *taken over all members of $\mathcal{S}_n(123)$ is given by*

$$\frac{1}{2}\left(4^{n-1} - \binom{2n}{n} + \binom{2n-2}{n-1}\right).$$

Remark: An equivalent form of the prior result in terms of the generating function for the total number of 23-1 over $\mathcal{S}_n(321)$ is given in [6, Corollary 2.3] and was found using different methods.

Applying reverse complements, one may replace 1-32 with 21-3 in the definition of $b_n(w)$. By symmetry, $b_n(w)$ is also seen to be the distribution on $\mathcal{S}_n(321)$ obtained by replacing ascents by descents and the pattern 1-32 by either 23-1 or 3-12 with regard to the p and t variables. Reasoning as in the prior section, the u variable in $u^{n-1}b_n\left(w; \frac{p}{u}, \frac{q}{u}, s, t\right)$ marks the number of adjacencies xy within a member of $\mathcal{S}_n(123)$ that serve as the first two letters in an occurrence of 21-3. Similar remarks apply to the equivalent distribution on $\mathcal{S}_n(321)$.

4. A RELATED DISTRIBUTION ON $\mathcal{S}_N(132)$

Let $c_n = c_n(p, q, s, t)$ denote the joint distribution on $\mathcal{S}_n(132)$ for the following four statistics marked by p, q, s and t, respectively: (i) number of adjacencies xy such that the x and y serve as the 2 and 3 in an occurrence of 1-23, (ii) movable letters, (iii) ascents, and (iv) occurrences of 1-23. Let $c_n^* = c_n^*(p, q, s, t)$ denote the restriction of c_n to those members of $\mathcal{S}_n(132)$ starting with n. Given $1 \le i \le n$, let $c_{n,i} = c_{n,i}(p, q, s, t)$ and $c_{n,i}^* = c_{n,i}^*(p, q, s, t)$ denote the distribution polynomials obtained from c_n and c_n^* respectively by restricting the sum in each case to those permutations belonging to $\mathcal{S}_{n,i}(132)$.

The $c_{n,i}$ and $c_{n,i}^*$ are determined by the following intertwined recurrences.

Lemma 5. *If $n \ge 2$, then*

$$c_{n,i}^* = (q-1)c_{n-1,i}^* + c_{n-1,i}, \qquad 1 \le i \le n-1, \tag{11}$$

and

$$c_{n,i} = c_{n,i}^* + \sum_{j=i}^{n-2} c_{j+1,i}^* \left(qsc_{n-j-1,1} + ps \sum_{\ell=2}^{n-j-1} c_{n-j-1,\ell}t^{\ell-1} \right), \ 1 \le i \le n-1, \tag{12}$$

where $c_{n,n}^* = \delta_{n,1}$ for $n \ge 1$ and

$$c_{n,n} = qsc_{n-1,1} + ps\sum_{\ell=2}^{n-1} c_{n-1,\ell}t^{\ell-1}, \qquad n \ge 2, \tag{13}$$

with $c_{1,1} = 1$.

Proof. Note that $c_{n,n}^* = 0$ if $n \ge 2$ with $c_{1,1}^* = 1$, by the definitions, and that (11) clearly holds if $n = 2$. Let $\mathcal{S}_{n,i}^* = \mathcal{S}_{n,i}^*(132)$ denote the subset of $\mathcal{S}_{n,i} = \mathcal{S}_{n,i}(132)$ whose members start with n. Note that the second letter within $\pi \in \mathcal{S}_{n,i}^*$ where $n \ge 3$ and $1 \le i \le n-1$ is movable if and only if it equals $n-1$, for otherwise an occurrence of 132 would arise when it is transposed with n. Since the n obviously cannot contribute to an occurrence of 1-23 within π, it follows that there are $qc^*_{n-1,i}$ possibilities if the second letter of π is movable and $c_{n-1,i} - c^*_{n-1,i}$ possibilities otherwise, which gives (11). Note that (12) holds if $i = n-1$, the sum on the right side being empty in this case, since the n would have to occur at the beginning in order to avoid 132 so that $\mathcal{S}_{n,n-1} = \mathcal{S}^*_{n,n-1}$. Suppose now $\pi \in \mathcal{S}_{n,i} - \mathcal{S}^*_{n,i}$, where $1 \leq i \leq n-2$. Then $\pi = \alpha n \beta$, where α, β are non-empty and β contains all letters in $[j]$ for some $j \in [i, n-2]$. (Note that $j = n-1$ is not permitted, and hence α is non-empty, since π does not belong to $\mathcal{S}^*_{n,i}$.) Then the section $n\beta$ of π corresponds to a member of $\mathcal{S}^*_{j+1,i}$ on the elements of the set $[j] \cup \{n\}$, which implies a contribution of $c^*_{j+1,i}$. The letters of α then correspond to a member of $\mathcal{S}_{n-j-1,\ell}$ for some ℓ. If $\ell = 1$, then n is movable (and accounts for an extra ascent), whence there are $qsc_{n-j-1,1}$ possibilities. On the other hand, if $2 \leq \ell \leq n-j-1$, then there are $\ell - 1$ occurrences of 1-23 in which the role of the 3 is played by n. Considering all $\ell > 1$ yields $\sum_{\ell=2}^{n-j-1} c_{n-j-1,\ell} \cdot pst^{\ell-1}$ additional possibilities for each j. Considering all possible j and ℓ then gives the summation formula on the right side of (12), as desired. Finally, if $\pi \in \mathcal{S}_{n,n}$ where $n > 1$, then making use of the same cases as before regarding the parameter ℓ leads to (13) and completes the proof. □

Let $c_n(w) = c_n(w; p, q, s, t)$ be given by $c_n(w) = \sum_{i=1}^{n} c_{n,i} w^i$ for $n \geq 1$ and $c^*_n(w) = \sum_{i=1}^{n} c^*_{n,i} w^i$.

Lemma 6. *If $n \geq 3$, then*

$$\begin{aligned} c_n(w) &= c^*_n(w) + q^2 s(w^n - w)c_{n-2}(1) + \frac{ps(w^n - w)}{t}(c_{n-1}(t) - qtc_{n-2}(1)) \\ &\quad + (q-p)sc^*_{n-1}(w) + q(q-p)s\sum_{j=1}^{n-2} c^*_j(w)c_{n-j-1}(1) + \frac{ps}{t}\sum_{j=1}^{n-1} c^*_j(w)c_{n-j}(t), \end{aligned} \tag{14}$$

where $c^*_n(w) = (q-1)c^*_{n-1}(w) + c_{n-1}(w)$ for $n \geq 2$ and $c_1(w) = c^*_1(w) = w$, $c_2(w) = qw + qsw^2$.

Proof. The initial conditions may be verified using the definitions and the formula for $c^*_n(w)$ follows immediately from (11). Multiplying both sides of (12) by w^i, summing over $1 \leq i \leq n-1$ and adding w^n times equation (13) yields for $n \geq 3$ the relation

$$\begin{aligned} c_n(w) &= c^*_n(w) + qsw^n c_{n-1,1} + \frac{psw^n}{t}(c_{n-1}(t) - tc_{n-1,1}) \\ &\quad + \sum_{i=1}^{n-2} w^i \sum_{j=i}^{n-2} c^*_{j+1,i}\left(qsc_{n-j-1,1} + ps\sum_{\ell=2}^{n-j-1} c_{n-j-1,\ell}t^{\ell-1}\right). \end{aligned} \tag{15}$$

Note $c_{n,1} = qc_{n-1} = qc_{n-1}(1)$ for $n \geq 2$, since a terminal 1 is movable and extraneous concerning either 132 or 1-23.

Working separately on the two sums in (15) then gives

$$\sum_{i=1}^{n-2} w^i \sum_{j=i}^{n-2} c^*_{j+1,i} c_{n-j-1,1} = \sum_{j=1}^{n-2} c_{n-j-1,1} \sum_{i=1}^{j} c^*_{j+1,i} w^i = \sum_{j=1}^{n-2} c_{n-j-1,1} c^*_{j+1}(w)$$
$$= q\sum_{j=1}^{n-3} c_{n-j-2}(1) c^*_{j+1}(w) + c^*_{n-1}(w) = q\sum_{j=1}^{n-2} c_{n-j-1}(1) c^*_j(w)$$
$$+ c^*_{n-1}(w) - qwc_{n-2}(1)$$

and

$$\sum_{i=1}^{n-2} w^i \sum_{j=i}^{n-2} c^*_{j+1,i} \sum_{\ell=2}^{n-j-1} c_{n-j-1,\ell} t^{\ell-1} = \sum_{i=1}^{n-3} w^i \sum_{j=i}^{n-3} c^*_{j+1,i} \sum_{\ell=2}^{n-j-1} c_{n-j-1,\ell} t^{\ell-1}$$
$$= \sum_{j=1}^{n-3} \sum_{\ell=2}^{n-j-1} c_{n-j-1,\ell} t^{\ell-1} \sum_{i=1}^{j} c^*_{j+1,i} w^i = \frac{1}{t} \sum_{j=1}^{n-3} c^*_{j+1}(w)(c_{n-j-1}(t) - tc_{n-j-1,1})$$
$$= \frac{1}{t} \sum_{j=1}^{n-3} c^*_{j+1}(w) c_{n-j-1}(t) - \sum_{j=1}^{n-3} c^*_{j+1}(w) c_{n-j-1,1}$$
$$= \frac{1}{t} \sum_{j=0}^{n-2} c^*_{j+1}(w) c_{n-j-1}(t) - c^*_{n-1}(w) - \frac{w}{t} c_{n-1}(t) - q\sum_{j=1}^{n-3} c^*_{j+1}(w) c_{n-j-2}(1)$$
$$= \frac{1}{t} \sum_{j=1}^{n-1} c^*_j(w) c_{n-j}(t) - q\sum_{j=1}^{n-2} c^*_j(w) c_{n-j-1}(1)$$
$$- \frac{w}{t}(c_{n-1}(t) - qtc_{n-2}(1)) - c^*_{n-1}(w).$$

Inserting these last two expressions into (15), and simplifying, yields (14). □

Let $h(x;w) = h(x;w;p,q,s,t)$ be defined by $h(x;w) = \sum_{n\geq 1} c_n(w)x^n$ with $h^*(x;w) = \sum_{n\geq 1} c^*_n(w)x^n$. Rewriting (14) in terms of generating functions, we have

$$\begin{aligned} h(x;w) = {} & qsw^2x^2 + h^*(x;w) + q^2sx^2(w^2h(wx;1) - wh(x;1)) \\ & + \frac{pswx}{t}(h(wx;t) - twx) - \frac{pswx}{t}(h(x;t) - tx) \\ & - pqswx^2(wh(wx;1) - h(x;1)) + (q-p)sx(h^*(x;w) - xw) \\ & + q(q-p)sxh^*(x;w)h(x;1) + \frac{ps}{t}(h^*(x;w)h(x;t) - twx^2), \end{aligned}$$

which reduces to

$$\begin{aligned} h(x;w) = {} & (p-q)(1-w)swx^2 + (1 + (q-p)sx)h^*(x;w) \\ & + q(q-p)swx^2(wh(wx;1) - h(x;1)) + \frac{pswx}{t}(h(wx;t) - h(x;t)) \\ & + q(q-p)sxh^*(x;w)h(x;1) + \frac{ps}{t}h^*(x;w)h(x;t), \end{aligned} \tag{16}$$

where $h^*(x;w) = \frac{wx+xh(x;w)}{1+(1-q)x}$.

Taking $t = w = 1$ in (16) yields

$$h(x;1;p,q,s,1) = (1+(q-p)sx)h^*(x;1;p,q,s,1) + (ps+q(q-p)sx)h^*(x;1;p,q,s,1)h(x;1;p,q,s,1),$$

where $h^*(x;1;p,q,s,1) = \frac{x+h(x;1;p,q,s,1)}{1+(1-q)x}$. Solving this last equation explicitly gives the following result.

Theorem 3. *We have*

$$h(x;1;p,q,s,1) = \frac{x(1-(p-q)sx)}{1-(ps+q)x+(q+1)(p-q)sx^2} C\left(\frac{sx^2(p-q(p-q)x)(1-(p-q)sx)}{(1-(ps+q)x+(q+1)(p-q)sx^2)^2}\right). \tag{17}$$

Remarks: Taking all parameters to be unity in (17), one has

$$h(x;1;1,1,1,1) = \frac{x}{1-2x} C\left(\frac{x^2}{(1-2x)^2}\right) = C(x) - 1,$$

as expected. Note that when $q = 0$ and $p = s = 1$ in (17), the generating function reduces to x via the Catalan transform $C\left(\frac{x^2(1-x)}{(1-x+x^2)^2}\right) = \frac{1-x+x^2}{1-x}$. This may be explained directly by observing that n is always movable within $\pi \in \mathcal{S}_n(132)$ for $n \geq 2$ if n is not the first letter, with $n-1$ movable if it is. Taking $p = 0$ and $q = s = 1$ in (17) gives $h(x;1;0,1,1,1) = \frac{1-x-2x^2-\sqrt{1-2x-3x^2}}{2x^2}$, which implies $|\mathcal{S}_n(132, 1\text{-}23)| = M_n$, where M_n denotes the n-th Motzkin number (see [17, A001006]). Finally, substituting (17) into (16), one can solve for $h(x;w;p,q,s,1)$ and find it explicitly.

Since the distribution in the q variable is the same as on $\mathcal{S}_n(213)$ discussed above, we consider here only the totals in the p and t variables.

Corollary 5. *The total number of (a) adjacencies xy such that the x and y are the last two letters in an occurrence of* 1-23 *and (b) occurrences of* 1-23 *over all members of $\mathcal{S}_n(132)$ for $n \geq 1$ is given respectively by*

$$(a)\ \frac{1}{2}\binom{2n+2}{n+1} - 2\binom{2n}{n} + \binom{2n-2}{n-1}, \qquad (b)\ 4^{n-1} + \binom{2n}{n} - \frac{1}{2}\binom{2n+2}{n+1}.$$

Proof. Differentiating (17) with respect to p, and setting $p = 1$, gives

$$\begin{aligned}
&\frac{\partial}{\partial p} h(x;1;p,1,1,1)\mid_{p=1} \\
&\quad = \frac{x^3}{(1-2x)^3} C'(w)\mid_{w=x^2/(1-2x)^2} \\
&\quad = \frac{x^3}{(1-2x)^3}\left(\frac{(1-2x)^3}{x^2\sqrt{1-4x}} - \frac{(1-2x)^2}{x^2} C\left(\frac{x^2}{(1-2x)^2}\right)\right) \\
&\quad = \frac{x}{\sqrt{1-4x}} - C(x) + 1 = \frac{1-4x+2x^2-(1-2x)\sqrt{1-4x}}{2x\sqrt{1-4x}}.
\end{aligned}$$

Extracting the coefficient of x^n in the last expression yields the first formula.

Let $h(x;w)$ denote here $h(x;w;1,1,1,1)$. Differentiating both sides of (16) with respect to w, where all other parameters are taken to be unity, gives

$$(1-x-xh(x;1))\frac{\partial}{\partial w}h(x;w) = x + xh(wx;1) + wx^2\frac{\partial}{\partial z}h(z;1)\mid_{z=wx},$$

which implies

$$\frac{\partial}{\partial w}h(x;w) = \frac{xC(x)}{\sqrt{1-4wx}}.$$

Now differentiating (16) with respect to t, setting $t=1$ (where all other parameters are unity), and making use of the expressions for $h(x;1)$ and $\frac{\partial}{\partial w}h(x;w)$ yields

$$\begin{aligned}\frac{\partial}{\partial t}h(x;1;1,1,1,t)\mid_{t=1} &= \frac{xC(x)}{\sqrt{1-4x}}\left(\frac{xC(x)}{\sqrt{1-4x}} - C(x) + 1\right)\\ &= \frac{x(C(x)-1)}{1-4x} + \frac{1-(1-x)C(x)}{\sqrt{1-4x}}\\ &= \frac{(1-2x-\sqrt{1-4x})^2}{4x(1-4x)}.\end{aligned}$$

Extracting the coefficient of x^n for $n \geq 1$ in the last expression gives the second formula above. □

Remarks: Differentiating (17) with respect to s, and setting $s=1$, yields

$$\frac{\partial}{\partial s}h(x;1;1,1,s,1)\mid_{s=1} = \frac{1-3x-(1-x)\sqrt{1-4x}}{2x\sqrt{1-4x}},$$

which implies that the total number of ascents over all members of $\mathcal{S}_n(132)$ is given by $\frac{1}{2}\binom{2n+2}{n+1} - \frac{3}{2}\binom{2n}{n} = \binom{2n-1}{n+1}$ for $n \geq 1$. By subtraction from $(n-1)C_n$, this gives the same number of descents over $\mathcal{S}_n(132)$, which is seen to apply also to the total in the s variable in Section 2. Equivalent formulas on $\mathcal{S}_n(231)$ were found earlier in [6, Corollary 3.2].

5. COMBINATORIAL PROOFS

Before providing combinatorial proofs of some of the prior results, we make some further remarks concerning the various distributions. Note that the distribution of μ on $\mathcal{S}_n(123)$ is distinct from that of μ on $\mathcal{S}_n(132)$, which follows from comparing the formulas above for $g(x;w)$ and $h(x;w)$ when all parameters are unity except q. Indeed, comparing the $n=4$ cases, one has $b_4(1,q,1,1) = 6q^3+4q^2+4q$, whereas $c_4(1,q,1,1) = 5q^3+6q^2+3q$. On the other hand, from Corollaries 2 and 3, it is seen that the sum of the μ values over all members of $\mathcal{S}_n(123)$ equals the sum over $\mathcal{S}_n(132)$ despite the distributions being different, and a combinatorial proof as to why is demonstrated below.

Comparing the formulas for $f(x;w)$ and $h(x;w)$ in Theorems 1 and 3, one has that the statistics marked by q and s are identically distributed on $\mathcal{S}_n(213)$ and $\mathcal{S}_n(132)$. However, the joint distribution of (q,s) is seen to be different. Taking reverse complements accounts for the behavior in the q variable, as already seen. Note that the s distribution being the same follows from the fact that the descent and ascent statistics on $\mathcal{S}_n(132)$ are identically distributed. To see this, let $\pi \in \mathcal{S}_n(132)$ be represented by $\pi = \alpha n \beta$, where α and β are permutations of $[i+1,n-1]$ and $[i]$, respectively, for some $0 \le i \le n-1$. Define f_n inductively on $\mathcal{S}_n(132)$ for $n \ge 0$ by setting $f_0(\varnothing) = \varnothing$ and $f_n(\pi) = f_i(\gamma) n f_{n-i-1}(\delta)$ for $n \ge 1$, where γ and δ are permutations of $[n-i,n-1]$ and $[n-i-1]$ that are isomorphic to β and α, respectively. One may verify that f_n is a bijection from $\mathcal{S}_n(132)$ to itself such that for all π the number of descents in π equals the number of ascents in $f_n(\pi)$ and vice versa.

Let $\nu_1(\pi)$ denote the number of adjacencies xy where x and y are the last two letters in an occurrence of 1-32 for $\pi \in \mathcal{S}_n(123)$ and let $\nu_2(\pi)$ denote the comparable number of adjacencies involving the pattern 1-23 for $\pi \in \mathcal{S}_n(132)$. Comparing the formulas for $g(x;w)$ and $h(x;w)$ from Theorems 2 and 3 in p yields the following result.

Corollary 6. *If* $n \ge 1$*, then*

$$\sum_{\pi \in \mathcal{S}_n(123)} p^{\nu_1(\pi)} = \sum_{\pi \in \mathcal{S}_n(132)} p^{\nu_2(\pi)}. \tag{18}$$

This result may also be explained using the Simion-Schmidt (S.S.) bijection from [16], which we will denote by Ψ. Recall that given $\pi \in \mathcal{S}_n(123)$, its image $\Psi(\pi)$ in $\mathcal{S}_n(132)$ has the same left-right minima (lr min) as π and occur in the same positions. The remaining entries are then inserted in increasing order so that each entry is placed in the leftmost unfilled position where it does not become an lr min. For example, if $\pi = \mathbf{894}7\mathbf{6}\mathbf{3}\mathbf{1}52 \in \mathcal{S}_9(123)$, then $\Psi(\pi) = \mathbf{894}5\mathbf{6}\mathbf{3}\mathbf{1}27 \in \mathcal{S}_9(132)$ where lr min are in bold. To show (18), first observe that the 1 in an occurrence of 1-32 within $\pi \in \mathcal{S}_n(123)$ or the 1 in an occurrence of 1-23 within $\tau \in \mathcal{S}_n(132)$ may be taken to be an lr min (indeed, in the former case, it is a requirement). Thus $\nu_1(\pi)$ and $\nu_2(\tau)$ give the number of adjacencies xy in π and τ respectively such that neither x nor y is an lr min. Since the S.S. bijection preserves positions of lr min, we then have $\nu_1(\pi) = \nu_2(\Psi(\pi))$ for all π, which implies (18).

One may verify that none of the individual parameters in Theorems 1 and 2 have identical distributions when taken separately. However, it should be noted that the sum of the ν_1 values taken over all members of $\mathcal{S}_n(123)$ is the same as the sum of ν_3 values over $\mathcal{S}_n(213)$, where ν_3 is defined similarly as before but with the pattern 2-31. This is a somewhat interesting result in view of the fact the underlying distributions are different. (Note that when $n = 4$, we have $a_4(p,1,1,1) = 6p + 8$ and $b_4(p,1,1,1) = p^2 + 4p + 9$.)

We now provide a bijective proof of the formula for the sum of the μ values taken over all members of $\mathcal{S}_n(132)$ implicit in Corollary 2.

Combinatorial proof for sum of μ values over $\mathcal{S}_n(132)$.

We argue combinatorially that the total number of movable letters within all the members of $\mathcal{S}_n(132)$ is given by $\binom{2n}{n} - 2\binom{2n-2}{n-1} = (2n-2)C_{n-1}$ for $n \geq 1$. We count separately the movable letters that correspond to lr min and those that do not and refer to such movable letters as being of type 1 and of type 2, respectively. Suppose that a non-initial entry x within $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n(132)$ is an lr min. If the closest lr min to the left of x is greater than $x+1$, and z denotes the predecessor of x, then transposing z and x results in an occurrence of 132 in π of the form $xz(x+1)$ since $x+1$ must occur somewhere to the right of x. Thus, if x is a type 1 movable letter, then $x+1$ is an lr min and this requirement is seen also to be sufficient.

We now show that there are $(n-1)C_{n-1}$ movable letters of type 1 within all of the members of $\mathcal{S}_n(132)$. To do so, we start with $\tau = \tau_1 \cdots \tau_{n-1} \in \mathcal{S}_{n-1}(132)$ and choose some position directly following any one of the entries of τ. Let i denote the closest lr min that occurs to the left of the chosen position. We then insert a copy of i into the chosen position and increase all elements of $[i+1, n-1]$, along with the original i, by one. Note that both i and $i+1$ are lr min in the resulting permutation of length n, which is seen to avoid 132. Since this operation is reversible, it follows that there are $(n-1)C_{n-1}$ movable letters of type 1.

We now count the type 2 letters. Note that a movable letter of type 2 cannot have a predecessor that is a non lr min, for otherwise an occurrence of 132 is introduced when the two letters are transposed where the role of 1 can be played by any lr min to the left. Furthermore, one may verify that transposing an entry that is not an lr min with a predecessor that is never introduces a 132. Thus, we seek to enumerate the total number of times that an lr min is directly followed by a non lr min. To do so, we first consider the following classes of lr min. Let us refer to an lr min a within $\pi \in \mathcal{S}_n(132)$ as being of type A if $a+1$ is also an lr min. A type B lr min will refer to one that is directly preceded by another lr min. For example, if $\pi = 768459312 \in \mathcal{S}_9(132)$, then the letters 6 and 3 are lr min of type A, 6 and 1 are of type B and 7 and 4 are lr min of neither type. We have the following result concerning the totals of the two types.

Lemma 7. *The total number of lr min of type A within all members of $\mathcal{S}_n(132)$ equals the total number of type B for all $n \geq 1$.*

Proof. Let $\mathcal{A}_n$ and $\mathcal{B}_n$ denote the set of "marked" members of $\mathcal{S}_n(132)$ wherein an lr min of type A or of type B is marked, respectively. We define inductively a bijection f_n between the sets $\mathcal{A}_n$ and $\mathcal{B}_n$ for all $n \geq 1$, which will imply the asserted equality. Since the $n = 1$ and $n = 2$ cases are trivial, we may assume $n \geq 3$. Let $\pi \in \mathcal{A}_n$. To define f_n, we consider several cases based on the position of n within π. If $\pi = \pi' n$, then let $f_n(\pi)$ be obtained by appending n to the permutation $f_{n-1}(\pi')$. If $\pi = n\pi'$ and $n-1$ is not the marked lr min, then let $f_n(\pi)$ be obtained by prepending n to $f_{n-1}(\pi')$. If $n-1$ is the marked lr min, then $n-1$ must be the second letter of π and is also of type B, in which case we simply let $f_n(\pi) = \pi$.

So assume $\pi = \alpha n \beta$, where both α and β are non-empty, and let x denote the marked lr min of π. If both x and $x+1$ belong to either α or β, then either let $f_n(\pi) = f_\ell(\alpha)n\beta$ or $f_n(\pi) = \alpha n f_{n-\ell-1}(\beta)$, where $\ell = |\alpha|$ and, in the first case, it is understood that both the input and output of the mapping f_ℓ is a permutation of the elements of $[n-\ell, n-1]$. Note that both the marked letter and its predecessor within a member ρ of $\mathcal{B}_n$ decomposed as above must belong to either α or β when both are non-empty and hence all such ρ lie within the range of f_n in this case.

Otherwise, we must have $x+1 \in \alpha$ and $x \in \beta$, which implies x is the first letter of β since it is an lr min of π and also the largest element of β as $\beta < \alpha$. That is, $\pi = \alpha n x \beta'$, where β' is a permutation of $[x-1]$. In this case, to obtain $f_n(\pi)$, we remove x, reduce all elements of π in $[x+1, n]$ by one, put n back at the beginning of the resulting permutation of $[n-1]$ and finally mark the second letter in the permutation of $[n]$ that is produced. For example, we have

$$\pi = 6578\mathbf{4}231 \in \mathcal{A}_8 \to 6578231 \to 5467231 \to f_8(\pi) = 8\mathbf{5}467231 \in \mathcal{B}_8,$$

where the marked lr min are indicated in bold. Note that this case accounts for all members of $\mathcal{B}_n$ starting with n, but not $n(n-1)$, in which the second letter is the marked lr min. Considering the various cases, one may verify that f_n yields the desired bijection between $\mathcal{A}_n$ and $\mathcal{B}_n$. □

Note further that there are $(2n-1)C_{n-1}$ lr min in $\mathcal{S}_n(132)$ altogether. To realize this, first observe that the standard bijection (see, e.g., [7]) between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ shows that the lr min statistic on the former has the same distribution as the peaks statistic on the latter, where $\mathcal{D}_n$ denotes the set of Dyck paths having semi-length n. That there are $(2n-1)C_{n-1}$ peaks taken over all members of $\mathcal{D}_n$, and hence the same number of lr min over $\mathcal{S}_n(132)$, can be seen upon inserting an upstep directly followed by a downstep at any one of the $2n-1$ possible positions along a path in $\mathcal{D}_{n-1}$, with the peak so obtained understood to be distinguished from all others in the resulting member of $\mathcal{D}_n$.

Hence, there are $(2n-2)C_{n-1}$ lr min altogether in $\mathcal{S}_n(132)$ that do not correspond to a terminal entry within a permutation (upon subtracting the C_{n-1} lr min that do, such entries corresponding to members of $\mathcal{S}_n(132)$ that end in 1). Since there are $(n-1)C_{n-1}$ lr min of type A in $\mathcal{S}_n(132)$ in all (being synonymous with movable letters of type 1 as seen), we have that there are the same number of type B, by Lemma 7. Thus, there are $(2n-2)C_{n-1} - (n-1)C_{n-1} = (n-1)C_{n-1}$ lr min that are directly followed by a non lr min, and hence the same number of movable letters of type 2. Combining the totals over $\mathcal{S}_n(132)$ for the two types of movable letters gives $(2n-2)C_{n-1}$, as desired. □

The arguments above may be extended to account for the formula for the sum of μ values in the 123-avoiding case.

Proof for sum of μ values over $\mathcal{S}_n(123)$.

We make use of the same terminology as in the preceding proof but applied to members of $\mathcal{S}_n(123)$. First note that the total number of type 2 movable letters is the same as in the 132 case, since the S.S. bijection is seen to preserve the number of times that a non lr min directly follows an lr min. Thus, to complete the proof, we must show that there are $(n-1)C_{n-1}$ type 1 movable letters altogether in $\mathcal{S}_n(123)$. Note that if the predecessor of an lr min x is not an lr min, then x is movable since the subsequence of non lr min is decreasing. On the other hand, if the predecessor y of x within $\pi \in \mathcal{S}_n(123)$ is an lr min, then x is movable if and only if no element greater than y occurs to the right of x. That is, π may be decomposed as $\pi = \alpha y \beta$, where α and β are permutations of $[y+1, n]$ and $[y-1]$, respectively, and $x < y$ is the first letter of β. Thus, there are $\sum_{i=1}^{n-1} C_i C_{n-1-i} = C_n - C_{n-1}$ type 1 movable letters where the predecessor is an lr min.

We now can obtain the number of movable type 1 letters from the total number of non-initial lr min within $\mathcal{S}_n(123)$, the latter being given by $(2n-1)C_{n-1} - C_n$. We first subtract from the total the number of lr min whose predecessor is also an lr min. By the S.S. bijection and Lemma 7, this number is $(n-1)C_{n-1}$. By subtraction, there are then $nC_{n-1} - C_n$ type 1 movable letters in all where the predecessor is not an lr min. We then add back to this the quantity $C_n - C_{n-1}$, which accounts for the remaining type 1 movable letters and gives $(n-1)C_{n-1}$, as desired. □

Proof for sum of ν_i values, $1 \leq i \leq 3$.

The arguments above also apply to the totals of the ν_i values given in Corollaries 2, 3 and 5 above. We first explain ν_1 and show that the total in this case is given by $nC_{n-1} - C_n$. Note that the sum of the ν_1 values over all members of $\mathcal{S}_n(123)$ equals the number of times both an entry and its predecessor within a permutation are non lr min. To find this, we subtract the number of times a non lr min follows an lr min from the total number of non lr min within $\mathcal{S}_n(123)$, the latter quantity being $nC_n - (2n-1)C_{n-1}$. The number of times that an lr min is followed by a non lr min is given by $(2n-1)C_{n-1} - (n-1)C_{n-1} - C_{n-1} = (n-1)C_{n-1}$, by the preceding proof and the fact that there are C_{n-1} lr min that have no successor. Thus, the number of times both an entry and its predecessor are non lr min equals $nC_n - (2n-1)C_{n-1} - (n-1)C_{n-1} = nC_n - (3n-2)C_{n-1} = nC_{n-1} - C_n$, as required. By Corollary 6, this is also the sum of ν_2 values over $\mathcal{S}_n(132)$. Finally, by prior observations and subtraction, we have that the sum of ν_3 over $\mathcal{S}_n(213)$ is given by $(n-1)C_n - (2n-2)C_{n-1} - (nC_n - (3n-2)C_{n-1}) = nC_{n-1} - C_n$. □

It would be interesting also to have bijective proofs of the formulas given above for the total number of occurrences of the various vincular patterns.

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