Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **15** (2021), 357–377.
https://doi.org/10.2298/AADM160719018S

SEPARATION OF CARTESIAN PRODUCTS OF GRAPHS INTO SEVERAL CONNECTED COMPONENTS BY THE REMOVAL OF EDGES

Simon Špacapan

Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq E(G)$ is an edge k-cut in G if the graph $G - S = (V(G), E(G) \setminus S)$ has at least k connected components. The generalized k-edge connectivity of a graph G, denoted as $\lambda_k(G)$, is the minimum cardinality of an edge k-cut in G. In this article we determine generalized 3-edge connectivity of Cartesian product of connected graphs G and H and describe the structure of any minimum edge 3-cut in $G \Box H$. The generalized 3-edge connectivity $\lambda_3(G \Box H)$ is given in terms of $\lambda_3(G)$ and $\lambda_3(H)$ and in terms of other invariants of factors G and H.

1. INTRODUCTION

We say that $S \subseteq E(G)$ is a λ_k*-set* in G if S is a minimum edge k-cut in G. An edge 2-cut is called an edge-cut and a λ_2-set a λ-set. In [**10**] the authors consider different ways to split a graph into three connected components by the removal of edges. Two ways are suggested: first is to isolate two vertices of minimum degree, x and y, by removing all edges indicident to x and y. This is called an *obvious separation* in [**10**]. The second is to remove edges of a λ-set S, thereby spliting the graph into components C_1 and C_2, and then remove either $\lambda(C_1)$ or $\lambda(C_2)$ edges from the resulting graph, depending on which one is smaller. This is called a *greedy separation*, and in this separation we look for the minimum $|S| + \lambda(C_1)$ resp. $|S| + \lambda(C_2)$ over all λ-sets S. The authors give an example of a graph where the obvious and the greedy separation require more than $\lambda_3(G)$ edges. They also give sufficient conditions for the greedy separation to be optimal (minimum).

2020 Mathematics Subject Classification. 05C40, 05C76.
Keywords and Phrases. Generalized connectivity, Cartesian product.

Edge and vertex connectivities of graph products have been discussed in several articles, and with the exception of vertex connectivity of direct products, the connectivity of all four standard graph products was determined. The connectivity of Cartesian products of graphs was determined in articles [**18, 12**]. It was proved that the edge connectivity of $G\Box H$ is

$$\lambda(G\Box H) = \min\{\lambda(G)|V(H)|, \lambda(H)|V(G)|, \delta(G)+\delta(H)\},$$

and that every minimum edge-cut in $G\Box H$ is equal to one of the following sets

$$S_1 = \{(x,y)(x',y) \mid xx' \in S_G\} \text{ or } S_2 = \{(x,y)(x,y') \mid yy' \in S_H\} \text{ or } S_3 = E(x,y),$$

where $S_G \subseteq E(G)$ resp. $S_H \subseteq E(H)$ is a minimum edge-cut in G resp. H, and $E(x,y)$ is the set of edges incident to a vertex (x,y) of minimum degree in $G\Box H$.

The edge connectivity of direct products is characterized in [**17**]. The connectivity of strong products is determined in [**4, 16**], and some further connectivity properties of strong products are given in [**7**] and [**15**]. Results on connectivity of lexicographic products are summerized in [**11**].

The connectivity of graph products was also studied in [**1**] and [**2**] where the authors determine connectivity properties of the $*$-product, and use these properties to construct networks with high fault-tolerance. The $*$-product was first defined in [**3**] and is a concept which is more general than the Cartesian product of graphs (i.e. Cartesian product of graphs is a special case of the $*$-product).

Let us also mention that several generalizations of connectivity are present in the literature, in particular the generalization introduced in [**9**] where for a set $S \subseteq V(G)$ of size k, the generalized local connectivity $\kappa_G(S)$ is defined as the maximum number of internally disjoint Steiner trees connecting set S. The generalized k-connectivity is then defined as the minimum local connectivity, where the minimum is taken over all sets S of size k (see [**14**], and references therin).

The minimum number of vertices that need to be removed in order to break a graph into k or more connected components was first studied by Chartrand et. al. in [**8**]. In this paper, instead of removing vertices, we study removal of edges. We prove, in Theorem 1, that $\lambda_3(G\Box H)$ is equal to

$$\min\{\lambda_3(G)|V(H)|, \lambda_3(H)|V(G)|, 2\delta(G)+\delta_2(H), 2\delta(H)+\delta_2(G), D(G,H)\}$$

where $\delta_2(G)$ is the minimum number of edges that need to be removed to isolate two vertices of G, and $D(G,H)$ is a function corresponding to some special types of edge 3-cuts in $G\Box H$.

2. GENERALIZED EDGE CONNECTIVITY OF CARTESIAN PRODUCT

The *Cartesian product* of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$, where vertices (x_1, y_1) and

(x_2, y_2) are adjacent in $G \Box H$ if and only if $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. For a vertex $y \in V(H)$ the *G-layer* G_y is the set

$$G_y = \{(x, y) \mid x \in V(G)\}.$$

Similarly, for an $x \in V(G)$ the *H-layer* H_x is the set

$$H_x = \{(x, y) \mid y \in V(H)\}.$$

For a vertex $x \in V(G)$ *the neighborhood* of x in G is the set

$$N_G(x) = \{y \in V(G) \mid xy \in E(G)\}.$$

There are several natural ways how to split the vertex set of the product $G \Box H$ into three connected components by removing some of its edges. Suppose that $S \subseteq E(G)$ is a λ_3-set in G. Then the set

$$S' = \{(x, y)(x', y) \mid xx' \in S, y \in V(H)\}$$

is an edge 3-cut in $G \Box H$. Since $|S'| = \lambda_3(G)|V(H)|$ we find that $\lambda_3(G \Box H) \leq \lambda_3(G)|V(H)|$. With similar arguments we also see that $\lambda_3(G \Box H) \leq \lambda_3(H)|V(G)|$. An edge 3-cut constructed in this way is called a *type 1* resp. *type 2* edge 3-cut. For a complete overwiev of all types of edge-cuts and their definitions see Fig. 2 at the end of this section.

To describe other natural ways how to split the product into three connected components let us define δ_{uv} and $\delta_2(G)$. For $u, v \in V(G)$ we define

$$\delta_{uv} = \begin{cases} 1 & \text{if } uv \in E(G) \\ 0 & \text{if } uv \notin E(G) \end{cases}$$

and let

$$\delta_2(G) = \min\{\deg(u) + \deg(v) - \delta_{uv} \mid u, v \in V(G), u \neq v\}.$$

It is straightforward that $\delta_2(G)$ is the minimum number of edges whose removal isolates two vertices of G. Let G be a graph and $X, Y \subseteq V(G)$ be sets. We define

$$E(X, Y) = \{xy \in E(G) \mid x \in X, y \in Y\}.$$

For a set $X \subseteq V(G)$ we write $E(X) = E(X, \bar{X})$, and in particular we use $E(x)$ to denote the set of edges incident to x, $E(x) = E(\{x\})$. Now assume that $(x, y), (x', y) \in V(G \Box H)$ are vertices such that $\deg_G(x) + \deg_G(x') - \delta_{xx'} = \delta_2(G)$ and $\deg_H(y) = \delta(H)$. The set

$$S = E(x, y) \cup E(x', y)$$

is the set of edges that isolates vertices (x, y) and (x', y) in the product, moreover $|S| = 2\delta(H) + \delta_2(G)$. An analogous construction gives an edge cut of size $2\delta(G) +$

$\delta_2(H)$ that also isolates two vertices. So we have $\lambda_3(G\Box H) \le 2\delta(H) + \delta_2(G)$, and similarly $\lambda_3(G\Box H) \le 2\delta(G) + \delta_2(H)$. An edge 3-cut that isolates two vertices is called a *type 3* edge-cut (see Fig. 2, and observe that the two isolated vertices might be contained in the same G-layer or H-layer).

For a graph G let $C(G)$ be the minimum size of a connected component of $G-S$ where S is a λ-set of G (here we take the minimum over all λ-sets of G). Fix a λ-set $S' \subseteq E(G)$ and a connected component U of $G-S'$ of size $C(G)$. Additionaly let $S'' \subseteq E(H)$ be a λ-set in H. Then the set

$$S = \{(x, y)(x', y) \mid xx' \in S', y \in V(H)\} \cup \{(x, y), (x, y') \mid x \in U, yy' \in S''\}$$

is a set that brakes the product into three parts, and $|S| = \lambda(G)|V(H)|+\lambda(H)C(G)$. An analogous construction gives us an edge 3-cut in the product of size $\lambda(H)|V(G)|+\lambda(G)C(H)$. We define

$$T(G, H) = \min\{\lambda(G)|V(H)| + \lambda(H)C(G), \lambda(H)|V(G)| + \lambda(G)C(H)\},$$

and it follows from the above discussion that $\lambda_3(G\Box H) \le T(G, H)$. An edge 3-cut constructed in this way is called a *type 5* resp. *type 4* edge 3-cut (see Fig. 2).

Before we describe yet another way of spliting the product into three components consider the following example (shown in Fig.). Let G be a graph obtained from three copies of complete graph on n vertices K_n, with n sufficiently large (for example $n = 10^6$ suffices), and two additional vertices x_1 and x_2. Suppose that there are 1000 edges between the first and the second copy of K_n and that x_1 has 1001 neighbors in the second copy of K_n and 99.500 neighbors in the third copy of K_n. Finally suppose that x_2 has 100.400 neighbors in the third copy of K_n, and let $H = C_{100}$ (the cycle of lenght 100). We see that $\delta(G) = 100.400, \lambda(G) = 1000$ and $\lambda_3(G) = 2001$. Therefore isolating two vertices of $G\Box H$ (in terminology of [**10**] an obvious separation) requires $\delta_2(H)+2\delta(G) = 2\delta(G\Box H)-1 = 200.803$ edges. Splitting the product on two parts by a minimum edg-cut and then spliting one of the two components on two parts (in terminology of [**10**] the greedy separation) requires $\lambda(G)|V(H)| + \delta(G\Box H) = 200.402$ edges, the separation by a canonical edge 3-cut requires $\lambda_3(G)|V(H)| = 200.100$ edges. However the minimum edge 3-cut has only $\deg(x_1) + 2 + 1001(|V(H)| - 1) = 199.602$ edges, where $\deg(x_1) + 2$ corresponds to the degree of a vertex (x_1, y) in $G\Box H$ and $1001(|V(H)| - 1)$ corresponds to edges, not incident to (x_1, y), in the canonical edge-cut of size $1001|V(H)|$.

We will call such an edge cut, an edge cut of *type 7*, to be precisely defined in the sequel. Observe that the above example demonstrates two things:

- A minimum edge cut of the product may be smaller than any edge-cut obtained by an obvious or a greedy separation.
- A minimum edge-cut of type 6 or 7 (as the one presented above) may contain a canonical edge-cut of the product which is not a minimum canonical edge-cut (in the above example a minimum canonical edge cut has $1000|V(H)|$ edges).

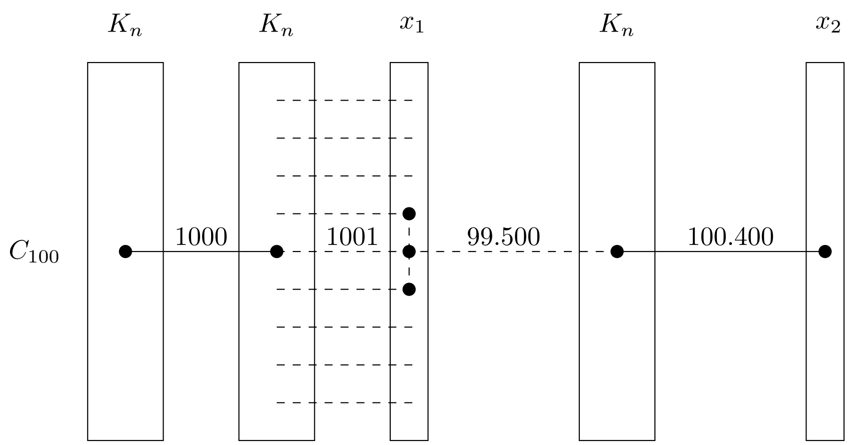

Figure 1: Edges of the minimum edge 3-cut are dashed.

This example of a minimum edge-cut brings us to the following definitions.

Let S be an edge-cut in G and let $x \in V(G)$ be a non-isolated vertex in $G - S$ (equivalently $\deg_{G-S}(x) > 0$). We define

$$\omega(S, x) = \deg(x) - |S \cap E(x)|\,,$$

observe that $\omega(S, x)$ is the number of edges that need to be removed to isolate x (after S was removed). If S is the edge-cut of size 1001 from the above example, then

$$\lambda_3(G \Box H) = |S||V(H)| + \omega(S, x_1) + \delta(H)\,.$$

We define invariants A and B as follows

$$A(G, H) = \min\{|S||V(H)| + \omega(S, x) + \delta(H)\}$$

and the minimum is taken over all edge-cuts S in G and vertices x with $\deg_{G-S}(x) > 0$. Similarly

$$B(G, H) = \min\{|S||V(G)| + \omega(S, y) + \delta(G)\}$$

here the minimum is taken over all edge-cuts S in H and vertices y with $\deg_{H-S}(y) > 0$. Finally we combine three definitions, the definition of A, B and T, to define $D(G, H)$ as follows

$$D(G, H) = \min\{A(G, H), B(G, H), T(G, H)\}\,.$$

Now we have defined all possible types of edge 3-cuts, and we will prove that every minimum edge 3-cut in $G \Box H$ belongs to one of these types. The types are shown in Fig. 2 where, for example, type 3 occures when two connected components are singletons, and the rest of the vertices are contained in the third connected

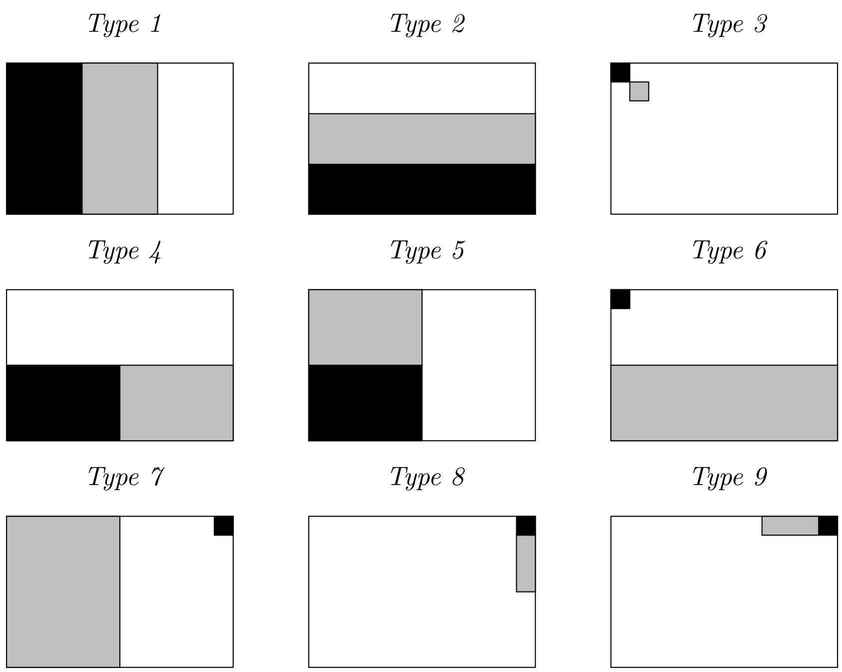

Figure 2: All possible types of minimum edge 3-cuts in $G\Box H$.

component (note that the two singletons might by contained in a common G-layer or H-layer). The figures suggest the deinfitions of other types. For example S is a type 6 edge 3-cut if

$$S = \{(x, y_1)(x, y_2) \,|\, x \in V(G), y_1 y_2 \in S_H\} \cup E(u, v),$$

for some edge-cut S_H in H, and where $E(u, v)$ is the set of edges incident to the vertex (u, v) in $G\Box H$. Type 8 occures when one connected component is a singleton, another connected component is contained in the same H-layer as the singleton, and the rest is contained in the third connected component. An example of the product $G\Box H$ where this type of edge 3-cut is a minimum edge 3-cut, is the product with factors G obtained from K_{10} and a vertex x that has one neighbor in K_{10}, and H obtained from a K_{10}, a K_{50} and a vertex y which has five neighbors in K_{10} and one neighbor in K_{50}. Separating the product into components $C_1 = \{(x, y)\}$, $C_2 = \{x\} \times K_{10}$ and $C_3 = V(G\Box H) - (C_1 \cup C_2)$ requires minimum number of edges, exactly 17.

Theorem 1. *Let G and H be connected graphs on at least four vertices. Then $\lambda_3(G\Box H)$ is equal to*

$$\min\{\lambda_3(G)|V(H)|, \lambda_3(H)|V(G)|, 2\delta(G) + \delta_2(H), 2\delta(H) + \delta_2(G), D(G, H)\}.$$

Moreover, every minimum edge 3-cut in $G\Box H$ is one of the 9 types shown in Figure 2.

Let us also mention that the condition $|V(G)|, |V(H)| \geq 4$ is necessary in the formulation of Theorem 1 as there exist examples of graph products where one factor has less than 4 vertices and a minimum edge 3-cut cannot be categorized in one of the nine types. Observe also the following:

- If a minimum edge 3-cut is of type 1, then $\lambda_3(G\Box H) = \lambda_3(G)|V(H)|$.
- If a minimum edge 3-cut is of type 2, then $\lambda_3(G\Box H) = \lambda_3(H)|V(G)|$.
- If a minimum edge 3-cut is of type 3, then $\lambda_3(G\Box H) = \delta_2(G) + 2\delta(H)$ or $\lambda_3(G\Box H) = \delta_2(H) + 2\delta(G)$.
- If a minimum edge 3-cut is of type 4,5,6 or 7, then $\lambda_3(G\Box H) = D(G, H)$.
- If a minimum edge 3-cut is of type 8, then $\lambda_3(G\Box H) = \delta_2(H) + 2\delta(G)$.
- If a minimum edge 3-cut is of type 9, then $\lambda_3(G\Box H) = \delta_2(G) + 2\delta(H)$.

In this work the generalized 3-edge connectivity of Cartesian products is solved. Before we proceed let us pose a question regarding the generalized k-edge connectivity of Cartesian products for $k > 3$, and suggest a possible continuation of this work. We define two operations on $G\Box H$ (and its subproducts) as follows:

- Operation 1: For an edge-cut $S_G \subseteq E(G)$ resp. $S_H \subseteq E(H)$ let operation 1 delete edges $\{(x, y)(x', y) \,|\, xx' \in S_G\}$ resp. edges $\{(x, y)(x, y') \,|\, yy' \in S_H\}$ from $G\Box H$.
- Operation 2: Let operation 2 delete all edges incident to a vertex in $G\Box H$.

When applying operation 1 the product $G\Box H$ is broken into two components, and both of them are subproducts of $G\Box H$. It follows from Theorem 1, that a minimum edge 3-cut is obtained by two applications of operation 1 or operation 2. However not every minimum edge 3-cut is obtained in this way, as type 8 and type 9 edge 3-cuts show (whenever we have a type 8 or 9 minimum edge 3-cut, there is also an edge 3-cut of same size (and thus minimum) of type 3). Results on generalized 3-edge connectivity of $G\Box H$ give rise to the following question.

Question 2. *Is for any connected graphs G and H a minimum edge k-cut in $G\Box H$ obtained by k applications of operation 1 or operation 2 to $G\Box H$?*

3. PRELIMINARY RESULTS

To construct an edge 3-cut S of type 4 we need an edge-cut S_H in H and an edge-cut S_G in G. The question is if we may assume that S_G resp. S_H are minnimum edge-cuts in G resp. H. Lemma 3 suggests that we may because if $|S| < \min\{\lambda_3(H)|V(G)|, 2\delta(G)+\delta_2(H), 2\delta(H)+\delta_2(G)\}$ then $|S| \geq \lambda(H)|V(G)| + \lambda(G)C(H)$.

Lemma 3. *Let G and H be connected nontrivial graphs, $S_H \subseteq E(H)$ be an edge-cut of H, C a connected component of $H - S_H$, and $S_G \subseteq E(G)$ an edge-cut of G. Then*

$$S = \{(x,y)(x,y') \,|\, x \in V(G), yy' \in S_H\} \cup \{(x,y)(x',y) \,|\, xx' \in S_G, y \in C\}$$

is an edge-cut of $G \Box H$ such that

$$|S| \geq \min\{\lambda_3(H)|V(G)|, 2\delta(G)+\delta_2(H), 2\delta(H)+\delta_2(G), \lambda(H)|V(G)| + \lambda(G)C(H)\}.$$

Proof. Note that S is an edge-cut with $|S| = |S_H||V(G)| + |S_G||C|$, and $(G \Box H) - S$ has three connected components (or more). If $|S_H| = \lambda(H)$ then $|S| \geq \lambda(H)|V(G)| + \lambda(G)C(H)$ follows from the definition of $C(H)$. So assume that $|S(H)| > \lambda(H) \geq 1$ and assume first that $|S_G| \geq 2$. Let C' be a connected component of $G - S_G$ such that $|C'| \leq |V(G)|/2$. If $|C| \geq 2$ we may choose $y_1, y_2 \in C$ and $x \in C'$ and compute $\deg_{G\Box H}(x,y_1) + \deg_{G\Box H}(x,y_2)$ to get an upper bound for $2\delta(G)+\delta_2(H)$. We have

$$\deg_{G\Box H}(x,y_1) + \deg_{G\Box H}(x,y_2) - \delta_{y_1y_2} \leq 2\left(\frac{|V(G)|}{2} - 1\right) + 2(|C|-1) - 1 + |S_H| + 2|S_G|$$

and therefore

$$2\delta(G) + \delta_2(H) \leq |V(G)| + 2|C| - 5 + |S_H| + 2|S_G| \leq |S_H||V(G)| + |S_G||C| = |S|$$

where the last inequality follows from

$$|V(G)| \geq \frac{|S_H|}{|S_H|-1} \quad \text{and} \quad |C| \geq \frac{2|S_G|-5}{|S_G|-2}\,.$$

If $|C| = 1$ then $|S_H| \geq \delta(H)$ and therefore

$$\begin{aligned} 2\delta(H) + \delta_2(G) &\leq 2\delta(H) + 2(|V(G)|-1) - 1 = 2\delta(H) + 2(|V(G)|-2) + 1 \\ &\leq 2|S_H| + (|V(G)|-2)|S_H| + |S_G| \leq |V(G)||S_H| + |C||S_G| = |S|. \end{aligned}$$

Let us now consider the case $|S_G| = 1$, and note that the above proof for $|C| = 1$ also works if $|S_G| = 1$. Therefore we may assume that $|C| \geq 2$. In this case $|S| = |S_H||V(G)| + |C|$. Choose two vertices x_1, x_2 of G, both in the smallest component of $G - S_G$ (or one in each component, if there is only one vertex in the smallest component) and compute $\deg(x_1)+\deg(x_2)-\delta_{x_1x_2}$ to get an upper bound for $\delta_2(G)$. Since $|S_G|=1$ we find that

$$\delta_2(G) \leq 2\left(\frac{|V(G)|}{2}-1\right)+|S_G| = |V(G)|-1\,.$$

Since $|C|\geq 2$ there is a vertex $y\in C$ such that at most $|S_H|/2$ edges of S_H are incident to y. Therefore $\delta_H \leq |S_H|/2+|C|-1$ and hence

$$2\delta(H)+\delta_2(G) \leq \deg_{G\Box H}(x_1,y)+\deg_{G\Box H}(x_2,y)-\delta_{x_1x_2} \leq |V(G)|+|S_H|+2|C|-3\,.$$

If $|S_H|\geq |C|$ then

$$2\delta(H)+\delta_2(G) \leq |V(G)|+|S_H|+2|C|-3 \leq |S_H||V(G)|+|C| = |S|\,,$$

follows from $|V(G)|\geq (2|S_H|-3)/(|S_H|-1)$. If $|S_H|<|C|$ then there is a vertex of $y\in C$ such that no edge in S_H is incident to y and therefore $\delta_H\leq |C|-1$ and so

$$2\delta(H)+\delta_2(G) \leq |V(G)|+2|C|-3\,.$$

If S_H is a minimal (with respect to inclusion) edge-cut in H (and we may assume, without loss of generality, that this is the case) then $\lambda_3(H)\leq |S_H|+\lambda(H)$. Since also $\lambda(H)<|S_H|$ we find that

$$\lambda_3(H)|V(G)| \leq |S_H||V(G)|+\lambda(H)|V(G)| \leq (2|S_H|-1)|V(G)|.$$

If $|S|<2\delta(H)+\delta_2(G)$ and $|S|<\lambda_3(H)|V(G)|$ a simple calculation gets us to the following contradiction

$$(|S_H|-1)|V(G)| > |C| > (|S_H|-1)|V(G)|+3\,.$$

□

Lemma 4. *Let G be a connected graph and S and edge 3-cut in G. Let C be a connected component of $G-S$ such that that $|C|>1$. Then $|C|+|S|>\lambda(G)+\delta(G)$ and if $|C|>2$ then $|C|+|S|>\lambda(G)+\delta(G)+1$.*

Proof. Let C, C_1 and C_2 be connected components of $G\Box H-S$. Observe that

$$|S\cap E(C,C_i)| \geq \left\lceil \frac{1}{2}|S\cap E(C,\bar{C})|\right\rceil\,,$$

for $i=1$ or $i=2$, and assume (wihout loss of generality) that this is true for $i=1$. Now we have

$$|S\cap E(C,C_1)|+|C| \geq \left\lceil \frac{1}{2}|S\cap E(C,\bar{C})|\right\rceil+|C|\,.$$

Since $|C|>1$ there is a vertex in C with at most $\lfloor \frac{1}{2}|S\cap E(C,\bar{C})|\rfloor$ neighbors in $\bar{C}$ and so

$$\left\lfloor \frac{1}{2}|S\cap E(C,\bar{C})|\right\rfloor+|C| > \delta(G)\,.$$

Note also that $|S \cap E(C_2, \bar{C}_2)| \geq \lambda(G)$ and hence

$$|S| + |C| = |S \cap E(C_2, \bar{C}_2)| + |S \cap E(C, C_1)| + |C| > \lambda(G) + \delta(G)\,,$$

which completes the proof of the first part of the lemma. For the second part observe that if $|C| > 2$ then there is a vertex in C with at most $\lfloor \frac{1}{3}|S \cap E(C, \bar{C})| \rfloor$ neighbors in $\bar{C}$ and so

$$\left\lceil \frac{1}{2}|S \cap E(C, \bar{C})| \right\rceil + |C| > \left\lfloor \frac{1}{3}|S \cap E(C, \bar{C})| \right\rfloor + |C| > \delta(G)\,,$$

with two strict inequlities above, we prove the second part of the claim similarly as the first part. □

Lemma 5. *Let G and H be connected graphs on at least four vertices, and let S be an edge 3-cut of $G \Box H$. If there is a connected component C of $G \Box H - S$, such that $C \subseteq H_a$ for some H-layer H_a with $\deg_G(a) > 1$ and $|C| > 1$, then $|S| > 2\delta(G) + \delta_2(H)$.*

Proof. The number of G-edges incident to a vertex in C is $|C| \deg_G(a)$. All these edges are in S. All H-edges with one endvertex in C and the other in $\bar{C}$ are in S. Suppose first that $2 < |C| < |V(H)|$. Since H is connected there are vertices $(a, y_0) \in C$ and $(a, y_0') \in \bar{C}$ such that $y_0 y_0' \in E(H)$. Since $|C| > 2$ there are vertices $u, v \neq y_0$ such that $(a, u), (a, v) \in C$. Note also that there is an H-layer $H_x, x \neq a$ such that $H_x - S$ is not connected, or there is a G-layer $G_y, y \notin p_H(C)$ such that $G_y - S$ is not connetced (here p_H denotes the projection from $V(G \Box H)$ to $V(H)$), for otherwise $G \Box H$ has only two connected components. Therefore there is at least one edge $e \in S$ in one of these layers. So we have

$$\begin{aligned} |S| &\geq |C| \deg_G(a) + E(\{u, v\}, \overline{p_H(C)}) + 2 \\ &\geq 2\deg_G(a) + (|C| - 2)\deg_G(a) + E(\{u, v\}, \overline{p_H(C)}) + 2 \end{aligned}$$

where the $+2$ above corresponds to edges $(a, y_0)(a, y_0')$ and e, both in S. Since $\deg_G(a) > 1$ we have $|S| > \deg_{G \Box H}(a, u) + \deg_{G \Box H}(a, v)$ if $u, v \notin E(H)$, and $|S| > \deg_{G \Box H}(a, u) + \deg_{G \Box H}(a, v) - 1$ if $uv \in E(H)$. In either case we have $|S| > 2\delta(G) + \delta_2(H)$.

Suppose that $C = \{(a, u), (a, v)\}$. If for all $y \notin \{u, v\}$, $G_y - S$ is connected, and for all $x \in V(G) - \{a\}$, $H_x - S$ is connected, then $G \Box H$ has only two connected components. Therefore one of these layers is not connected and thus contains an edge of S. We claim that S contains at least two edges that are not incident to (a, u) or (a, v), and note that if this is true, then $|S| > 2\delta(G) + \delta_2(H)$. If there is an $x \in V(G) - \{a\}$ such that $H_x - S$ is not connected, then one edge of S is in the H_x-layer. If there is no other edges of S (that is not incident to a vertex in C) then all other G and H-layers (except $H_a - S, H_x - S, G_u - S$ and $G_v - S$) are connected. But then x is adjacent only to a (for otherwise we have a G-edge, not incident to a vertex in C, that is in S) and hence $\delta(G) = 1$. Since $\deg_G(a) \geq 2 > \delta(G)$ we find that $|S| > 2\delta(G) + \delta_2(H)$. The other case is that there is an $y \in V(H) - \{u, v\}$ such that $G_y - S$ is not connected, and all other G and H-layers (except $H_a - S, G_y - S, G_u - S$ and $G_v - S$) are connected. Since $|V(H)| > 3$ we find that all H-layers, except H_a, are contained in the same connected component (for otherwise all G-layers of $G \square H - S$ are disconnected, and we have two edges in S not incident to a vertex in C). Let C and C' be connected components contained in H_a. If $|C'| > 1$, then the number of edges in S is at least $\deg_{G \square H}(a, u) + \deg_{G \square H}(a, v) - 2 + 2 > 2\delta(G) + \delta_2(H)$, where $\deg_{G \square H}(a, u) + \deg_{G \square H}(a, v) - 2$ corresponds to the neighborhood of C, and $+2$ corresponds to two G neighbors of C'. Assume now $C' = \{(a, w)\}$. If w is adjacent to a vertex $y \notin \{u, v\}$ we see again that $|S| \geq \deg_{G \square H}(a, u) + \deg_{G \square H}(a, v) - 2 + 2$, where $+2$ corresponds to one G-neighbor and one H-neighbor of (a, w) (the H-neighbor is (a, y)). If w is adjacent only to u or v, then one of these two vertices has a neighbor in the complement of $\{u, v, w\}$, and assume without loss of generality that this is u. Then $|S| \geq \deg_{G \square H}(a, v) + \deg_{G \square H}(a, w) - 1 + 2$ where $+2$ corresponds to a G-neighbor and an H-neighbor of (a, u). $\square$

4. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1. The proof is devided in several parts, and each of the parts treats one of the cases that might appear. The discussion from section 2, where all types of edge 3-cuts are described, proves that

$$\min\{\lambda_3(G)|V(H)|, \lambda_3(H)|V(G)|, 2\delta(G) + \delta_2(H), 2\delta(H) + \delta_2(G), D(G, H)\}.$$

is an upper bound for $\lambda_3(G \square H)$. To complete the proof we have to prove that the above minimum is also a lower bound. Let $S \subseteq E(G \square H)$ be an edge-cut that splits the product into three connected components. We shall prove that

$$|S| \geq \min\{\lambda_3(G)|V(H)|, \lambda_3(H)|V(G)|, 2\delta(G) + \delta_2(H), 2\delta(H) + \delta_2(G), D(G, H)\}.$$

Consider the graph $(G \square H) - S$ and suppose that there is no connected G-layer or H-layer of $(G \square H) - S$. In this case $|S| \geq \lambda(G)|V(H)| + \lambda(H)|V(G)| > T(G, H)$. So we may assume that there is at least one connected G-layer or H-layer of $(G \square H) - S$. There are several cases that might appear.

(i) There is at least one connected G-layer of $(G \square H) - S$ and there is no connected H-layer of $(G \square H) - S$.

  (a) All connected G-layers of $(G \square H) - S$ are contained in one connected component, say C_1.

  (b) All connected G-layers of $(G \square H) - S$ are contained in two connected components, say C_1 and C_2.

  (c) Not (a) or (b).

(ii) There is at least one connected G-layer of $(G \square H) - S$ and at least one connected H-layer of $(G \square H) - S$.

Case (i): Assume first that there is at least one connected G-layer of $(G\Box H)-S$ and that there is no connected H-layer of $(G\Box H)-S$ (or vice-versa, in which case an analogous proof works). Let C_1, C_2 and C_3 be connected components of $(G\Box H)-S$ and let

$$Y = \{y \in V(H) \,|\, G_y - S \text{ is connected}\}$$

represent the set of connected G-layers in $(G\Box H)-S$. Define

$$A = \bigcup_{y\in Y} G_y \text{ and } B = V(G\Box H) - A.$$

Case (a). Assume that $A \subseteq C_1$. Let $x_0 \in V(G)$ be such that $|S_{x_0}| \leq |S_x|$ for every $x \in V(G)$. Then we have

$$|S| = \sum_{x\in V(G)} |S_x| + \sum_{y\in V(H)} |S_y| \geq |S_{x_0}||V(G)| + \lambda(G)|\bar{Y}|.$$

Note that if the equality holds then $|S_x| = |S_{x_0}|$ for every $x \in V(G)$ (therefore, in case of equality, x_0 may be any vertex in G), and every layer $G_y, y \in \bar{Y}$ has exactly $\lambda(G)$ edges in S (and so it has exactly two connected components). Since no H-layer of $(G\Box H)-S$ is connected and $C_2, C_3 \subseteq B$, we find that $H_{x_0} - S_{x_0}$ has a connected component $C_{x_0} \subseteq B$, and so $|C_{x_0}| \leq |\bar{Y}|$. Let S' be the edge 3-cut in $G\Box H$ of size $|S_{x_0}||V(G)| + \lambda(G)|C_{x_0}|$ (constructed as in Lemma 3).

$$S' = \{(x,y)(x,y') \,|\, x \in V(G), yy' \in S_{x_0}\} \cup \{(x,y)(x',y) \,|\, xx' \in S_G, y \in p_H(C_{x_0})\}$$

where $S_G \subseteq E(G)$ is a λ-set in G. Observe that $|S'| = |S_{x_0}||V(G)| + \lambda(G)|C_{x_0}| \leq |S|$, moreover if the equality holds then $|C_{x_0}| = |\bar{Y}|$ and therefore $p_H(C_{x_0}) = \bar{Y}$. Since x_0 may be any vertex in G, we find that every H-layer has a component C such that $p_H(C) = \bar{Y}$. Since also $G_y - S$ has exactly two connected components for every $y \in \bar{Y}$, we find that S is a type 4 edge 3-cut. By Lemma 3 we have

$$\begin{aligned} |S| &\geq |S_{x_0}||V(G)| + \lambda(G)|C_{x_0}| = |S'| \\ &\geq \min\{\lambda_3(H)|V(G)|, 2\delta(G) + \delta_2(H), 2\delta(H) + \delta_2(G), T(G,H)\}, \end{aligned}$$

which concludes the proof of case (a).

Case (b). Assume $C_1 \cap A \neq \emptyset, C_2 \cap A \neq \emptyset$ and $C_3 \cap A = \emptyset$. Suppose that there is an H-layer H_{x_0} such that $|H_{x_0} \cap C_3| > 1$. Let us denote $H^i_{x_0} = H_{x_0} \cap C_i$ for $i \in [3] = \{1,2,3\}$. Note that by assumption $H^i_{x_0} \neq \emptyset$ for $i \in [3]$. By Lemma 4 we have $|H^3_{x_0}| + |S_{x_0}| > \lambda(H) + \delta(H)$ and if $|H^3_{x_0}| > 2$ then $|H^3_{x_0}| + |S_{x_0}| > \lambda(H) + \delta(H) + 1$. We will prove that $|S| > \lambda(H)|V(G)| + \delta(G\Box H)$ (and observe that this implies $|S| > T(G,H)$). Let $X \subseteq V(G)$ represent the set of H-layers that have nonempty intersection with C_3

$$X = \{x \in V(G) \,|\, H_x \cap C_3 \neq \emptyset\}.$$

Note that $X \neq V(G)$ (since otherwise $|S| \geq \lambda_3(H)|V(G)|$) and $X \neq \emptyset$ (since $C_3 \neq \emptyset$). Observe also that $|S_x| \geq \lambda(H)+1$ for every $x \in X$ because there are three connected components in the H_x layer. Let $(x_0, y_0) \in C_3$, and suppose that $|X| \geq 2$. Then

$$\begin{aligned}
|S| &= \sum_{x \in V(G)} |S_x| + \sum_{y \in V(H)} |S_y| \\
&\geq |S_{x_0}| + \sum_{x \in X \setminus \{x_0\}} |S_x| + \sum_{x \in \bar{X}} |S_x| + \sum_{y \in \bar{Y} \setminus \{y_0\}} |S_y| + |S_{y_0}| \\
&> \lambda(H) + \delta(H) - |H^3_{x_0}| + (|X|-1)(\lambda(H)+1) \\
&+ |\bar{X}|\lambda(H) + (|\bar{Y}|-1)\lambda(G) + |S_{y_0}| \\
&= \lambda(H)|V(G)| + (|X|-1) + \delta(H) - |H^3_{x_0}| + (|\bar{Y}|-1)\lambda(G) + |S_{y_0}| \\
&\geq \lambda(H)|V(G)| + (|X|-1) + \delta(H) - |\bar{Y}| + |\bar{Y}| - 1 + |E(\{x_0\}, \bar{X})| \\
&\geq \lambda(H)|V(G)| + \delta(G) + \delta(H) - 1,
\end{aligned}$$

the last inequality follows from $|X|-2+|E(\{x_0\},\bar{X})| \geq \delta(G)-1$. This proves that $|S| \geq \lambda(H)|V(G)|+\delta(G\Box H)$ and if the equality holds, then $|H^3_{x_0}| = |\bar{Y}|$ and $|S_{y_0}| = |E(\{x_0\},\bar{X})|$ (and $\lambda(G)=1$). So assume (for the purpose of a contradicition) that $|H^3_{x_0}| = |\bar{Y}|$ and $|S_{y_0}| = |E(\{x_0\},\bar{X})|$. Now we claim that $\delta(G) < (|X|-1) + |E(\{x_0\},\bar{X})|$, and note that this is equivalent to $\delta(G)-1 < (|X|-2)+|E(\{x_0\},\bar{X})|$ and therefore (if the claim is true) there are two strict inequalities in the above estimate for $|S|$, which implies that $|S| > \lambda(H)|V(G)|+\delta(G\Box H)$. It remains to show that $\delta(G) < (|X|-1)+|E(\{x_0\},\bar{X})|$. If x_0 is the only vertex in X with neighbors in $\bar{X}$, then $\delta(G) \leq \deg(x_1) \leq |X|-1 < (|X|-1)+|E(\{x_0\},\bar{X})|$, where $x_1 \in X$ is any vertex different from x_0. Otherwise there is a vertex $x_1 \in X$, with a neighbor in $\bar{X}$. We may choose y_0 so that $(x_0,y_0) \in C_3$ and $(x_1,y_0) \in C_3$ (note that we have assumed that $|H^3_{x_0}| = |\bar{Y}|$). So we have $|S_{y_0}| \geq |E(\{x_0\},\bar{X})|+|E(\{x_1\},\bar{X})| > |E(\{x_0\},\bar{X})|$, a contradiction. This completes the case $|X| \geq 2$.

Let $X = \{x_0\}$, and suppose $|C_3| = 2$. Then all edges adjacent to any vertex in C_3 are in S, except the edge between both vertices in C_3. Recall that no H-layer is connected so every H-layer different from H_{x_0} has at least one edge in S. Let $C_3 = \{(x_0,u),(x_0,v)\}$. Since u and v are adjacent we see that

$$|S| \geq \deg_{G\Box H}(x_0,u) + \deg_{G\Box H}(x_0,v) - 2 + 3 > \delta_2(H) + 2\delta(G)\,,$$

(here $+3$ is because of (at least) three edges of S in H-layers different from H_{x_0}).

If $|C_3| = |H^3_{x_0}| > 2$ then, according to Lemma 4, we have $|S_{x_0}| + |H^3_{x_0}| > \lambda(H)+\delta(H)+1$. Therefore S is greater than

$$\lambda(H)+\delta(H)+1-|H^3_{x_0}|+(|V(G)|-1)\lambda(H)+|H^3_{x_0}|\delta(G) \geq \lambda(H)|V(G)|+\delta(G)+\delta(H).$$

We conclude $|S| > \lambda(H)|V(G)| + \delta(G\Box H) \geq T(G,H) \geq D(G,H)$.

Assume now that there is no H-layer with two or more vertices in C_3. If $|C_3| = |X| = 2$, then $|\bar{X}| \geq 2$ because $|V(G)| \geq 4$. So we have

$$\begin{aligned} |S| &\geq |X|\delta(H) + |E(X,\bar{X})| + |\bar{X}|\lambda(H) \\ &\geq 2\delta(H) + |E(X,\bar{X})| + |\bar{X}|\,. \end{aligned}$$

Here $E(X,\bar{X})$ corresponds to G-edges of S and $|X|\delta(H)+|\bar{X}|\lambda(H)$ to H-edges of S. Since $|E(X,\bar{X})| + |\bar{X}| \geq |E(X,\bar{X})| + 2 > \delta_2(G)$ we find that $|S| > 2\delta(H) + \delta_2(G)$. Suppose that $|C_3| = |X| \geq 3$, and let $x_0 \in X$ be such that $|E(x_0,\bar{X})| \geq |E(x,\bar{X})|$ for every $x \in X$. Let $a, b \neq x_0$ be any vertices in X. Since in every layer $H_x, x \in X$ there are at least $\max\{\delta(H), 2\}$ edges in S we find that

$$\begin{aligned} |S| &\geq |X|\max\{\delta(H),2\} + |E(X,\bar{X})| + |\bar{X}|\lambda(H) \\ &\geq 2\delta(H) + 2(|X|-2) + |E(X,\bar{X})| + |\bar{X}| \\ &\geq 2\delta(H) + 2(|X|-2) + |E(\{a,b\},\bar{X})| + |E(x_0,\bar{X})| + |\bar{X}| \\ &\geq 2\delta(H) + 2(|X|-2) + |E(\{a,b\},\bar{X})| + 2 \end{aligned}$$

Since $2(|X|-2) + |E(\{a,b\},\bar{X})| + 2 > \delta_2(G)$ we arrive to the same conclusion $|S| > 2\delta(H) + \delta_2(G)$.

Finally let $|C_3| = |X| = 1$ and assume that $C_3 = \{(a, y_0)\}$. Recall that for an edge-cut $S \subseteq E(H)$ and a vertex $y \in V(H)$ we have defined

$$\omega(S,y) = \deg(y) - |S \cap E(y)|\,.$$

Without loss of generality assume that

$$|E(C_1 \cap H_a, \bar{C_1} \cap H_a)| \leq |E(C_2 \cap H_a, \bar{C_2} \cap H_a)|.$$

Suppose that for every $x \neq a$ we have $|S_x| \geq |E(C_1 \cap H_a, \bar{C_1} \cap H_a)|$. Let us denote by S' the projection of $E(C_1 \cap H_a, \bar{C_1} \cap H_a)$ to H, that is

$$S' = \{y_1y_2 \,|\, (a,y_1) \in C_1, (a,y_2) \notin C_1\}.$$

With this notation we have $\omega(S', y_0) + |S'| = |S_a|$ and therefore

$$|S| \geq |V(G)||S'| + \omega(S',y_0) + \deg_G(a) \geq B(G,H) \geq D(G,H)\,.$$

If the equality holds, then no G-layer, except G_{y_0} (which contains exactly $\deg_G(a)$ edges of S), contains an edge of S, and therefore $\bar{Y} = \{y_0\}$. If $C_1 \cap G_{y_0} = \emptyset$ or $C_2 \cap G_{y_0} = \emptyset$, then S is a type 6 edge 3-cut. So suppose that both of thses sets are nonempty and that $(a_1, y_0) \in C_1 \cap G_{y_0}$ and $(a_2, y_0) \in C_2 \cap G_{y_0}$. We claim that in this case $|S| > \delta_2(G) + 2\delta(H)$ or $|S| > D(G,H)$. Note first that there is no edge with one endvertex in $C_1 \cap G_{y_0}$ and the other in $C_2 \cap G_{y_0}$ for otherwise (as shown above) $|S| > D(G,H)$.

Let $Y_i = \{y \in Y \mid G_y \subseteq C_i\}$ for $i = 1, 2$. With this notation we have

$$|S| \geq \deg_H(y_0) + |N_H(y_0) \cap Y_1| + |N_H(y_0) \cap Y_2| + |V(G)| - 3 + \deg_G(a)\,,$$

where $\deg_H(y_0)$ corresponds to H-neighbors of (a, y_0), $|N_H(y_0) \cap Y_1|$ corresponds to H-neighbors of (a_2, y_0), $|N_H(y_0) \cap Y_2|$ corresponds to H-neighbors of (a_1, y_0), $|V(G)| - 3$ edges of S are placed in H-layers different from H_a, H_{a_1} and H_{a_2}, and finally $\deg_G(a)$ corresponds to G-neighbors of (a, y_0). Since $|N_H(y_0) \cap Y_1| + |N_H(y_0) \cap Y_2| = \deg_H(y_0)$, and since there is no edge with one endvertex in $C_1 \cap G_{y_0}$ and the other in $C_2 \cap G_{y_0}$ we find that $|S| > \deg_{G\Box H}(a_1, y_0) + \deg_{G\Box H}(a_2, y_0)$ if $\deg_G(a) > 2$. Otherwise $\deg_G(a) \leq 2$ and therefore we may choose a_1 and a_2 so that $aa_1 \notin E(G)$ or $aa_2 \notin E(G)$ (because $|V(G)| \geq 4$), in either case we have $|S| > \deg_{G\Box H}(a_1, y_0) + \deg_{G\Box H}(a_2, y_0)$.

Suppose that there is an $x \neq a$ such that $|S_x| < |S'|$, and let $x_0 \in V(G)$ be such that $|S_{x_0}| \leq |S_x|$ for every $x \in V(G)$. Then

$$|S| > |S_{x_0}||V(G)| + \omega(S', y_0) + \deg_G(a) + |\bar{Y}| - 1$$

where $\deg_G(a) + |\bar{Y}| - 1$ corresponds to edges in S that are in layers $G_y, y \in \bar{Y}$ (each of these layers is not connected, and so it has at least one edge in S, moreover G_{y_0} has at least $\deg_G(a)$ edges in S). We claim that $\omega(S', y_0) + |\bar{Y}| - 1 \geq \omega(p_H(S_{x_0}), y_0)$, where $p_H(S_{x_0})$ denotes the projection of S_{x_0} to H. To see this note that this is equivalent to

$$(1)\qquad |p_H(S_{x_0}) \cap E(y_0)| + |\bar{Y}| - 1 \geq |S' \cap E(y_0)|\,.$$

Before we prove the claim observe that $|E(C_1 \cap H_a, \bar{C}_1 \cap H_a)| \leq |E(C_2 \cap H_a, \bar{C}_2 \cap H_a)|$ is equivalent to $|E(y_0) - S'| \geq |S' \cap E(y_0)|$. Suppose first that $(x_0, y_0) \in C_2$. If $yy_0 \in S' \cap E(y_0)$ then $(a, y) \in C_1$. Now if $y \notin \bar{Y}$, then $y \in Y$, and so $(a, y) \in C_1$ implies $(x_0, y) \in C_1$ (recall that Y corresponds to connected G-layers) and thus $yy_0 \in p_H(S_{x_0}) \cap E(y_0)$. Therefore $|p_H(S_{x_0}) \cap E(y_0)| + |\bar{Y}| - 1 \geq |S' \cap E(y_0)|$. Now suppose that $(x_0, y_0) \in C_1$. If $yy_0 \in E(y_0) - S'$ then $(a, y) \in C_2$, and if $y \notin \bar{Y}$, then $y \in Y$, and so $(a, y) \in C_2$ implies $(x_0, y) \in C_2$. Therefore $yy_0 \in p_H(S_{x_0}) \cap E(y_0)$. The claim follows from $|E(y_0) - S'| \geq |S' \cap E(y_0)|$. This proves inequality (1) and therefore

$$|S| > |S_{x_0}||V(G)| + \omega(p_H(S_{x_0}), y_0) + \deg_G(a) \geq D(G, H)\,.$$

Case (c). If $C_i \cap A \neq \emptyset$ for every $i \in [3]$, then we have $|S| \geq \lambda_3(H)|V(G)|$, and if the equality holds then S contains no G-edges and so every G-layer of $G\Box H$ is connected. Consequently S is a type 2 edge 3-cut.

Case (ii): If there is at least one connected G-layer and at least one connected H-layer of $(G\Box H) - S$, then these two layers are contained in the same connected component, say C_1 (see Fig. 3). Let

$$
\begin{aligned}
Y_2 &= \{y \in V(H) - Y \mid G_y \subseteq C_1 \cup C_2\} \\
Y_3 &= \{y \in V(H) - Y \mid G_y \subseteq C_1 \cup C_3\} \\
Y_{2,3} &= \{y \in V(H) \mid G_y \cap C_2 \neq \emptyset, G_y \cap C_3 \neq \emptyset\} .
\end{aligned}
$$

Recall also that

$$Y = \{y \in V(H) \mid G_y - S \text{ is connected}\}$$

and note that all such layers are contained in C_1. Let

$$X_1 = \{x \in V(G) \mid H_x - S \text{ is connected}\} ,$$

and again note that all such layers are contained in C_1. Suppose that $Y_{2,3} \neq \emptyset$. Then there are vertices $(a, y_0) \in C_2$ and $(b, y_0) \in C_3$ for some $y_0 \in Y_{2,3}$. Let us count the edges in S. All H-neighbors of (a, y_0) that are in C_3 or C_1 contribute one edge to S. The number of such neighbors is at least $|Y_3 \cap p_H(N_H(y_0))| + |Y \cap p_H(N_H(y_0))|$. Similarly all H-neighbors of (b, y_0) that are in C_2 or C_1 contribute one edge to S and the number of such neighbors is at least $|Y_2 \cap p_H(N_H(y_0))| + |Y \cap p_H(N_H(y_0))|$. All G-neighbors of (a, y_0) or (b, y_0) that are in C_1 contribute one edge to S, the number of these edges is at least $|E(\{a, b\}, X_1)|$. For every $x \in \bar{X}_1 - \{a, b\}$ we have the following: if $(x, y_0) \in C_2$ the edge $(b, y_0)(x, y_0)$ is in S (if $bx \in E(G)$), and if $(x, y_0) \in C_3$ the edge $(a, y_0)(x, y_0)$ is in S (if $ax \in E(G)$), and if $(x, y_0) \in C_1$ then edges $(a, y_0)(x, y_0)$ and $(b, y_0)(x, y_0)$ are in S (if $ax \in E(G)$ resp. $bx \in E(G)$). So we have $|N_G(a) \cap N_G(b) \cap \bar{X}_1)|$ G-edges in S. Note that if there is a vertex

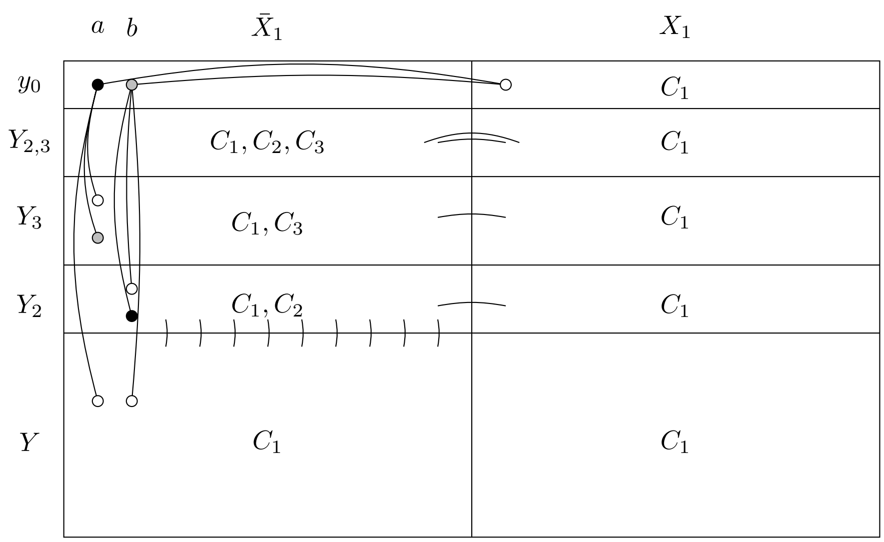

Figure 3: Vertices in C_1 are white, in C_2 are black, and in C_3 are gray.

$(x, y_0) \in C_1$ (where $x \in \bar{X}_1 - \{a, b\}$) adjacent to (a, y_0) or (b, y_0), then we have strictly more than $|N_G(a) \cap N_G(b) \cap \bar{X}_1)|$ edges in S. In every G-layer G_y with $y \in Y_{2,3} - \{y_0\}$ there are at least two G-edges of S (because these layers have three connected components), and in every G-layer G_y with $y \in Y_2 \cup Y_3$ there is at least one G-edge of S. Finally, in every H-layer H_x, where $x \in \bar{X}_1 - \{a, b\}$, there is at least one H-edge of S. When we sum all edges listed above we get

$$\begin{aligned} |S| &\geq 2(|Y_2 \cap N_H(y_0)| + |Y_3 \cap N_H(y_0)|) + 2|Y \cap N_H(y_0)| + 2(|Y_{2,3}| - 1) \\ &+ |E(\{a, b\}, X_1)| + |\bar{X}_1| - 2 + |N_G(a) \cap N_G(b) \cap \bar{X}_1)| \,. \end{aligned}$$

If (a, y_0) and (b, y_0) are adjacent then S has an additional edge (the edge between these two vertices). Now we have

$$\delta(H) \leq \deg(y_0) \leq |Y_2 \cap N_H(y_0)| + |Y_3 \cap N_H(y_0)| + |Y \cap N_H(y_0)| + |Y_{2,3}| - 1 \,.$$

If a and b are not adjacent then we have

$$\delta_2(G) \leq \deg(a) + \deg(b) \leq |E(\{a, b\}, X_1)| + |\bar{X}_1| - 2 + |N_G(a) \cap N_G(b) \cap \bar{X}_1)| \,,$$

and if the equality holds, then $\bar{X}_1 - \{a, b\} \subseteq N_G(a) \cup N_G(b)$. It follows that $|S| \geq \delta_2(G) + 2\delta(H)$ (note that if a and b are adjacent, then $\delta_2(G) \leq \deg(a) + \deg(b) - 1$, and so also in this case we have $|S| \geq \delta_2(G) + 2\delta(H)$).

We prove next, if the equality holds, then S is a type 3 or type 9 edge 3-cut. So assume $|S| = \delta_2(G) + 2\delta(H)$. Then according to the above observation $\bar{X}_1 - \{a, b\} \subseteq N_G(a) \cup N_G(b)$. Then for every vertex (x, y_0), where $x \in \bar{X}_1 - \{a, b\}$, we have $(x, y_0) \notin C_1$, since otherwise we have strictly more than $|N_G(a) \cap N_G(b) \cap \bar{X}_1)|$ edges in S (as noted before), and recall that these edges have one endvertex in $\{a, b\} \times \{y_0\}$ and the other in $(\bar{X}_1 - \{a, b\}) \times \{y_0\}$. So it follows $\bar{X}_1 \times \{y_0\} \subseteq C_2 \cup C_3$. Now observe that $E((\bar{X}_1 - \{a, b\}) \times \{y_0\}, X_1 \times \{y_0\}) = \emptyset$, since any edge in this set must be in S, and it was not listed above. This is equivalent to $E(\bar{X}_1 - \{a, b\}, X_1) = \emptyset$. We will first assume that $\bar{X}_1 - \{a, b\} \neq \emptyset$. Now the choice of vertices a and b at the beginning was random, with the only requirement that $a, b \in \bar{X}_1$, and $(a, y_0) \in C_2, (b, y_0) \in C_3$. So the condition $E((\bar{X}_1 - \{a, b\}), X_1) = \emptyset$ must be true for any such choice of a and b. This is possible only if there is exactly one vertex of $\bar{X}_1 \times \{y_0\}$ in C_2 and all other vertices are in C_3, and all edges of $E(\bar{X}_1 \times \{y_0\}, X_1 \times \{y_0\})$ are incident to the single vertex in C_2, or vice-versa (exactly one vertex in C_3 and the rest in C_2). So from now on assume that (a, y_0) is the only vertex in $G_{y_0} \cap C_2$ and that $(\bar{X}_1 - \{a\}) \times \{y_0\} \subseteq C_3$. Note also that (b, y_0) can be randomly chosen from $(\bar{X}_1 - \{a\}) \times \{y_0\}$. We have noted before that in every H-layer H_x, where $x \in \bar{X}_1 - \{a, b\}$, there is at least one H-edge of S. However if $x \in \bar{X}_1 - \{a\}$ and $C_2 \cap H_x \neq \emptyset$, then $H_x - S$ has three connected components and therefore there are at least two edges of S in H_x, which is a contradiction to our assumption $|S| = \delta_2(G) + 2\delta(H)$. It follows that $C_2 \subseteq H_a$. Since $E(\bar{X}_1 - \{a\}, X_1) = \emptyset$ we find that $\deg_G(a) \geq 2$, otherwise G is not connected. If $|C_2| > 1$ then Lemma 5 applies and so $|S| > \delta_2(G) + 2\delta(H)$, a contradiction. Therefore $|C_2| = 1$ and so

$$|S| \geq |E(a, y_0)| + |E((b, y_0), \{b\} \times Y)| + |\bar{Y}| - 1 + |\bar{X}_1| - 2$$

because all edges in $G\Box H$ incident to (a,y_0) are in S ((a,y_0) is the only vertex in C_2), and H-edges with one endvertex in (b,y_0) and the other in $\{b\}\times Y$ are in S, and because every $G_y, y\in \bar{Y}-\{y_0\}$, and every $H_x, x\in \bar{X}_1-\{a,b\}$ contains an edge of S. Since b has no neighbors in X_1 we see that

$$\begin{aligned}|S| &\geq |E(a,y_0)|+|E((b,y_0),\{b\}\times Y)|+|\bar{Y}|-1+|\bar{X}_1|-2\\ &\geq \deg_{G\Box H}(a,y_0)+\deg_{G\Box H}(b,y_0)\geq \delta_2(G)+2\delta(H)\end{aligned}$$

if a and b are not adjacent (recall that a is the only vertex in $\bar{X}_1$ with neighbors in X_1, and so b has no neighbors in X_1), and

$$\begin{aligned}|S| &\geq |E(a,y_0)|+|E((b,y_0),\{b\}\times Y)|+|\bar{Y}|-1+|\bar{X}_1|-2\\ &\geq \deg_{G\Box H}(a,y_0)+\deg_{G\Box H}(b,y_0)-1\geq \delta_2(G)+2\delta(H)\end{aligned}$$

if a and b are adjacent. If the equality holds, then in both cases the following is true

- (i) Every vertex in $\bar{Y}$ is adjacent to y_0.
- (ii) Every vertex in $(\bar{X}_1-\{a\})\times\bar{Y}$ is in C_3
- (iii) No vertex in $\bar{Y}-\{y_0\}$ is adjacent to a vertex in Y.

If (i) is not true, then $|E((b,y_0),\{b\}\times Y)|+|\bar{Y}|-1+|\bar{X}_1|-2>\deg_{G\Box H}(b,y_0)$ if a and b are not adjacent, and so $|S|>\delta_2(G)+2\delta(H)$. If a and b are adjacent, then $|E((b,y_0),\{b\}\times Y)|+|\bar{Y}|-1+|\bar{X}_1|-1>\deg_{G\Box H}(b,y_0)$ and so again $|S|>\delta_2(G)+2\delta(H)$. Claim (ii) follows from (i) and the fact that $(\bar{X}_1-\{a\})\times\{y_0\}\subseteq C_3$, as shown before. Claim (iii) follows from (ii), because any edge uv with $u\in\bar{Y}-\{y_0\}$ and $v\in Y$ implies $(b,u)(b,v)\in S$, and this edge was not listed above, and so in this case we have $|S|>\delta_2(G)+2\delta(H)$. Assuming (i) and (iii) we find that there is an edge with one endvertex in y_0 and the other in Y (because H is connected) and so $\deg_H(y_0)>\deg_H(y)$ for every $y\in\bar{Y}, y\neq y_0$. It follows $\deg_{G\Box H}(a,y_0)+\deg_{G\Box H}(b,y_0)>\deg_{G\Box H}(a,y)+\deg_{G\Box H}(b,y)\geq\delta_2(G)+2\delta(H)$ and so $|S|>\delta_2(G)+2\delta(H)$, provided that such y exists. If $\bar{Y}=\{y_0\}$ and $|C_2|=1$, then S is a type 9 edge 3-cut.

We still have to consider the case $\bar{X}_1-\{a,b\}=\emptyset$. As before let $\bar{X}_1=\{a,b\}$ and $y_0\in\bar{Y}$. In this case we have

$$\begin{aligned}|S| \geq\ & |E(X_1,\bar{X}_1)|+2(|Y_{2,3}|-1)+|Y_2|+|Y_3|+2|E(\{y_0\},Y)|\\ +\ & |N_H(a,y_0)\cap(\{a\}\times\bar{Y})\cap(C_1\cup C_3)|+|N_H(b,y_0)\cap(\{b\}\times\bar{Y})\cap(C_1\cup C_2)|\end{aligned}$$

where $|E(X_1,\bar{X}_1)|$ corresponds to G-edges with on endvertex in $\{(a,y_0),(b,y_0)\}$ and the other in $X_1\times\{y_0\}$, every H-layer H_y with $y\in Y_{2,3}$ has at least two edges in S and every H-layer H_y with $y\in Y_2\cup Y_3$ has at least one edges in S, this corresponds to $|Y_2|+|Y_3|$, finally $2|E(\{y_0\},Y)|$ corresponds to H-edges with one endvertex in $\{(a,y_0),(b,y_0)\}$ and the other in $\{a,b\}\times Y$. Last two terms correspond to H-neighors of (a,y_0) and (b,y_0) that are in $\{a,b\}\times\bar{Y}$. It is straightforward to check (and it follows from the above inequality) that $|S| \geq \deg_{G\Box H}(a, y_0) + \deg_{G\Box H}(b, y_0)$ and if the equality holds then y_0 is adjacent to every vertex in $\bar{Y}$ and no vertex from $\bar{Y}$, except y_0, is adjacent to a vertex in Y. Therefore, if $\bar{Y} \neq \{y_0\}$, there is a $y \in \bar{Y}$ with $\deg_H(y) < \deg_H(y_0)$ and so $|S|$ is at least

$$\deg_{G\Box H}(a, y_0) + \deg_{G\Box H}(b, y_0) > \deg_{G\Box H}(a, y) + \deg_{G\Box H}(b, y) \geq \delta_2(G) + 2\delta(H),$$

if a and b are not adjacent (if they are we argue similarly). In this case S is a type 3 edge 3-cut.

Until now we have, in case (ii), assumed that $Y_{2,3} \neq \emptyset$. So let us now assume that $Y_{2,3} = \emptyset$, this means that there is no G-layer with nonempty intersection with C_2 and C_3, and by symmetry we may assume that there is also no such H-layer. Let

$$\begin{aligned} Y_2 &= \{y \in V(H) - Y \mid G_y \subseteq C_1 \cup C_2\} \\ Y_3 &= \{y \in V(H) - Y \mid G_y \subseteq C_1 \cup C_3\} \\ X_2 &= \{x \in V(G) - X_1 \mid H_x \subseteq C_1 \cup C_2\} \\ X_3 &= \{x \in V(G) - X_1 \mid H_x \subseteq C_1 \cup C_3\}. \end{aligned}$$

It follows from the definition of the above sets that $X_2 \times Y_3 \subseteq C_1$ and $X_3 \times Y_2 \subseteq C_1$. Let $(x_2, y_2) \in C_2 \subseteq X_2 \times Y_2$ and $(x_3, y_3) \in C_3 \subseteq X_3 \times Y_3$. Observing the neighbors of these two vertices we have

$$\begin{aligned} S \geq\ & |X_2| + |X_3| - 2 + |Y_2| + |Y_3| - 2 + |N_G(x_2) \cap (X_3 \cup X_1)| \\ +\ & |N_G(x_3) \cap (X_2 \cup X_1)| + |N_H(y_2) \cap (Y_3 \cup Y)| + |N_H(y_2) \cap (Y_3 \cup Y)|. \end{aligned}$$

This is explained by the following:

- Every G-layer, except G_{y_2} and G_{y_3}, contains an edge of S,
- Every H-layer, except H_{x_2} and H_{x_3}, contains an edge of S,
- All G-neighbors of (x_2, y_2) that are in $(X_3 \cup X_1) \times \{y_2\}$ contribute an edge to S,
- All G-neighbors of (x_3, y_3) that are in $(X_2 \cup X_1) \times \{y_3\}$ contribute an edge to S,
- All H-neighbors of (x_2, y_2) that are in $\{x_2\} \times (Y_3 \cup Y)$ contribute an edge to S,
- All H-neighbors of (x_3, y_3) that are in $\{x_3\} \times (Y_2 \cup Y)$ contribute an edge to S.

If follows that $|S| \geq \deg_{G\Box H}(x_2, y_2) + \deg_{G\Box H}(x_3, y_3)$ and if the equality holds, then X_2, X_3, Y_2 and Y_3 induce complete graphs, moreover $X_2 \times Y_2 \subseteq C_2$ and

$X_3 \times Y_3 \subseteq C_3$ (for otherwise there are additional edges in S, that are not listed above). Now if $|X_2| \neq 1$ or $|X_3| \neq 1$ or $|Y_2| \neq 1$ or $|Y_3| \neq 1$ we may choose (x_2, y_2) or (x_3, y_3) so that there are additional (to the ones listed above) edges in S and so $|S| > \deg_{G\Box H}(x_2, y_2) + \deg_{G\Box H}(x_3, y_3)$. Therefore $|X_2| = |X_3| = |Y_2| = |Y_3| = 1$ and so $|C_2| = 1$ and $|C_3| = 1$, hence S s a type 3 edge 3-cut. This completes the last remaining case.

Acknowledgments. This work was supported by the Ministry of Education of Slovenia [grant numbers P1-0297, J1-9109].

REFERENCES

1. C. Balbuena, P. García-Vázquez, X. Marcote: *Reliability of interconnection networks modeled by a product of graphs.* Networks, **48** (2006), 114–120.
2. C. Balbuena, M Cera, A. Diánez, P. García-Vázquez, X. Marcote: *On the edge-connectivity and restricted edge-connectivity of a product of graphs.* Discrete Appl. Math., **155** (2007), 2444–2455.
3. J.-C. Bermond, C. Delorme, G. Farhi: *Large graphs with given degree and diameter. II:* J. Combin. Theory Ser. B, **36** (1984), 32–48.
4. B. Brešar, S. Špacapan: *Edge-connectivity of strong products of graphs.* Discuss. Math. Graph Theory **27** (2007),333–343.
5. B. Brešar, S. Špacapan: *On the connectivity of the direct product of graphs.* Australas. J. Combin., **41** (2008), 45–56.
6. X. Cao, Š. Brglez, S. Špacapan, E. Vumar: *On edge connectivity of direct products of graphs.* Inform. Process. Lett.,**111** (2011), 899–902.
7. R. M. Casablanca, M. Cera, P. García-Vázquez, J.C. Valenzuela-Tripodoro: *Superconnectivity of networks modeled by the strong product of graphs.* Appl. Anal. Discrete Math., **9** (2015), 347–356.
8. G. Chartrand, S. Kappor, L. Lesniak, D. Lick: *Generalized connectivity in graphs.* Bull. Bombay Math. Colloq., **2** (1984), 1–6.
9. G. Chartrand, F. Okamoto, P. Zhang: *Rainbow trees in graphs and generalized connectivity.* Networks **55** (2010), 360–367.
10. D. L. Goldsmith, B. Manvel, V. Faber: *Separation of graphs into three components by the removal of edges.* J. Graph Theory, **4** (1980), 219–218.
11. R. Hammack, W. Imrich, S. Klavžar: *Handbook of product graphs. Second edition.* CRC Press, Boca Raton, FL, 2011.
12. S. Klavžar, S. Špacapan: *On the edge-connectivity of Cartesian product graphs.* Asian-Eur. J. Math. **1** (2008), 93–98.
13. X. Li, Y. Mao, Y. Sun: *On the generalized (edge-)connectivity of graphs.* Australas. J. Comb., **58** (2014), 304–319.

14. X. Li, Y. Mao: *Generalized connectivity of graphs.* Springer Briefs in Mathematics, Springer, 2016.

15. J. Ou, W. Zhao: *On restricted edge connectivity of strong product graphs* Ars Combin., **123** (2015), 55–64.

16. S. Špacapan: *Connectivity of strong products of graphs.* Graphs Combin. **26** (2010), 457–467.

17. S. Špacapan: *A characterization of the edge connectivity of direct products of graphs, Discrete Math.*, **313** (2013), 1385–1393.

18. S. Špacapan: *Connectivity of Cartesian products of graphs.* Appl. Math. Lett., **21** (2008), 682–685.

Simon Špacapan,
University of Maribor, FME,
Smetanova 17, 2000 Maribor, Slovenia
E-mail: *simon.spacapan@um.si*

(Received 19. 07. 2016.)
(Revised 12. 05. 2021.)