

**OSCILLATION CRITERIA OF THIRD-ORDER
NONLINEAR NEUTRAL DELAY DIFFERENCE
EQUATIONS WITH NONCANONICAL OPERATORS**

*G. Ayyappan, G. E. Chatzarakis, T. Gopal and E. Thandapani**

In this paper, we present some new oscillation criteria for nonlinear neutral difference equations of the form

$$\Delta(b(n)\Delta(a(n)\Delta z(n))) + q(n)x^\alpha(\sigma(n)) = 0$$

where $z(n) = x(n) + p(n)x(\tau(n))$, $\alpha > 0$, $b(n) > 0$, $a(n) > 0$, $q(n) \geq 0$ and $p(n) > 1$. By summation averaging technique, we establish new criteria for the oscillation of all solutions of the studied difference equation above. We present four examples to show the strength of the new obtained results.

1. INTRODUCTION

This paper is concerned with the oscillatory behavior of solutions of the third-order nonlinear neutral delay difference equation

$$(1) \quad \Delta(b(n)\Delta(a(n)\Delta z(n))) + q(n)x^\alpha(\sigma(n)) = 0, \quad n \geq n_0$$

where $z(n) = x(n) + p(n)x(\tau(n))$, and $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 is a positive integer. We use the following assumptions:

(H_1) $\{a(n)\}$, $\{b(n)\}$ and $\{p(n)\}$ are positive real sequences with $p(n) \geq p_1 > 1$ for all $n \geq n_0$, where p_1 is a constant;

*Corresponding author. E. Thandapani

2020 Mathematics Subject Classification. 39A10.

Keywords and Phrases. Nonlinear difference equation, Third-order, Neutral, Noncanonical Operators, Oscillation.

- (H_2) $\{q(n)\}$ is a nonnegative real sequence and does not vanish eventually;
- (H_3) $\{\sigma(n)\}$ and $\{\tau(n)\}$ are strictly increasing sequences of integers with $\tau(n) \leq n-1$, $\sigma(n) \leq n-1$ and $\lim_{n \rightarrow \infty} \sigma(n) = \lim_{n \rightarrow \infty} \tau(n) = \infty$;
- (H_4) α is a ratio of odd positive integers;
- (H_5) $h(n) = \tau^{-1}(\sigma(n)) < n$.

We make use of the following operators in our subsequent discussion:

$$D_0 z(n) = z(n), \quad D_1 z(n) = a(n)\Delta z(n), \quad D_2 z(n) = b(n)\Delta(a(n)\Delta z(n)), \\ D_3 z(n) = \Delta(b(n)\Delta(a(n)\Delta z(n)))$$

and further assume that $D_3 z(n)$ is of noncanonical type, that is,

$$(2) \quad \sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} < \infty.$$

The real sequence $\{x(n)\}$ is said to be a solution of (1) if it is defined and satisfies (1) for all $n \in \mathbb{N}(n_0)$. A nontrivial solution of (1) is said to be oscillatory if the terms of the sequence $\{x(n)\}$ are neither eventually all positive nor eventually all negative, and nonoscillatory otherwise. As usual, we say that (1) has property A if any solution $\{x(n)\}$ of (1) is either oscillatory or satisfies $\lim_{n \rightarrow \infty} x(n) = 0$.

Determining oscillation criteria for neutral type difference equations attracted a large portion of research interest in recent years. The oscillatory and asymptotic properties of equations of type (1) with $p(n) \equiv 0$ were extensively investigated in the literature, see e.g., [1, 2, 4, 6, 7, 8, 13, 18] and the references contained therein. Most of the papers deal with the examination of so called canonical type equations, where conditions opposite to (2) hold, namely

$$(3) \quad \sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty.$$

Further, most of the results obtained in [1, 2, 9, 10, 12, 14, 16, 17] for neutral type third order difference equations ensure that every solution is either oscillatory or tends to zero monotonically.

In [15], the authors considered equation of the form (1) under condition (3) with $0 \leq p(n) \leq p < \infty$ and established conditions that guarantee every solution of (1) is oscillatory. To the best of authors' knowledge, there is no result regarding property A or the oscillation of all solutions of (1) under the conditions (H_1)–(H_2). Motivated by this observation, we attempt to obtain some new oscillation results for equation (1) under the condition (2) which ensures that all solutions of (1) are oscillatory. Four examples are presented to show the strength of the main results. It should be noted that the research in this paper was motivated by the recent results in [3, 5] established for differential equations.

2. MAIN RESULTS

For convenience, we use the following notation. For $n \geq N \in \mathbb{N}(n_0)$, we define

$$\begin{aligned}
 A(n) &= \sum_{s=N}^{n-1} \frac{1}{a(s)}, \quad B(n) = \sum_{s=N}^{n-1} \frac{1}{b(s)}, \quad C(n) = \sum_{s=N}^{n-1} \frac{B(s)}{a(s)}, \\
 \Pi_1(n) &= \sum_{s=n}^{\infty} \frac{1}{a(s)}, \quad \Pi_2(n) = \sum_{s=n}^{\infty} \frac{1}{b(s)}, \quad \Pi(n) = \sum_{s=n}^{\infty} \frac{\Pi_1(s+1)}{b(s)}, \\
 C(n, N) &= \sum_{s=N}^{n-1} \frac{1}{a(s)} \sum_{t=s}^{n-1} \frac{1}{b(t)}.
 \end{aligned}$$

Furthermore, in view of (H_5) , we see that the sequence $\{h(n)\}$ is nondecreasing with $\lim_{n \rightarrow \infty} h(n) = \infty$ which follows from $h(n) \geq \sigma(n)$ and

$$\Delta h(n) = \tau^{-1}(\sigma(n+1)) - \tau^{-1}(\sigma(n)) > 0,$$

since $\tau^{-1}(n)$ and $\sigma(n)$ are strictly increasing.

We begin with the following lemma which provides classification of nonoscillatory solutions of (1).

Lemma 1. *Assume that $(H_1) - (H_5)$ hold and $\{x(n)\}$ is an eventually positive solution of (1). Then*

$$(4) \quad z(n) \geq x(n) \geq \frac{1}{p(\tau^{-1}(n))} \left(z(\tau^{-1}(n)) - \frac{z(\tau^{-1}(\tau^{-1}(n)))}{p(\tau^{-1}(\tau^{-1}(n)))} \right)$$

and the corresponding sequence $z(n)$ satisfies one of the following cases

- $z(n) \in T_1 \iff z(n) > 0, D_1 z(n) < 0, D_2 z(n) < 0,$
- $z(n) \in T_2 \iff z(n) > 0, D_1 z(n) < 0, D_2 z(n) > 0,$
- $z(n) \in T_3 \iff z(n) > 0, D_1 z(n) > 0, D_2 z(n) > 0,$
- $z(n) \in T_4 \iff z(n) > 0, D_1 z(n) > 0, D_2 z(n) < 0,$

eventually.

Proof. Let $N \in \mathbb{N}(n_0)$ be such that $x(\sigma(n)) > 0$ and $x(\tau(n)) > 0$ for all $n \geq N$. By the definition of $z(n)$, we have $z(n) > x(n) > 0$ and

$$x(n) = \frac{z(\tau^{-1}(n)) - x(\tau^{-1}(n))}{p(\tau^{-1}(n))} \geq \frac{1}{p(\tau^{-1}(n))} \left(z(\tau^{-1}(n)) - \frac{z(\tau^{-1}(\tau^{-1}(n)))}{p(\tau^{-1}(\tau^{-1}(n)))} \right)$$

for $n \geq N$. Clearly, $D_3z(n)$ is nonincreasing for all $n \geq N$, since

$$D_3z(n) = -q(n)x^\alpha(\sigma(n)) \leq 0.$$

Hence $\{D_1z(n)\}$ and $\{D_2z(n)\}$ have the same sign eventually, which implies that the cases T_1 to T_4 are possible for $z(n)$. The proof of the lemma is complete. \square

In the next lemma, we state and prove a criterion for the nonexistence of positive increasing solutions of (1). In the proof, we use the following fact

$$(5) \quad \lim_{n \rightarrow \infty} \frac{C(\tau^{-1}(n))}{C(n)} = \lim_{n \rightarrow \infty} \frac{A(\tau^{-1}(n))}{A(n)} = 1,$$

which follows from (2).

Lemma 2. *Assume that $(H_1) - (H_5)$ hold. If*

$$(6) \quad \sum_{s=n_0}^{\infty} \frac{\Pi_2(s)q(s)}{p^\alpha(h(s))} = \infty,$$

then $T_3 \cup T_4 = \phi$.

Proof. Let condition (6) be satisfied but $z(n) \in T_3 \cup T_4$. Choose $N \in \mathbb{N}(n_0)$ such that $x(n) > 0$, $x(\sigma(n)) > 0$ and $x(\tau(n)) > 0$ for $n \geq N$.

First assume that $z(n) \in T_3$. Since $\{D_3z(n)\}$ is decreasing, we have

$$D_1z(n) \geq \sum_{s=N}^{n-1} \frac{1}{b(s)} D_2z(s) \geq B(n)D_2z(n).$$

Thus,

$$\Delta \left(\frac{D_1z(n)}{B(n)} \right) = \frac{B(n)D_2z(n) - D_1z(n)}{b(n)B(n)B(n+1)} \leq 0.$$

Therefore, $\left\{ \frac{D_1z(n)}{B(n)} \right\}$ is nonincreasing for all $n \geq N$ and also this fact implies

$$z(n) \geq \sum_{s=N}^{n-1} \frac{B(s)}{a(s)B(s)} D_1z(s) \geq \frac{D_1z(n)}{B(n)} C(n)$$

for all $n \geq N$. Hence, $\left\{ \frac{z(n)}{C(n)} \right\}$ is nonincreasing for all $n \geq N$, since

$$\Delta \left(\frac{z(n)}{C(n)} \right) = \frac{C(n)D_1z(n) - z(n)B(n)}{a(n)C(n)C(n+1)} \leq 0.$$

From $\tau^{-1}(\tau^{-1}(n)) > \tau^{-1}(n)$, we have

$$(7) \quad z(\tau^{-1}(\tau^{-1}(n))) \leq \frac{C(\tau^{-1}(\tau^{-1}(n)))}{C(\tau^{-1}(n))} z(\tau^{-1}(n)).$$

Using (7) in (4), we obtain

$$x(n) \geq \frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))} \left(1 - \frac{C(\tau^{-1}(\tau^{-1}(n)))}{C(\tau^{-1}(n))p(\tau^{-1}(\tau^{-1}(n)))} \right), \quad n \geq N.$$

In view of (H_2) and (5), there is an integer $N_1 \geq N$ such that for any constant $\varepsilon \in (0, p_1 - 1)$ and $n \geq N_1$, we get

$$\frac{C(\tau^{-1}(\tau^{-1}(n)))}{C(\tau^{-1}(n))p(\tau^{-1}(\tau^{-1}(n)))} \leq \frac{1 + \varepsilon}{p_1},$$

which implies

$$(8) \quad x(n) \geq \frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))} \left(1 - \frac{1 + \varepsilon}{p_1} \right) > 0.$$

Using (8) in (1), we obtain

$$(9) \quad \begin{aligned} 0 &\geq D_3 z(n) + \left(1 - \frac{1 + \varepsilon}{p_1} \right)^\alpha \frac{q(n)}{p^\alpha(h(n))} z^\alpha(h(n)) \\ &\geq D_3 z(n) + M^\alpha \left(1 - \frac{1 + \varepsilon}{p_1} \right)^\alpha \frac{q(n)}{p^\alpha(h(n))}, \end{aligned}$$

where we have used the fact that $z(n)$ and $h(n)$ are increasing and set $M = z(\tau^{-1}(\sigma(N_1))) < z(h(n))$. Summing up (9) from N_1 to n , we obtain

$$(10) \quad D_2 z(n + 1) \leq D_2 z(N_1) - M^\alpha \left(1 - \frac{1 + \varepsilon}{p_1} \right)^\alpha \sum_{s=N_1}^n \frac{q(s)}{p^\alpha(h(s))}.$$

Now, from (2) and (6), it follows that

$$\sum_{n=n_0}^{\infty} \frac{q(n)}{p^\alpha(h(n))} = \infty,$$

which, in view of (10), contradicts the positivity of $D_2 z(n)$.

Next, assume that $z(n) \in T_4$ for $n \geq N_1$. From the monotonicity of $D_1 z(n)$, we get

$$z(n) \geq \sum_{s=N_1}^{n-1} \frac{1}{a(s)} D_1 z(s) \geq A(n) D_1 z(n).$$

Thus

$$\Delta \left(\frac{z(n)}{A(n)} \right) = \frac{A(n) D_1 z(n) - z(n)}{a(n) A(n) A(n + 1)} \leq 0,$$

which implies $\left\{ \frac{z(n)}{A(n)} \right\}$ is nonincreasing for all $n \geq N_1$. Hence

$$z(\tau^{-1}(\tau^{-1}(n))) \leq \frac{A(\tau^{-1}(\tau^{-1}(n)))}{A(\tau^{-1}(n))} z(\tau^{-1}(n)).$$

Again, in view of (5) we obtain (9), which holds for any $\varepsilon > 0$ and $n \geq N_2 \geq N_1$. Summing up (9) from N_2 to $n - 1$, we get

$$-\Delta(D_1 z(n)) \geq M^\alpha \left(1 - \frac{1 + \varepsilon}{p_1}\right)^\alpha \frac{1}{b(n)} \sum_{s=N_2}^{n-1} \frac{q(s)}{p^\alpha(h(s))}.$$

Summing up the above inequality from N_2 to $n - 1$, we obtain

$$D_1 z(n) \leq D_1 z(N_2) - M^\alpha \left(1 - \frac{1 + \varepsilon}{p_1}\right)^\alpha \sum_{s=N_2}^{n-1} \frac{1}{b(s)} \sum_{t=N_2}^{s-1} \frac{q(t)}{p^\alpha(h(t))}.$$

Letting $n \rightarrow \infty$, by changing the order of summation, and using (6), one obtains

$$\begin{aligned} 0 \leq \lim_{n \rightarrow \infty} D_1 z(n) &\leq D_1 z(N_2) - M^\alpha \left(1 - \frac{1 + \varepsilon}{p_1}\right)^\alpha \sum_{s=N_2}^{n-1} \frac{1}{b(s)} \sum_{t=N_2}^{s-1} \frac{q(t)}{p^\alpha(h(t))} \\ &= D_1 z(N_2) - M^\alpha \left(1 - \frac{1 + \varepsilon}{p_1}\right)^\alpha \sum_{s=N_2}^{\infty} \frac{\Pi_2(s)q(s)}{p^\alpha(h(s))} = -\infty, \end{aligned}$$

which is a contradiction. The proof of the lemma is complete. \square

Theorem 1. Assume that $(H_1) - (H_5)$ hold. If

$$(11) \quad \sum_{s=n_0}^{\infty} \frac{\Pi(s)q(s)}{p^\alpha(h(s))} = \infty,$$

then (1) has property A.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (1). With no loss of generality, we can assume $x(n) > 0$, $x(\sigma(n)) > 0$ and $x(\tau(n)) > 0$ for all $n \geq N_1$ where $N_1 \in \mathbb{N}(n_0)$ is large enough. By Lemma 1, we see that $z(n) \in T_i$, $i = 1, 2, 3, 4$ for all $n \geq N_1$. First, in view of (2), we see that condition (11) yields

$$\sum_{s=n_0}^{\infty} \frac{\Pi_2(s)q(s)}{p^\alpha(h(s))} = \sum_{s=n_0}^{\infty} \frac{q(s)}{p^\alpha(h(s))} = \infty.$$

Therefore, by Lemma 2, we have $T_3 = T_4 = \phi_1$ and so $z(n) \in T_1$ or $z(n) \in T_2$. Since $z(n)$ is decreasing in (4), we have

$$(12) \quad x(n) \geq \frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))} \left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(n)))}\right) \geq \left(1 - \frac{1}{p_1}\right) \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}.$$

Therefore, there is a constant $M > 0$ such that

$$\lim_{n \rightarrow \infty} z(n) = M < \infty.$$

If $M > 0$, there is an integer $N_2 \geq N_1$, such that $z(n) \geq M$ for $n \geq N_2$. Hence from (12), we have

$$x(n) \geq M \frac{(p_1 - 1)}{p_1} \frac{1}{p(\tau^{-1}(n))}, \quad n \geq N_2.$$

Using this in (1), we obtain

$$(13) \quad D_3 z(n) + M^\alpha \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \frac{q(n)}{p^\alpha(h(n))} \leq 0, \quad n \geq N_2.$$

If $z(n) \in T_1$, then by summing up (13) from N_2 to $n - 1$ one gets

$$-\Delta(D_1 z(n)) \geq M^\alpha \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \frac{1}{b(n)} \sum_{s=N_2}^{n-1} \frac{q(s)}{p^\alpha(h(s))}.$$

Summing up the above inequality from N_2 to $n - 1$, we have

$$(14) \quad -\Delta z(n) \geq M^\alpha \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \frac{1}{a(n)} \sum_{s=N_2}^{n-1} \frac{1}{b(s)} \sum_{t=N_2}^{s-1} \frac{q(t)}{p^\alpha(h(t))}.$$

Summing up (14) from N_2 to $n - 1$, letting $n \rightarrow \infty$ and by changing the order of summation in the resulting inequality, and taking (11) into account, we obtain

$$\begin{aligned} M = \lim_{n \rightarrow \infty} z(n) &\leq z(N_2) - M^\alpha \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \sum_{n=N_2}^{\infty} \frac{1}{a(s)} \sum_{s=N_2}^{n-1} \frac{1}{b(s)} \sum_{t=N_2}^{s-1} \frac{q(t)}{p^\alpha(h(t))} \\ &= z(N_2) - M^\alpha \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \sum_{n=N_2}^{\infty} \frac{\Pi(n)q(n)}{p^\alpha(h(n))} = -\infty, \end{aligned}$$

which is a contradiction. Hence, $M = 0$.

If we take $z(n) \in T_2$, then by summing up (13) from N_2 to $n - 1$, and using (11), we get

$$D_2 z(n) \leq D_2 z(N_2) - M^\alpha \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \sum_{s=N_2}^{\infty} \frac{q(s)}{p^\alpha(h(s))} \rightarrow -\infty \text{ as } n \rightarrow \infty,$$

which contradicts the positivity of $D_2 z(n)$ and so $M = 0$. Since $z(n) \geq x(n)$, we obtain $\lim_{n \rightarrow \infty} x(n) = 0$. The proof of the theorem is complete. \square

Lemma 3. Assume that $(H_1) - (H_5)$ and (6) hold. If for $0 < \alpha \leq 1$ and for any $N \in \mathbb{N}(n_0)$ large enough

$$(15) \quad \limsup_{n \rightarrow \infty} \sum_{s=N}^{n-1} \left[\frac{\Pi(s+1)q(s)}{p^\alpha(h(s))M_1^{1-\alpha}} - \left(\frac{p_1}{p_1 - 1}\right)^\alpha \frac{\Pi_1(s+1)}{4\Pi(s+1)b(s)} \right] > \left(\frac{p_1}{p_1 - 1}\right)^\alpha$$

for any $M_1 > 0$, then $T_1 \cup T_3 \cup T_4 = \phi$.

Proof. Suppose $z(n) \in T_1 \cup T_3 \cup T_4$. Choose $N \in \mathbb{N}(n_0)$ such that $x(n) > 0$, $x(\tau(n)) > 0$, and $x(\sigma(n)) > 0$ for all $n \geq N$.

First assume that $z(n) \in T_1$. From Theorem 1, we see that (12) holds and using this in (1) yields

$$(16) \quad D_3 z(n) + \left(\frac{p_1 - 1}{p_1} \right)^\alpha \frac{q(n)}{p^\alpha(h(n))} z^\alpha(h(n)) \leq 0.$$

Define the sequence

$$(17) \quad w(n) = \Pi_1(n) D_1 z(n) + z(n).$$

From

$$z(n) \geq - \sum_{s=n}^{\infty} \frac{1}{a(s)} D_1 z(s) \geq -\Pi_1(n) D_1 z(n)$$

and

$$\Delta w(n) = \frac{\Pi_1(n+1)}{b(n)} D_2 z(n) < 0,$$

one can see that $\{w(n)\}$ is an eventually strictly decreasing positive sequence for all $n \geq N_1 \geq N$. Using (17) in (16), we obtain

$$(18) \quad \Delta \left(\frac{b(n)}{\Pi_1(n+1)} \Delta w(n) \right) + \left(\frac{p_1 - 1}{p_1} \right)^\alpha \frac{q(n)}{p^\alpha(h(n))} w^\alpha(h(n)) \leq 0.$$

Also

$$(19) \quad w(n) \geq - \sum_{s=n}^{\infty} \frac{\Pi_1(s+1)}{b(s)} \frac{b(s)}{\Pi_1(s+1)} \Delta w(s) \geq - \frac{b(n)}{\Pi_1(n+1)} \Pi(n) \Delta w(n).$$

Consider the sequence $\{\mu(n)\}$ defined by

$$(20) \quad \mu(n) = \frac{b(n) \Delta w(n)}{\Pi_1(n+1) w(n)}, \quad n \geq N_1.$$

Then $\mu(n) < 0$ for all $n \geq N_1$. From (19), it is clear that

$$(21) \quad -1 \leq \Pi(n) \mu(n) < 0.$$

Combining (20) and (18), we have

$$(22) \quad \begin{aligned} \Delta \mu(n) &\leq - \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \frac{q(n)}{p^\alpha(h(n))} \frac{w^\alpha(h(n))}{w(n+1)} - \frac{b(n) (\Delta w(n))^2}{\Pi_1(n+1) w(n) w(n+1)} \\ &\leq - \frac{(p_1 - 1)^\alpha}{p_1^\alpha} \frac{q(n)}{p^\alpha(h(n))} M_1^{\alpha-1} - \frac{\Pi_1(n+1)}{b(n)} \mu^2(n), \end{aligned}$$

where we have used that $w(n)$ is decreasing and $w(n) \leq M_1$ for $M_1 > 0$.

Multiplying (22) by $\Pi(n+1)$ and then summing the resulting inequality from N_1 to $n-1$, we have

$$\begin{aligned} \Pi(n)\mu(n) &\leq \Pi(N_1)\mu(N_1) - \sum_{s=N_1}^{n-1} \frac{\Pi_1(s+1)}{b(s)}\mu(s) - \left(\frac{p_1-1}{p_1}\right)^\alpha \\ &\quad \sum_{s=N_1}^{n-1} \frac{\Pi(s+1)q(s)}{p^\alpha(h(s))}M_1^{\alpha-1} - \sum_{s=N_1}^{n-1} \frac{\Pi(s+1)\Pi_1(s+1)}{b(s)}\mu^2(s) \\ &\leq \Pi(N_1)\mu(N_1) - \left(\frac{p_1-1}{p_1}\right)^\alpha \sum_{s=N_1}^{n-1} \frac{\Pi(s+1)q(s)}{p^\alpha(h(s))}M_1^{\alpha-1} \\ &\quad - \sum_{s=N_1}^{n-1} \frac{\Pi(s+1)\Pi_1(s+1)}{b(s)} \left(\left(\mu(s) + \frac{1}{2\Pi(s+1)} \right)^2 - \frac{1}{4\Pi^2(s+1)} \right) \\ &\leq - \left(\frac{p_1-1}{p_1}\right)^\alpha \sum_{s=N_1}^{n-1} \left[\frac{\Pi(s+1)q(s)}{p^\alpha(h(s))}M_1^{\alpha-1} - \left(\frac{p_1}{p_1-1}\right)^\alpha \frac{\Pi_1(s+1)}{4b(s)\Pi(s+1)} \right]. \end{aligned}$$

Hence, in view of (21), this inequality contradicts (15). Thus, $T_1 = \phi$. By Lemma 2, $T_3 = T_4 = \phi$ due to (6). This completes the proof. \square

In the next result, we state and prove a lemma which eliminates solutions belonging to T_2 .

Lemma 4. Assume that $(H_1) - (H_5)$ hold and $0 < \alpha \leq 1$. If

$$(23) \quad \limsup_{n \rightarrow \infty} \sum_{s=h(n)}^n \frac{q(s)C^\alpha(h(n), h(s))}{p^\alpha(h(s))} \begin{cases} > \left(\frac{p_1}{p_1-1}\right) & \text{if } \alpha = 1 \\ = \infty & \text{if } \alpha < 1, \end{cases}$$

then $T_2 = \phi$.

Proof. Assume that $z(n) \in T_2$. Choose $N \in \mathbb{N}(n_0)$ large enough so that $x(n) > 0$, $x(\tau(n)) > 0$, and $x(\sigma(n)) > 0$ for all $n \geq N$. Using (12) in (1), one gets

$$(24) \quad D_3z(n) + \left(\frac{p_1-1}{p_1}\right)^\alpha \frac{q(n)}{p^\alpha(h(n))}z^\alpha(h(n)) \leq 0.$$

From the monotonicity of $D_2z(n)$, we can easily see that

$$(25) \quad -D_1z(j) \geq D_1z(i) - D_1z(j) = \sum_{s=j}^{i-1} \frac{D_2z(s)}{b(s)} \geq D_2z(i) \sum_{s=j}^{i-1} \frac{1}{b(s)}$$

for $i \geq j \geq N$. Summing up (25) from j to $i-1$, we obtain

$$(26) \quad z(j) \geq D_2z(i) \sum_{s=j}^{i-1} \frac{1}{a(s)} \sum_{t=s}^{i-1} \frac{1}{b(t)} = C(i, j)D_2z(i).$$

Letting $j = h(s)$ and $i = h(n)$, $n \geq s \geq N$ in (26), we get

$$(27) \quad z(h(s)) \geq C(h(n), h(s))D_2z(h(n)).$$

Now, summing up (24) from $h(n)$ to n and using (27), we have

$$\begin{aligned} D_2z(h(n)) &\leq D_2z(h(n)) - D_2z(n+1) \geq \left(\frac{p_1-1}{p_1}\right)^\alpha \sum_{s=h(n)}^n \frac{q(s)}{p^\alpha(h(s))} z^\alpha(h(s)) \\ &\geq \left(\frac{p_1-1}{p_1}\right)^\alpha (D_2z(h(n)))^\alpha \sum_{s=h(n)}^n \frac{q(s)C^\alpha(h(n), h(s))}{p^\alpha(h(s))}. \end{aligned}$$

Dividing the last inequality by $(D_2z(h(n)))^\alpha$, we obtain

$$(28) \quad [D_2z(h(n))]^{1-\alpha} \geq \left(\frac{p_1-1}{p_1}\right)^\alpha \sum_{s=h(n)}^n \frac{q(s)C^\alpha(h(n), h(s))}{p^\alpha(h(s))}.$$

Since $0 < \alpha \leq 1$ and $D_2z(h(n))$ is positive and decreasing, by taking lim sup of (28), we are led to a contradiction with (23). This completes the proof. \square

Using Lemmas 2 to 4, we obtain the following main result of the paper.

Theorem 2. *Assume $(H_1) - (H_5)$ hold and $0 < \alpha \leq 1$. If conditions (6), (15) and (23) hold, then all solutions of (1) are oscillatory.*

3. EXAMPLES

In this section, we present four examples to show the importance of the main results.

Example 1. *Consider the third-order neutral difference equation*

$$(29) \quad \Delta(n(n+1)\Delta((n+1)(n+2)\Delta(x(n) + p_1x(\tau(n)))) + \lambda nx(\sigma(n))) = 0, \quad n \geq 1.$$

This equation is of the form (1). Thus $b(n) = n(n+1)$, $a(n) = (n+1)(n+2)$, $p(n) = p_1 > 1$, $q(n) = \lambda n$, $\lambda > 0$, $\alpha = 1$ and the delay functions $\sigma(n)$ and $\tau(n)$ satisfy (H_3) . Simple computation shows that $A(n) = \frac{1}{n+1}$, $\Pi(n) = \frac{1}{2n(n+1)}$. Using this, one can easily see that condition (11) reduces to

$$\sum_{n=1}^{\infty} \frac{\lambda}{2p(n+1)} = \infty.$$

Hence all assumptions of Theorem 1 are satisfied, and therefore (29) has property A.

Example 2. Consider the nonlinear third-order neutral difference equation

$$(30) \quad \Delta(n(n+1)\Delta((n+1)(n+2)\Delta(x(n) + (n+1)x(n-1)))) + \lambda n^4 x^3(n-2) = 0,$$

for $n \geq 1$. This equation is of the form (1). Thus $b(n) = n(n+1)$, $a(n) = (n+1)(n+2)$, $p(n) = n+1$, $q(n) = \lambda n^4$, $\lambda > 0$, $\alpha = 3$, $\sigma(n) = n-2$, $\tau(n) = n-1$ and $h(n) = n-1$. Simple calculation shows that $A(n) = \frac{1}{n+1}$, $\Pi(n) = \frac{1}{2n(n+1)}$ and $p(h(n)) = n$. Using this, we see that condition (11) reduces to

$$\sum_{n=1}^{\infty} \frac{\lambda n^4}{2n^4(n+1)} = \frac{\lambda}{2} \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

Thus, all conditions of Theorem 1 are satisfied, and therefore (30) has property A.

Example 3. Consider the linear third-order neutral difference equation

$$(31) \quad \Delta(2^n \Delta(2^n \Delta(x(n) + (n+1)x(n-1)))) + \lambda n 4^n x(n-2) = 0, \quad n \geq 1.$$

This equation is of the form (1). Thus $b(n) = a(n) = 2^n$, $p(n) = n+1$, $q(n) = \lambda n 4^n$, $\lambda > 0$, $\alpha = 1$, $\sigma(n) = n-2$, $\tau(n) = n-1$ and $h(n) = n-1$. Here

$$\Pi(n) = \frac{4}{3} \left(\frac{1}{4^n} \right), \quad C(h(n), h(s)) = \left(\frac{16}{3}(4^{-n}) - 16(2^{-n-s}) + \frac{32}{3}(4^{-s}) \right).$$

Since condition (11) is satisfied, by Theorem 1, (31) has property A. By simple calculations, conditions (15) and (23), take the form

$$\lambda > \frac{9}{4} \frac{p_1^2}{(p_1 - 1)}$$

and

$$\lambda > \frac{p_1^2}{4(p_1 - 1)},$$

respectively. By Theorem 2, we conclude that (31) is oscillatory if $\lambda > \frac{9}{4} \frac{p_1^2}{(p_1 - 1)}$.

Example 4. Consider the nonlinear third-order neutral difference equation

$$(32) \quad \Delta(2^n \Delta(2^n \Delta(x(n) + p_1 x(n-1)))) + \lambda n 4^n x^{\frac{1}{3}}(n-2) = 0, \quad n \geq 1.$$

This equation is of the form (1). Thus $b(n) = a(n) = 2^n$, $p(n) = p_1 > 1$, $q(n) = \lambda n 4^n$, $\lambda > 0$, $\alpha = \frac{1}{3}$, $\sigma(n) = n-2$, $\tau(n) = n-1$ and $h(n) = n-1$. Here

$$\Pi(n) = \frac{4}{3} \left(\frac{1}{4^n} \right), \quad C(h(n), h(s)) = \left(\frac{16}{3}(4^{-n}) - 16(2^{-n-s}) + \frac{32}{3}(4^{-s}) \right).$$

Since condition (11) is satisfied, by Theorem 1, (32) has property A. By simple calculations, conditions (15) and (23), take the form

$$\limsup_{n \rightarrow \infty} \sum_{s=1}^{n-1} \left[\frac{\lambda n}{3M_1^{\frac{2}{3}} p^{\frac{1}{4}}} - \frac{3}{4} \left(\frac{p_1}{p_1 - 1} \right)^{\frac{1}{3}} \right] = \infty$$

and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{s=n-1}^n \frac{\lambda s 4^s}{p_1^{\frac{1}{3}}} \left(\frac{16}{3} 4^{-n} - 16(2^{-n-s}) + \frac{32}{3} 4^{-s} \right)^{\frac{1}{3}} \\ = \limsup_{n \rightarrow \infty} \frac{\lambda}{p_1^{\frac{1}{3}}} (n-1) 4^{\frac{2n-1}{3}} = \infty, \end{aligned}$$

respectively. By Theorem 2, we conclude that (32) is oscillatory.

4. CONCLUSION

In this paper, we have established some new oscillation criteria for (1) when $p(n) > 1$ and $\alpha \in (0, 1]$. The main results ensure that all solutions are only oscillatory. Therefore our main results are new and complement to those in [9, 10, 11, 12, 14, 16, 17, 15]. It is also interesting to extend the results of this paper when $-1 < p(n) < 1$ or $\{p(n)\}$ is oscillatory.

Acknowledgements. The authors thank the reviewers for their useful suggestions and corrections, which led to the improvement of the content of the paper.

REFERENCES

1. R.P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.
2. R.P. AGARWAL, M. BOHNER, S.R. GRACE AND D.O' REGAN, *Discrete Oscillation Theory*, Hindawi Publ. Corp., New York, 2005.
3. G.E. CHATZARAKIS, J. DZURINA AND I. JADLOVSKA, *Oscillatory properties of third-order neutral delay differential equations with noncanonical operators*, Mathematics, **7(12)** (2019), 1177.
4. P. DINAKAR, S. SELVARANGAM AND E. THANDAPANI, *Oscillation of non-linear third order perturbed functional difference equations*, Nonauton. Dyn. Syst., **6**(2019), 57-64.
5. J. DZURINA AND I. JADLOVSKA, *Oscillation of third-order differential equations with noncanonical operators*, Appl. Math. Comput., **336** (2018), 394-402.
6. S.R. GRACE, R.P. AGARWAL AND M.F. AKTAS, *Oscillation criteria for third order nonlinear difference equations*, Fasc. Math., **42** (2009), 39-51.
7. S.R. GRACE, R.P. AGARWAL AND J.R. GRAEF, *Oscillation criteria for certain third order nonlinear difference equations*, Appl. Anal. Discrete Math., **3** (2009), 27-38.
8. J.R. GRAEF AND E. THANDAPANI, *Oscillatory and asymptotic behavior of solutions of third order delay difference equations*, Funkcial. Ekvac., **42** (1999), 355-369.

9. S. MEHAR BANU, S. NALINI AND R. ARUL, *Oscillation of third order nonlinear difference equations with several neutral terms*, Int. J. Math. Anal. Appl., **6** (2018), 213-221.
10. S. MEHAR BANU AND M. NAZREEN BANU, *Oscillatory behavior of half-linear third order delay difference equations*, IAENG Int. J. Appl. Math., (to appear).
11. S.H. SAKER, *Oscillation and asymptotic behavior of third order nonlinear neutral delay difference equations*, Dynam. Systems Appl., **15** (2006), 549-567.
12. S.H. SAKER, *New oscillation criteria for third order nonlinear neutral difference equations*, Math. Slovaca, **61** (2011), 579-600.
13. S.H. SAKER, J.O. ALZABUT AND A. MUKHEIMER, *On the oscillatory behavior for a certain class of third order nonlinear delay difference equations*, Electron. J. Qual. Theo. Diff. Equ., **67** (2010), 1-16.
14. A.A. SOBILO AND M. MIGDA, *On the oscillation of solutions of third order linear difference equations of neutral type*, Math. Bohemica, **130** (2005), 19-33.
15. S. SELVARANGAM, M. MADHAN, E. THANDAPANI AND S. PINELAS, *Improved oscillation conditions for third order neutral type difference equations*, Electron J. Differential Equations, **2017** (2017), 90. Pp.1-13.
16. E. THANDAPANI, M. VIJAYA AND T. LI, *ON THE OSCILLATION OF THIRD ORDER HALF-LINEAR NEUTRAL TYPE DIFFERENCE EQUATIONS*, Electron. J. Qual. Theory Differ. Equ., **76** (2011), 1-13.
17. E. THANDAPANI AND S. SELVARANGAM, *Oscillation of third order half-linear neutral difference equations*, Math. Bohemica, **138** (2013), 87-104.
18. K.S. VIDHYA, C. DHARUMAN, E. THANDAPANI AND S. PINELAS, *Oscillation theorems for third order nonlinear delay difference equations*, Math. Bohemica, **144** (2019), 25-37.

G. Ayyappan

Department of Mathematics,
Periyar University College of Arts and Science,
Pappireddipatti - 636 905, Tamilnadu, India
E-mail: ayyapmath@gmail.com

(Received 06. 02. 2020.)

(Revised 10. 04. 2021.)

G. E. Chatzarakis

Department of Electrical and Electronic Engineering Educators,
School of Pedagogical and Technological Education,
Marousi 15122, Athens, Greece
E-mail: gea.xatz@aspete.gr, geaxatz@otenet.gr

T. Gopal

Department of Mathematics,
Periyar University,
Salem - 636 011, Tamilnadu, India
E-mail: gopalmaths@gmail.com

E. Thandapani

Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chennai-600005, India
E-mail: ethandapani@yahoo.co.in