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NOTE ON SOME CLASSES OF HOLOMORPHIC FUNCTIONS RELATED TO JACK'S AND SCHWARZ'S LEMMA

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In this paper we discuss holomorphic mappings f of the unit disc $\mathbb{U}$ and corresponding index defined as $I_f(z) = \frac{zf'(z)}{f(z)}$. We are interested in finding bounds on the growth of functions f and related issues, if there are known some properties of I_f on $\mathbb{U}$. Our main tool in accomplishing this connection is Jack's lemma. As a special case, we got estimates on the growth of some classes of α-starlike functions as well as some interesting generalisations.

1. INTRODUCTION

The first author has started to consider some projects related to Schwarz lemma with collaborators Svetlik, Khalfallah and Ornek [**5, 13, 19, 11**]. Here we continue this investigation and outline some possibilities for further research. Concerning the content of this paper (with respect to application of Jack's lemma), we first obtained results in section 2 related to the growth of some classes of functions which include starlike functions and extend some Örnek's result. Second, we study the approach presented in section 2 and develop a more general mapping method for estimates based on Jack's lemma (see section 4, Theorem 5, and Theorem 7), which we call approach via Jack's lemma, to study the growth of some classes of holomorphic functions. It should be noted that we derive Theorem 6 from Theorem 5. Theorems 7 and 8 are further results related to the subordination principle. From these results it becames clear that the growth estimate of starlike functions, Theorem 1, can be considered as a special case of application of our general method.

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In addition we study the growth of holomorphic functions in Theorem 9. In Section 6, we give a short review of some results related to the subject.

Before we start to present our result, let's clarify some necessary notations.

Definition 1. 1. By $\mathbb{C}$ we denote the complex plane, by $\mathbb{U}$ the unit disk, by $B(a,r)$ the disk with center at a with radius r, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

2. If U and V are domains in $\mathbb{C}$ by $\mathrm{Hol}(U,V)$ (or $O(U,V)$) we denote the family of all holomorphic functions $f: U \to V$. For $\mathrm{Hol}(U,U)$ we simple write $\mathrm{Hol}(U)$.

3. If $F \in \mathrm{Hol}(\mathbb{U})$ with $F(0)=0$, we say that F is a Schwarz function on $\mathbb{U}$.

4. By $\mathbb{P}^+$ we denote the right hand half plane $\{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. Let $(\mathcal{P})$ denote the class of functions $p(z)$ which are holomorphic in the unit disc $\mathbb{U}$ and satisfy $\mathrm{Re}\, p(z) > 0$ for $|z|<1$. Let $(\mathcal{P}_a)$ denote the class of functions $p(z)$ which are holomorphic in the unit disc $\mathbb{U}$ and satisfy $p(0)=a$, $\mathrm{Re}\, p(z)>0$ for $|z|<1$.

5. Set $I_f(z) = \frac{zf'(z)}{f(z)}$ and $C[f](z) = 1 + \frac{zf''(z)}{f'(z)}$. We also use $S[f](z)$ instead of $I_f(z)$.

6. Set $M_f(r) = \max\{|f(re^{it})| : 0 \le t \le 2\pi\}$.

7. Let V be a set (in particular a domain) in $\mathbb{C}$. By $O_I(V)$ we denote the class of all holomorphic function f on $\mathbb{U}$ for which $I_f(\mathbb{U}) \subset V$. For a given constant a we say that $f \in O_I^a(V)$ if $f \in O_I(V)$ and $f(0)=a$.

This functional I_f plays important role in the theory of starlike functions, Jack's lemma and has some properties similar to logarithmic residum.

It is defined for example if V is an open set and f is holomorphic on V and $f : V \setminus \{0\} \to \mathbb{C}^*$; and in particular if $0 \in V$ and f has zero at 0 of order $n \ge 1$, then $I_f(0) = n$. This functional appears in definition of starlike function and Jack's Lemma.

If $a \neq 0$, $0 < r < |a|$ and $f : B(a,r) \to \mathbb{C}^*$ is holomorphic, then we can write $z = re^{i\theta}$ in polar form on $B(a,r)$ and there is a continuous branch $\theta^*(r,\theta)$ of $\arg f(re^{i\theta})$ and holds the following formula

$$D_\theta \theta^*(r,\theta) = \mathrm{Re}\, I_f(z). \tag{1}$$

For a fixed r we can consider $\theta^*(r,\theta)$ as a function of θ. From (1) it follows:

(I1) If $I_f : B(a,r_0) \to \mathbb{P}^+$, then $\theta^*(r,\theta)$ increases strictly.

From (I1) it follows:

(I2) If f is holomorphic on $\mathbb{U}$, $I_f : \mathbb{U} \to \mathbb{P}^+$ and f has a simple zero at 0, then f is injective and $f(\mathbb{U})$ is a starlike domain.

Concerning the growth of starlike functions Example 1 (Section 2 below) clarifies the situation and we recall it here.

Set $X_a = \{w : \mathrm{Re}\, w > a/2\}$, $L_s(z) = \frac{z}{(1-z)^s}$ and $L = L_1$. Then

(I3) $I_{L_s} = 1 + sL$ and for $s > 0$ it maps $\mathbb{U}$ onto $X_{2\alpha}$, where $\alpha = \alpha(s) = 1 - s/2$.

Note here that (I3) plays an important role in estimates concerning the growth of starlike functions and that L_s is a typical (with extremal growth) starlike function for $0 < s \leq 2$. L_s is not a starlike function for $s > 2$. It is interesting to draw the images of circles of radius $r \in (0,1)$ under L_s for $s > 2$.

Let us continue with the following Lemma

Lemma 1 (Jack's Lemma)**.** *Let $f(z)$ be a non-constant and holomorphic function in the unit disc $\mathbb{U}$ with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$, $r < 1$, at the point z_0, then*

$$I_f(z_0) = \frac{z_0 f'(z_0)}{f(z_0)} = k,$$

where $k \geq 1$ is a real number.

Lemma 2 (Jack's Lemma version 2)**.** *Let $f(z)$ be a non-constant and holomorphic function in the unit disc $\mathbb{U}$ with $f(0) = 0$. If $f(\mathbb{U})$ has points out of $\mathbb{U}$, then*

(a) there is a point $z_0 \in \mathbb{U}$ such that $|f'(z_0)| > 1$, and

(b) there is a point $z_1 \in \mathbb{U}$ such that $w_0 = f(z_1) \in \mathbb{T}$ and $I_f(z_1) = k$, where $k \geq 1$ is a real number.

Proof. Let $M_r = M_r(f)$, $0 < r < 1$, denote the maximum of $|f|$ on T_r and $M = M_f$ denote the supremum of $|f|$ on $\mathbb{U}$. By hypothesis one has $M > 1$. Suppose $|f|$ attain the maximum on T_r at z_r. By the boundary version of Schwarz lemma, for every $0 < r < 1$, we have $|f'(z_r)| \geq M_r/r$. Further there is $r_0 \in (0,1)$ such that for $r \in (r_0, 1)$, $M_r/r \geq M_1 > 1$ and therefore $|f'(z_0)| \geq M_1 > 1$.

□

In connection with Jack's lemma we suggest the following approach.

Suppose that ϕ is a holomorphic function defined on domain G and $f : \mathbb{U} \to G$ holomorphic function. Recall that we set $I_f(z) = \frac{zf'}{f}$ and $w = f(z)$. If $F = \phi \circ f$, then $F'(z) = \phi'(w)f'(z)$, we consider the following function

$$I_f^{\phi}(z) := I_f(\phi)(z) = \frac{zF'}{F} = \frac{z\phi'(w)}{\phi(w)} f'(z) = I_{\phi}(w) I_f(z). \tag{2}$$

Next let f be a holomorphic injective function. Then by the formula (2), $I_{f^{-1}}(w)I_f(z) = 1$, where $w = f(z)$.

Check that $I_{(fg)} = I_f + I_g$ and if $f = z^n g$, then $I_f = n + I_g$.

In this note we consider some estimate of Schwarz lemma type for functions f defined by means I_f based on Jack's lemma.

In [**18**] the author considered essentially the class $N(a)$ of holomorphic functions f on the unit disk satisfying $f(0) = a$ and $|\frac{zf'(z)}{f(z)}| < q := \frac{2a}{1+a^2}$.

It seems natural for given $q > 0$ and $a \in \mathbb{C}$ to consider the class $N(a,q)$ of holomorphic functions f on the unit disk satisfying

$$f(0) = a \quad \text{and} \quad \left|\frac{zf'(z)}{f(z)}\right| < q.$$

Thus, $N(a,q)$ is a simple notation for $O_I^a(B(0,q(a)))$.

Note that here q is independent of a. If $a = 0$ then for $s_n = bz^n$, $n \geq 1$ an natural number, $I_{s_n} = n$ and therefore s_n belongs to $N(0,q)$ iff $q > n$.

If $0 < q \leq 1$ the class $N(0,q)$ contains only constant functions.

If $|I_f(z)| \leq q$ and there is z_0 such that $|I_f(z_0)| = q$, then by maximum principle $I_f(z) = c$. If we set $g = \ln f$, then $g' = c/z$. Hence formally $g = c \ln z + c_1$, i.e. $f = c_2 z^c$ and therefore $c \geq 0$ is a natural number.

(A-1) If $f \in O_I(\overline{B(0,q)})$ and $|I_f|$ attains a local maximum at some points z_0, then $q = n$ and $f = cz^n$ for some natural integer n.

In order to present the subject of this paper we need:

(A-2) *Monodromy Theorem.* If a complex function f is analytic in a disk contained in a simply connected domain D and f can be analytically continued along every polygonal arc in D, then f can be analytically continued to a single-valued analytic function on all of D.

2. GROWTH OF α-STARLIKE FUNCTION

The subject related to Jack's lemma has been discussed by Örnek in several papers (see the literature). For $p \in \mathbb{N}$, where $p \geq 1$, let $\mathcal{A}(p)$ denote the class of all analytic functions of the form: $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$ which are holomorphic in the unit disc $\mathbb{U}$.

Let denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ which are holomorphic in the unit disc $\mathbb{U}$ by $\mathcal{A}$. Also, we give the following defintions:

Definition 2.

1. A function $f \in \mathcal{A}$ belongs to the class $S(\alpha)$ if $\mathrm{Re}S[f](z) > \alpha$, $z \in \mathbb{U}$.

2. A function $f \in \mathcal{A}(p)$ belongs to the class $S_p(\alpha)$ if $\mathrm{Re}S[f](z) > \alpha$, $z \in \mathbb{U}$.

3. A function $f \in \mathcal{A}$ is called starlike of order α, $0 \leq \alpha < 1$, if it is univalent and $\mathrm{Re}S[f](z) > \alpha$, $z \in \mathbb{U}$. We denote this class by $S^*(\alpha)$.

4. By $C^*(\alpha)$ we denote the family of convex univalent functions of order α.They are described by the analytic condition $\mathrm{Re}C[f](z) > \alpha$. Set $S^* = S^*(0)$ and $C^* = C^*(0)$.

If $f \in S^*(\alpha)$ and F is defined by $F(z) = f(z^n)$, n the natural number, then $F \in S(\alpha + n)$.

Example 1. Here we recall some comments from introduction. Set $X_a = \{w : \mathrm{Re}\, w > a/2\}$, $L_s(z) = \frac{z}{(1-z)^s}$ and $L = L_1$. Then

(I3) $I_{L_s} = 1 + sL$ and for $s > 0$ maps $\mathbb{U}$ onto $X_{2\alpha}$, where $\alpha = \alpha(s) = 1 - s/2$.

Note here that (I3) plays an important role in estimates concerning the growth of starlike functions and that L_s is a typical (with extremal growth) starlike function for $0 < s \le 2$. L_s is not a starlike function for $s > 2$. It is interesting to draw the images of circles of radius $r \in (0, 1)$ under L_s for for $s > 2$.

It is also convenient to introduce $s(\alpha) := 2(1 - \alpha)$ and $\beta_0 = \beta_0(\alpha) = 1/s(\alpha)$. We will prove the following result concerning growth of function which belong to the class $S_p(\alpha)$.

Theorem 1. *If f belongs to $S(\alpha)$ (in particular $S^*(\alpha)$), $0 < \alpha < 1$, and $1/\beta_0 = 2(1 - \alpha) = s(\alpha)$, then*

(i)

$$|f(z)| \le \frac{|z|}{(1 - |z|)^{1/\beta_0}}.$$

(ii) $|f''(0)| \le 2/\beta_0$.

It is clear that (i) holds if $\beta \le \beta_0$ i.e. $1/\beta \ge 2(1 - \alpha)$. In particular, we get Ornek's result:
if f belongs $S^*(1/2)$, then (i') $|f(z)| \le \frac{|z|}{(1-|z|)}$ and $|f''(0)| \le 2$.

For convex functions (i') holds. Since convex functions are in $S^*(1/2)$, this result is a generalization of corresponding one for convex functions.

Proof. Set $\tau := \beta(1 - \alpha)$ and

$$h = h_\beta = \left(\frac{z}{f}\right)^\beta - 1.$$

Then $f = z(1 + h)^{-1/\beta}$ and

$$\frac{h'}{1+h} = \beta\left(\frac{1}{z} - \frac{f'}{f}\right), \quad \text{i.e.} \quad \mathrm{Re}\left(z\frac{h'}{1+h}\right) = \beta(1 - S[f])$$

and therefore

$$\mathrm{Re}\left(z\frac{h'}{1+h}\right) = \beta(1 - S[f]) \le \beta(1 - \alpha) = \tau. \tag{3}$$

Now we show that h satisfies the condition of the Schwarz lemma: $h(0) = 0$ and h maps $\mathbb{U}$ into itself. For every r_0 there is z_0 such that $|z_0| = r_0$ and $\max_{|z|\le r_0} |h(z)| = |h(z_0)| = R$, and by Jack's lemma $z_0 h'(z_0) = k_0 h(z_0)$, $k_0 \ge 1$.

Using 3, we find $k_0 \mathrm{Re}(\frac{h}{1+h}) < \tau$ at z_0. If we set $w = h(z_0)$ we can rewrite 3 in the form

$$\mathrm{Re}\left(\frac{w}{1+w}\right) < \tau. \tag{4}$$

Since $A^0(w) = \frac{w}{1+w}$ maps the circle of radius 1 onto the line $\mathrm{Re}\zeta = \frac{1}{2}$, we conclude that A^0 maps $\mathbb{U}$ onto the half plane S defined by $\mathrm{Re}\zeta < \frac{1}{2}$. Hence in particular, if we suppose that $\beta(1-\alpha) \leq 1/2$, we conclude that $w' = A(w) \in S$ and therefore $w = A^{-1}(w') \in \mathbb{U}$ i.e. $R < 1$.

Set $g = (1+h)^{-1}$. If $\tau = \beta(1-\alpha) \leq 1/2$, then h satisfies the conditions of Schwarz lemma and therefore $|h(z)| \leq |z|$, $z \in \mathbb{U}$. Hence, we first have $|1+h(z)| \geq 1 - |h(z)| \geq 1 - |z|$ and therefore $|g(z)| \leq (1-|z|)^{-1}$. Next since $f = z(1+h)^{-1/\beta} = zg^\beta$ we find (i):

$$|f(z)| \leq \frac{|z|}{(1-|z|)^\beta}.$$

(ii) follows from $h'(0) = -\beta a_2$ and the Schwarz lemma.

□

2.1 An explanation of the above proof

Now we will give an explanation of the above proof. We hope that it will be useful for understanding of general method which we consider in Section 4.

Set $X_s = \{z : \mathrm{Re}\, z > -s\}$,$Y_s = \{z : \mathrm{Re}\, z < -s\}$ and define B_c by $B_c(z) = \frac{2cz}{1-z}$. Then B_c maps the unit disk $\mathbb{U}$ onto X_c. If f belongs to $S(\alpha)$ foe some $\alpha > 0$ then $f(0) = 0$ and f has no other zeros in $\mathbb{U}$. Suppose that it has zero at 0 of first order. Thus $f = zg$, where $g(0) \neq 0$. Next $I_f = 1 + I_g$ and therefore $\mathrm{Re}\, I_g > p := \alpha - 1$. Since g/c satisfies the same equation we can suppose that $g(0) = 1$ and consider $H = g - 1$. For any real number q there is a branch F of $(f/z)^q$ such that $F(0) = 1$ and $I_F = q(I_f - 1)$. Hence (1) $\mathrm{Re}\, I_F > l := q(\alpha - 1)$. Set $F = 1 + h$. We will apply Jack's lemma on h but according the above approach we substitute in inequality (1) instead of $h(z)$ and zh' respectively w and kw. The mapping $B_{k/2}$ maps the exterior of the unit disk $\mathbb{E} = \{|z| > 1\}$ onto $Y_{k/2} \subset Y_{1/2}$ for $k \geq 1$. Thus if we choose $l \geq -1/2$ we have a contradiction with (1). Hence $q(\alpha - 1) \leq 1/2$.

3. FURTHER PARTICULAR RESULTS OF INTRODUCTORY CHARACTER

Here, we outline the mapping method which give estimates based on Jack's lemma (see section 4) on some examples. We first have proved Proposition 1 and then Theorem 2 which improved the estimate in Proposition 1.

Concerning Örnek notation we set $a = z_1$ and without loss of generality we can suppose that a is in $[0,1]$. Further set $O_a = \frac{1+a^2}{1-a^2}$, $R_a = \frac{2a}{1-a^2}$ and the disk $K_a = B(O_a; R_a)$. Denote by $O(\mathbb{U}, K_a)$ the family of holomorphic functions f from the unit disk $\mathbb{U}$ into K_a with $f(0) = 1$.

Then using Jack's lemma we can prove that

Proposition 1. *$N(1, q(a))$ is a subclass of $H_a = O(\mathbb{U}, K_a)$.*

Proof. Set $p(z) = \frac{1+az}{1-az}$ and $\psi_a(z) = \frac{1+az}{1-az}$.

We see that ψ_a is injective on $\mathbb{C}$, ψ_a maps $\mathbb{U}$ onto K_a, $\psi_a'(z) = \frac{2a}{(1-az)^2}$ and $|\psi_a'(z)| \geq q_1(a) := \frac{2a}{(1+a)^2}$ for $|z| \geq 1$.

Let G_s be the image of $\mathbb{U}$ under I_{ψ_s}. We have

1. $I_{\psi_a}(z) = \frac{2az}{1-(az)^2}$, $I_{\psi_a}(1/z) = -I_{\psi_a}(z)$,
2. I_{ψ_a} is injective on $\mathbb{U}$,
3. I_{ψ_a} maps $\mathbb{U}$ onto G_a, and
4. $|I_{\psi_a}(z)| \geq q(a) := \frac{2a}{1+a^2}$ for $|z| = 1$.

Let G_s be the image of $\mathbb{U}$ under I_{ψ_s} and $\gamma_s = bG_s$. Let us prove that

(A-1) for every $s > a$, f maps $\mathbb{U}$ into K_s.

If (A-1) is not true that there is $r_0 \in (0,1)$ and a point z_0 such that $f(B(0,r_0) \subset K_s$, $|z_0| = r_0$ and $w_0 = f(z_0) \in bK_s$.

Next if $z_s = \psi_s^{-1}(w_0)$, by Jack's lemma, $I_f(z_0) = kI_{\psi_s}(z_s)$, $k \geq 1$. Since $|I_{\psi_s}(z_s)| \geq q(s) > q(a)$, we get $I_f(z_0) > q(a)$ which is a contradiction.

Hence it follows that f maps $\mathbb{U}$ into K_a.

Thus using Jack's lemma can we prove that $N(1, q(a))$ is subclass of $H_a = O(\mathbb{U}, K_a)$ and therefore $N(a, q(a))$ is subclass of aH_a and since $\psi_a'(0) = 2a$ by subordination principle $|f'(0)| \leq 2a^2$.

□

For example if (A1) $\psi(z) = ae^{qz}$, then $\psi'(z) = aqe^{qz}$, $I_\psi(z) = qz$ and $\psi'(0) = aqe^{q0} = aq$. If $q \leq \pi$, then ψ is injective on $\mathbb{U}$.

Theorem 2. *Let $\psi(z) = ae^{qz}$, $a \neq 0$. Suppose that $g \in N(a,q)$. Then*

(a) $g(\mathbb{U}) \subset \psi(\mathbb{U})$,

(b) $|g'(0)| \leq |\psi'(0)| = aq$,

(c) $\sup\{|g'(0)| : g \in N(a,q)\} = |\psi'(0)|$.

Proof. Suppose that $g \in N(a,q)$ and set $H = \psi^{-1} \circ g$. The function ψ^{-1} is not single-valued and therefore H is not single-valued. Since locally ψ has the inverse h which can be analytically continued along every polygonal arc in $\mathbb{U}$ and therefore there is a single-valued analytic branch function of H which we denote by h. Let us prove that $g(\mathbb{U}) \subset \psi(\mathbb{U})$.

If it is not true, then $h(\mathbb{U})$ has points out of $\mathbb{U}$. Without loss of generality we can suppose that h is holomorphic on the closed unit disk and let $|h|$ attains maximum at z_0 on unit circle. If we set $w_0 = g(z_0)$ and $z^0 = h(z_0)$ it is clear that $|z^0| > 1$ and $w_0 = \psi(z^0)$.

Next by the formula (2) and Jack's lemma, $I_{\psi^{-1}}(w_0)I_g(z_0) = k$, $k \geq 1$, and since $I_{\psi^{-1}}(w_0)I_\psi(z^0) = 1$ therefore $I_g(z_0) = kI_\psi(z^0)$. Hence since $I_\psi(z^0) = qz^0$, we conclude that $|I_g(z_0)| > q$ and it leads to a contradiction since $|I_g(z_0)| \geq q$ by definition of $N(a,q)$.

By Schwarz lemma, $|g'(0)| \leq |\psi'(0)| = aq$. In general if $q > 0$, there is a holomorphic branch of $f^{-1} \circ g$. $\square$

Note that by an application of Jack's lemma the class $N(0,q)$ contains only constant functions if $q > 1$.

Set $J_f(z) = \frac{f'(z)}{f(z)}$ and for $q > 0$ let $J_a(q)$ denote the class of all holomorphic functions f in $\mathbb{U}$ for which $f(0) = a$ and $|J_f(z)| \leq q$, $z \in \mathbb{U}$. We write J_a instead of $J_a(1)$.

(A) If f has zero of order $n \geq 1$, then J_f has a pole at 0. So, for $f \in J_a(q)$, a must not be zero, also, f has no zeros in $\mathbb{U}$. In that case, there exist branch $\ln f$ of logarithm to function f and since $(\ln f)' = \frac{f'}{f}$, we have $|(\ln f)'| \leq q$. Let us suppose that there is $z_0 \in \mathbb{U}$ such that $|\frac{f'(z_0)}{f(z_0)}| = q$. Then, by maximum principle, there is $\alpha \in [0, 2\pi)$ such that $\frac{f'(z)}{f(z)} = e^{i\alpha} q$ for every $z \in \mathbb{U}$. That precisely means that $f(z) = ae^{e^{i\alpha} qz}$, for every $z \in \mathbb{U}$.

(B) Note that Khalfallah also has observed that (2) below holds. If $f \in O_I^a(B(0, q(a)))$, $a \neq 0$, then I_f is a bounded analytic function on the unit disc and its value at zero is zero. Thus by the classical Schwarz lemma, we deduce that (1) $|I_f(z)| \leq q(a) = \frac{2a}{1+a^2}|z|$, and hence

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2a}{1+a^2}, \quad z \in \mathbb{U}.$$

In particular we obtain

(2) $|f'(0)| \leq \frac{2a}{1+a^2}$.

Note here if $a \neq 0$ and if f belongs $N(a,q)$ then f has no zeros in the unit disc and therefore (as above in (B)) we conclude:

(C) $N(a,q) = J_a(q)$.

Note here that in particular

$$\left|\frac{f'(z)}{f(z)}\right| \leq q := \frac{2a}{1+a^2},$$

is equivalent to $|(\ln f)'| \leq q$.

Perhaps we can characterize those functions geometrically; see starlike functions.

So if f belongs to $J_a(q)$ we have condition

(i) $|(\ln f)'| \leq q$.

Note that for $A(z) = A_{q,a}(z) = ae^{qz}$, $J_A = q$, and $f \in J_a(q)$ iff $g \in J_a(1)$, where $g(z) = f(z/q)$.

From (i) it follows that

(D) $f \in J_a(q)$ iff $f = A_{q,a} \circ F$, where F is Schwarz function on $\mathbb{U}$.

Theorem 3. *(i) Suppose that $q > 0$ and $a > 0$. Then*

(a) If $f \in J_a(q)$, then $ae^{-q|z|} \le |f(z)| \le M(z) := ae^{q|z|}$.

(b) If $f(1) = c = M(1) = ae^q$ and complex derivative $f'(1)$ exists, then $f'(1) = kc\frac{1-b}{1+b}$, where $b = a/c$ and $1 \le k \le q\frac{1+b}{1-b}$.

Proof. If $f \in J_a(q)$, then $f \in N(a, q)$. Now we have $f(\mathbb{U}) \subset \psi(\mathbb{U})$, where $\psi(z) = ae^{qz}$. Let us denote $k(z) = qz$. Since $f \ne 0$ in $\mathbb{U}$, there is a branch of function $h = \ln \circ f_1$ in disk $\mathbb{U}$, where $f_1 = f/a$. Let us denote $\psi_1 = \psi/a$. From $f_1(\mathbb{U}) \subset \psi_1(\mathbb{U})$, we have $h(\mathbb{U}) \subset k(\mathbb{U})$, where $k = \ln \circ \psi_1$ defined above is injective. By subordination we have $h(U_r) \subset k(U_r)$, so $-qr < \operatorname{Re} h(U_r) < qr$. Passing to limit gives us $-q|z| < \operatorname{Re} h(z) < q|z|$, where $|z| = r$. Since $|e^z| = e^{\operatorname{Re} z}$, we get $ae^{-q|z|} \le |f(z)| \le ae^{q|z|}$. Suppose that $0 \le a \le 1$ and set $T_b(z) = \frac{z-b}{1-bz}$. Then $T_b(1) = 1$, $T_b'(1) = \frac{1+b}{1-b}$.

Define $c = M(1) = ae^q$ and $g = f/c$. Note that g maps the unit disk into self.

Suppose that $f(1) = c$ and consider $F = T_b \circ g$. Then $g(1) = 1$ and by an application of Jack's lemma we have $F'(1) = kF(1)$, $k \ge 1$. Hence since $F(1) = 1$ we conclude first that $F'(1) = k$ and therefore $k = F'(1) = T_b'(1)g'(1) = \frac{1+b}{1-b}g'(1)$, we find $f'(1) = kc\frac{1-b}{1+b}$, $k \ge 1$. It follows from $f'(1) \le qf(1) = qc$, that $k \le q\frac{1+b}{1-b}$. □

Theorem 4. *Suppose that f belongs to $N(0, 1)$ and f has zero at 0 of order n. Then*

(a)
$$|f(z)| \le |g(0)|e^{(n+1)|z|}|z|^n = M(z),$$
where $g(0) = \frac{f^{(n)}(0)}{n!}$.

(b) If there is complex derivative at 1 and $f(1) = M(1) = |g(0)|e^{(n+1)}$, then $f'(1) = kf(1)$, $k \ge n$, and in particular $|f'(1)| \ge n|g(0)|e^{(n+1)}$.

Proof. If f belongs to $N(0, 1)$ and f has zero at 0 of order n, then $f = z^n g$, where g is a holomorphic function on $\mathbb{U}$ which has no zeros and $|n + I_g| \le 1$. Hence $|g(z)| \le |g(0)|e^{(n+1)|z|}$. Since $g(0) = \frac{f^{(n)}(0)}{n!}$, we find $|f(z)| \le |g(0)|e^{(n+1)|z|}|z|^n = M(z)$.

An application of Jack's lemma yields (b).

□

4. MAPPING METHOD FOR ESTIMATES BASED ON JACK'S LEMMA

In this manuscript we also consider more specific setting. Let G be a simply connected domain, a in G and ψ conformal map of $\mathbb{U}$ onto G with $\psi(0) = a$. Further let $\phi = \psi^{-1}$ and $F = \phi \circ f$. Set $w = f(z)$ and consider

$$I_f^{\psi}(z) = I_f(\psi)(z) = \frac{zF'}{F} = \frac{z\phi'(w)}{\phi(w)} f'(z) = I_\phi(w) I_f(z).$$

Denote by $O_a(\mathbb{U}, G)$ the family of holomorphic functions f from the unit disk $\mathbb{U}$ into G with $f(0) = a$.

4.1 Subordination principle, I_f and Starlike case

Set $U_r = B(0, r)$. If function f_r is holomorphic on disc U_r. Then function f defined with $f(z) = f_r(rz)$ is defined on the unit disk. For this function we have $I_f(z) = \frac{zf'(z)}{f(z)} = \frac{rzf_r'(rz)}{f_r(rz)} = I_{f_r}(rz)$. So, Jack's lemma can be formulated for function holomorphic on disk U_r instead od unit disk.

Theorem 5. *Suppose that*

(a) f and ψ are holomorphic on $\mathbb{U}$, and

(b) ψ is locally injective on $\mathbb{U}$,

(c) I_ψ is injective and $G_r := I_\psi(U_r)$ is starlike domain with respect to 0 for r close to 1.

If $I_f(\mathbb{U}) \subset I_\psi(\mathbb{U})$, then

(i-1) $f(\mathbb{U}) \subset \psi(\mathbb{U})$.

Proof. If $f(0) \neq 0$, then by the subordination principle

(A-1) $I_f(U_r) \subset I_\psi(U_r)$.

Let ψ be continuous on $\overline{\mathbb{U}}$ and locally injective. Suppose that (i-1) is not true. Then there is r near to 1 such that $f(U_r)$ is not subset of $\psi(U_r)$. Further there is $z_r \in U_r$ such that $f(U_m) \subset G_r := \psi(U_r)$ and $w_r = f(z_r) \in bG_r$, where $m = |z_r|$.

There is a branch h of $\psi^{-1} \circ f$ on U_m. If $\zeta_r = h(z_r)$, then $|\zeta_r| = r$ and by Jack

(A-2) $a_r = I_f(z_r) = kI_\psi(\zeta_r)$,$k \geq 1$. Set $A_r = I_\psi(\zeta_r)$. Then by (A-1), $a_r \in G_r$

If I_ψ is injective and G_r starlike then $k = 1$ and $A_r = a_r$ and it is a contradiction. Hence (i-1) holds true.

□

If g is holomorphic on $\mathbb{U}$ and $g(0) = 0$ we consider equation (1) $I_f = g$. If $h = g/z$, then $f'/f = h$ and therefore $(\ln f)' = h$. The solution of this equation satisfies $\ln f = H$, where H is a primitive function of h and therefore the solutions of (1) are given by $f = e^H$. If H_0 is a primitive of h with $H_0(0) = 0$, then $f = ce^{H_0}$, where $c \neq 0$.

We denote $S_0 = \{z \in \mathbb{C} : |\operatorname{Re} z| < 1\}$ and also $S_q = qS_0$. It is well know, that function tan maps strip $S_{\pi/2}$ injectivily onto $D = \mathbb{C} \setminus ((-\infty i, -i] \cup [i, i\infty))$. We can choose branch (in notation (A1) arctan) of multi valued inverse to function tan, which maps set D injectivily onto $S_{\pi/2}$.

It can be also checked that arctan maps unit disk onto strip $S_{\pi/4}$.

Consider a particular function $\omega = \omega_q = \frac{4}{\pi} q \arctan z$ which maps the unit disc onto strip S_q and let Θ be defined on $\mathbb{U}$ such that $\Theta'(z) = \frac{\omega(z)}{z}$, and $\Theta(0) = 0$.

Then $\psi(z) = \psi_q(z) = a\exp(\Theta(z))$ is a solution of equation $I_\psi = \omega_q$.

(B1) $\psi(z) = a\exp(\Theta(z))$.

Now we prove that

(A-1) ω is starlike.

It is possible to prove this on different manner, but we prefer direct analytic proof by computing I_{ω_0}.

Let $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$. If $\operatorname{tg}(x+iy) = a+ib$ then $\operatorname{tg}(x-iy) = a-ib$. Now, we have

$$\tan 2x = \tan[(x+iy)+(x-iy)] = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)} = \\ = \frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)} = \frac{2a}{1-a^2-b^2}.$$

Since $2x$ belongs to $S_{\pi/2}$ we can apply function arctan(defined at (A1)) to both sides of last equation. From that we get $x = \frac{1}{2}\arctan\frac{2a}{1-a^2-b^2}$. Further more

$$i\tanh 2y = \tan 2iy = \tan[(x+iy)-(x-iy)] = \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)} \\ = \frac{2ib}{1+a^2+b^2}.$$

So, we have $y = \frac{1}{2}\operatorname{atanh}\frac{2b}{1+a^2+b^2}$. Finaly, we get formula

$$\arctan(a+ib) = \frac{1}{2}\arctan\frac{2a}{1-a^2-b^2} + \frac{1}{2} i \operatorname{arcth}\frac{2b}{1+a^2+b^2},$$

holds for $a, b \in \mathbb{R}$ such that $a^2+b^2 < 1$.

Now, let us define $\omega_0(z) = \arctan(z)$ for $z = x+iy$ and $|z| < 1$. Then,

$$I_{\omega_0}(z) = \frac{z}{(1+z^2)\arctan z} = \\ = \frac{x\arctan\frac{2x}{1-x^2-y^2}(1+x^2+y^2) + y\operatorname{atanh}\frac{2y}{1+x^2+y^2}(1-x^2-y^2)}{2|1+z^2|^2|\arctan z|^2} + \\ + i\frac{y\arctan\frac{2x}{1-x^2-y^2}(1-x^2-y^2) - x\operatorname{arcth}\frac{2y}{1+x^2+y^2}(1+x^2+y^2)}{2|1+z^2|^2|\arctan z|^2}.$$

Set $A = x\arctan\frac{2x}{1-x^2-y^2}(1+x^2+y^2)$ and $B = y\operatorname{atanh}\frac{2y}{1+x^2+y^2}(1-x^2-y^2)$.

Since A and B are even function respectively with respect to x and y, we conclude that $A > 0$ for $x \neq 0$ and $B > 0$ for $y \neq 0$. Hence $A + B > 0$ for $(x, y) \neq (0, 0)$.

Thus

(i1) $\operatorname{Re} I_{\omega_0}(z) > 0$ for $z \neq 0$.

From $I_{\omega_0}(z) = \frac{z}{(1+z^2)\arctan z}$ it follows that

(i2) $I_{\omega_0}(0) = 1$.

Further from (i1) and (i2) it follows that $\operatorname{Re} I_\omega(z) > 0$ for all $z \in \mathbb{U}$ and therefore ω satisfies conditions of Theorem 5.

Finally, from Theorem 5 we derive the following.

Theorem 6. *(i) Let $\psi(z) = \psi_{(a,q)}(z) = a\exp(\Theta(z))$, $a \neq 0$, be a function given by* (B1).

(ii) Let f be a function holomrohic on the unit disc $\mathbb{U}$, such that $f(0) = a$ and $-q < \operatorname{Re} I_f(z) < q$, for all $z \in \mathbb{U}$.
If we suppose hypotheses (i) and (ii) hold, then

(a) $f(\mathbb{U}) \subset \psi(\mathbb{U})$ and

(b) $|f'(0)| \leq aq\frac{4}{\pi}$,

(c) Θ is injective convex function,

(d) $\psi(-|z|) \leq |f(z)| \leq \psi(|z|)$ when $z \in \mathbb{U}$.

Proof. Set $H = \psi^{-1} \circ f$. The function ψ^{-1} is not single-valued and therefore H is not single-valued. Since locally ψ has the inverse, H can be analytically continued along every polygonal arc in $\mathbb{U}$ and therefore there is single-valued analytic branch function of H which we denote by h. Since $\omega = I_\psi$ satisfies conditions of Theorem 5, by Theorem 5, we have $f(\mathbb{U}) \subset \psi(\mathbb{U})$. Now an application of Schwarz lemma to h yields (b).

(c) Further $C[\Theta] = S[z\Theta'] = 1 + S[\Theta'] = S[\omega]$ and therefore $\operatorname{Re} C[\Theta] > 0$. Since Θ' has no zeros in $\mathbb{U}$, we can conclude that Θ is an injective convex function.

For convenience of the reader we add more details of the proofs of (c). Let $0 < r < 1$. We define curve β_r such that $\beta_r(t) = \Theta(re^{it})$. Formula $\operatorname{Re} S[z\Theta'](re^{it}) > 0$ for all $0 \leq t < 2\pi$ geometrically means that argument of tangents to the curve β_r is (locally) strictly increasing function in t. From this we have that the change of argument of function $z\Theta'$ on the curves $\gamma_r(t) = re^{it}, 0 \leq t \leq 2\pi$ is equal $2\pi n$, where $n \in \mathbb{N}$. If, for some r we get $n \geq 2$ by the argument principle we would have that number of zeros of functions Θ' inside the curve γ_r must be $n - 1 \geq 1$. Since $\Theta'(z) = \frac{\operatorname{arctg} z}{z}$ has no zeros in $\mathbb{U}$, we have contradiction. So, $\Delta_{\gamma_r} Argz\Theta' = 2\pi$, which means that $\beta_r = \Theta(\gamma_r)$ is a Jordan curve, enclosing a convex region.

This precisely means that

(I) set $\Theta(U_r)$ is convex for every $0 < r < 1$.

If $x \in (-1, 1)$ we get $\Theta(x) = \int\limits_0^x \frac{\arctan t}{t} \mathrm{d}t \in \mathbb{R}$. If $y \in (-1, 1)$, we have that $\Theta(iy) = \int\limits_0^y \frac{\arctan it}{t} \mathrm{d}t = \int\limits_0^y \frac{i \operatorname{atanh} t}{t} \mathrm{d}t = iv$, where $v = \int\limits_0^y \frac{\operatorname{atanh} t}{t} \mathrm{d}t \in \mathbb{R}$.

(d) Proof of this part can be based on a version of Theorem 8, but we prefer a direct proof. Let $0 < r < 1$. We define curve β_r such that $\beta_r(t) = \Theta(re^{it})$. Next, we prove that tangents to curves β_r are vertical at points on the real axis. This can be seen from next equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta_r(t)|_{t=0} = ire^{it}\Theta'(re^{it})|_{t=0} = ire^{it}\frac{\arctan(re^{it})}{re^{it}}|_{t=0} = ir\frac{\arctan r}{r} = i\arctan r.$$

Analogously, $\frac{\mathrm{d}}{\mathrm{d}t}\beta_r(t)|_{t=\pi} = ire^{it}\frac{\arctan re^{it}}{re^{it}}|_{t=\pi} = -i\arctan r$.

Combined with (I) this gives that $\Theta(-r) \le \operatorname{Re}\Theta(re^{it}) \le \Theta(r)$. Subsequently, $\Theta(-r) < \operatorname{Re}\Theta(z) < \Theta(r)$ for every $z \in U_r$.

Since $\psi \ne 0$, we have $f \ne 0$ on $\mathbb{U}$. Now, we can define a branch of function $h = \ln\circ f_1$, where $f_1 = f/a$. Let us denote $\psi_1 = \psi/a$. Now $\Theta = \ln\circ\psi_1$. From $f(\mathbb{U}) \subset \psi(\mathbb{U})$, we can deduce that $h(\mathbb{U}) \subset \Theta(\mathbb{U})$. Since Θ is injective, we have $h(U_r) \subset \Theta(U_r)$. This means that $\Theta(-r) < \operatorname{Re} h(z) < \Theta(r)$, for $z \in U_r$.

Hence finally, we have $\psi(-|z|) \le |f(z)| \le \psi(|z|)$ on the unit disk.

□

4.2 Subordination principle 2

Definition 1. Suppose that a is given. Let ψ be a holomorphic function defined on domain D which contains $\overline{\mathbb{U}}$, $\psi(0) = a$ and set $m(\psi) = \inf\{|I_\psi(z)| : z \in D \setminus \overline{\mathbb{U}}$. Suppose that a germ of ψ^{-1} has holomorphic continuation along any curve in G.

Denote by $O_a^1(\psi)$ the family of all holomorphic functions f in $O_a(\mathbb{U}, G)$ for which

(1) $|I_f(z)| < m(\psi)$ for $z \in \mathbb{U}$.

Note that in this definition ψ determines G and in particular $\psi(\mathbb{U})$.

Let h be branch of $\psi^{-1} \circ f$ and $z' = h(z)$. Since by Jack's lemma, (1) implies that $f(\mathbb{U}) \subset \psi(\mathbb{U})$ there are points z in $\mathbb{U}$ such that $|I_f(z)| \ge |I_\psi(z')|$, and hence by (1) we conclude that $|I_\psi(z')| < m(\psi)$.

The equation $I_\psi = z^n$ has solution $\psi = ce^{z^n/n}$.

Theorem 7. *If $f \in O_a^1(\psi)$, then*

(a) $|f'(0)| \le |\psi'(0)|$,

(b) $\sup\{|f'(0)| : f \in O_a(\psi)\} = |\psi'(0)|$.

For a set $M \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$ we define $\operatorname{Star}(M, z_0) = \cup_{z\in M}[z, z_0]$ and

$\operatorname{Star}(M, \infty) = \cup_{z\in M}[z, \infty)$, where $[z, \infty) = \{sz : s \ge 1\}$.

Note that it is convenient to write shortly S instead of Star.

Denote by $\mathbb{E} = \{z : |z| > 1\}$ exterior of the closed unit disk. For $\epsilon > 0$ set $S(I_\psi(A(1, 1+\epsilon)), \infty) = \cup_{s>1} sI_\psi(A(1, 1+\epsilon))$ and $S(I_\psi(\mathbb{E}), \infty) = \cup_{s>1} sI_\psi(E)$, where $A = A(1, 1+\epsilon) = \{z \in \mathbb{C} : 1 < |z| < 1+\epsilon\}$ denotes the ring domain centered at 0.

Suppose that

(h1) $f(\mathbb{U}) \subset G = G(\psi, \epsilon) := \psi(B(0, 1 + \epsilon)$,

(h2) $I_f : \mathbb{U} \to \mathbb{C} \setminus S(I_\psi(A(1, 1 + \epsilon)), \infty)$, and

(h3) A germ of ψ^{-1} has holomorphic continuation along any curve in G.

Suppose that ψ, $\epsilon > 0$, a in $G = G(\psi, \epsilon)$ are given and $\psi(0) = a$.

Denote $O_a(\psi, \epsilon)$ the family of holomorphic functions f on $\mathbb{U}$ which satisfy the previous hypothesis and $f(0) = a$.

If there is $\epsilon > 0$ such that $f \in O_a(\psi, \epsilon)$ we say that $f \in O_a(\psi)$.

Theorem 8. *If $f \in O_a(\psi)$, then*

(a) $f(\mathbb{U}) \subset \psi(\mathbb{U})$,

(b) $|f(z)| \leq M_\psi(r)$, *where* $r = |z|$.

Proof. (a) Suppose that $f \in O_a(\psi)$ and set $H = \psi^{-1} \circ f$. The function ψ^{-1} is not single-valued in general and therefore H is not single-valued. Since by (h3) and monodromy theorem locally ψ^{-1} has the inverse then a germ of H can be analytically continued along every polygonal arc in $\mathbb{U}$ and therefore there is single-valued analytic branch function of H which we denote by h. Let us prove that $f(\mathbb{U}) \subset \psi(\mathbb{U})$.

If it is not true, then $h(\mathbb{U})$ has points out of $\mathbb{U}$. Without lost of generality we can suppose that h is holomorphic on the closed unit disk and let $|h|$ attains maximum at z_0 on the unit circle. If we set $w_0 = g(z_0)$ and $z^0 = h(z_0)$ it is clear that $|z^0| > 1$ and $w_0 = f_0(z^0)$.

(b) Set $w = f(z)$ and $z' = h(z)$. Then $w = f(z) = \psi(z')$ and by Schwarz lemma, $|z'| = |h(z)| \leq |z|$. Hence an application of Maximum principle yields (b).

Next by the formula (2) and Jack's lemma, $I_{f_0^{-1}}(w_0)I_g(z_0) = k$, $k \geq 1$, and since $I_{f_0^{-1}}(w_0)I_{f_0}(z^0) = 1$ therefore

$I_f(z_0) = kI_\psi(z^0)$. It is a contradiction with $(h2)$.

By Schwarz lemma, $|f'(0)| \leq |\psi'(0)|$. □

Example 2. Set $M(z) = M_a(z) = \frac{a}{1-z}$, $X_a = \{w : Rew > a/2\}$, $X_a^l = \{w : Rew < a/2\}$.

(i1) We have that $I_M(z) = L(z) = \frac{z}{1-z}$, $S(M) = X_{-1}^l$.

If I_f maps $\mathbb{U}$ into X_b, $b \geq -1$ and $f(0) = a$, $a \neq 0$, then we distinct two cases. If $a \leq 0$, then $f(\mathbb{U}) \subset X_a^l$. Also, if $a > 0$, then $f(\mathbb{U}) \subset X_a$.

The equation $I_f = L$ has solution $f = M_c(z) = \frac{c}{1-z}$.

(i2) Set $A(z) = 1 - z$, $L_s(z) = \frac{z}{(1-z)^s}$ and $L = L_1$. Then $I_A = -L$ maps $\mathbb{U}$ onto X_1^l, $I_{L_s} = 1 + sL$ and for $s > 0$ maps $\mathbb{U}$ onto $X_{2\alpha}$, where $\alpha = 1 - s/2$.

If $f \in S_\alpha$, $0 > \alpha$, then using (i2) we can prove that $|f(z)| \le L_s(|z|)$. Thus L_s is an extremal function.

Set $S(I_\psi(\mathbb{T}), \infty) = \cup_{s>1} s I_\psi(\mathbb{T})$. If f satisfies (h1), (h3) and
(h-2) $I_f : \mathbb{U} \to \mathbb{C} \setminus S(I_\psi(\mathbb{T})$ we say that $f \in O_a^2(\psi)$. We note
(i-1) $f(\mathbb{U}) \subset \psi(\mathbb{U})$.

Proposition 2. *If $f \in O_a^2(\psi)$ and (h-2), then (i-1) holds true.*

Proof. Suppose there is a $z_0 \in \mathbb{U}$ such that $f(B(0, r_0) \subset \psi(\mathbb{U})$, where $r_0 = |z_0|$, and $w_0 = f(z_0) \in b\psi(\mathbb{U})$. This ensures that then there is a point $\zeta_0 \in \mathbb{T}$ such that $w_0 = \psi(\zeta_0)$. There is a branch h of $\psi^{-1} \circ f$. By Jack $I_f(z_0) = kI_\psi(\zeta_0)$,$k \ge 1$, which is a contradiction with (h'2) . □

5. GROWTH OF HOLOMORPHIC MAPS

In the following examples we give explicit formula for conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ and use it to compute the hyperbolic density of a strip and a half-plane.

Example 3. Let $\mathbb{S}_1 = \{w : |\mathrm{Re}\, w| < \pi/4\}$. It is easy to check that tan maps $\mathbb{S}_1$ onto $\mathbb{U}$. Let $B(w) = \frac{\pi}{4} w$ and $f_0 = \tan \circ B$, i.e. $f_0(w) = \tan(\frac{\pi}{4} w)$. Then f_0 maps $\mathbb{S}_0$ onto $\mathbb{U}$. Further set $A_0(z) = \frac{1+z}{1-z}$, and let $\phi = i\frac{2}{\pi} \ln A_0$, that is $\phi = \phi_0 \circ A_0$, where $\phi_0 = i\frac{2}{\pi}\ln$. Let $\hat{\phi}$ be defined by $\hat{\phi}(z) = -\phi(iz)$. Note that ϕ maps $I_0 = (-1, 1)$ onto y-axis and $\hat{\phi}$ maps I_0 onto itself, and that $\hat{\phi} = \frac{4}{\pi}\arctan$ is the inverse of f_0. Hence (i1): $\hat{\phi}'(0) = \frac{4}{\pi}$ and if f is a conformal map of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(0) = 0$, then $|f'(0)| = \frac{4}{\pi}$.

If $\hat{u} = \mathrm{Re}\hat{\phi}$, then (i2) $\hat{u} = \frac{2}{\pi}\arg(\frac{1+iz}{1-iz})$ and $\hat{u}$ maps $I_0 = (-1, 1)$ onto itself.

Recall $M_f(r) = \max\{|f(re^{it})| : 0 \le t \le 2\pi\}$. We have the following theorem.

Theorem 9. *(a) Let F be a holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$ with $F(0) = 0$. Then $M_F(r) \le \frac{2}{\pi} \ln \frac{1+r}{1-r}$, $0 < r < 1$.*

(b) Let F be a holomorphic map from $\mathbb{B}_n$ into $\mathbb{S}_0$ with $F(0) = 0$. Then $M_F(r) \le \frac{2}{\pi} \ln \frac{1+r}{1-r}$, $0 < r < 1$.

The proof of this result is based on the following proposition.

Proposition 3. $M_\phi(r) = \frac{2}{\pi} \ln \frac{1+r}{1-r}$.

Proof. a) There are various ways to find $M_\phi(r)$, but we prefer to demonstrate it by a direct computation.

Let $z = re^{i\theta}$, $w = \rho e^{i\varphi}$, $w = A_0(z) = \frac{1+z}{1-z}$, $\phi = \ln \circ A_0$ and $M(r, \theta) = |\phi(z)|^2$. Since $\ln w = \ln \rho + i\varphi$

$$\rho^2 = |A_0(z)|^2 = \frac{1 + 2r\cos\theta + r^2}{1 - 2r\cos\theta + r^2}, \quad \text{and} \quad A_0(z) = \frac{1 - r^2 + 2ir\sin\theta}{|1 - z|^2},$$

we find $\tan\varphi = \frac{2r\sin\theta}{1-r^2}$ and $\varphi = \arctan(\frac{2r\sin\theta}{1-r^2})$. Hence

$$M(r,\theta) = \arctan^2 \frac{2r\sin\theta}{1-r^2} + \frac{1}{4}\ln^2 \frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}, 0<r<1, 0\le\theta\le\frac{\pi}{2},$$

and by computation

$$\frac{\partial M}{\partial\theta}(r,\theta) = \frac{2}{r}\frac{2(1-r^2)\cos\theta\arctan\frac{2r\sin\theta}{1-r^2} - (1+r^2)\sin\theta\ln\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}}{4\sin^2\theta + (r-\frac{1}{r})^2}.$$

Let us to prove
(i) $\frac{\partial M}{\partial\theta}(r,\theta)\le 0, 0<r<1, 0\le\theta\le\frac{\pi}{2}$.
Hence it follows that $\max\limits_{0\le\theta\le\frac{\pi}{2}} M(r,\theta) = M(r,0)$.
It is clear that $\operatorname{sgn}\frac{\partial M}{\partial\theta}(r,\theta) = \operatorname{sgn} N(r,\theta)$, where

$$N(r,\theta) = 2(1-r^2)\cos\theta\arctan\frac{2r\sin\theta}{1-r^2} - (1+r^2)\sin\theta\ln\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}.$$

Since $N(0,\theta)=0, 0\le\theta\le\frac{\pi}{2}$ it is enough to prove that
(ii) $\frac{\partial N}{\partial r}(r,\theta)\le 0, 0<r<1, 0\le\theta\le\frac{\pi}{2}$.
Finally, by computation

$$\frac{\partial N}{\partial r}(r,\theta) = -2r\left(2\cos\theta\arctan\frac{2r\sin\theta}{1-r^2} + \sin\theta\ln\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}\right),$$

$0<r<1, 0\le\theta\le\frac{\pi}{2}$. Since $\frac{\partial N}{\partial r}(r,\theta) = -2rA(r,\theta)$, where $A\ge 0$ on $I := \{(r,\theta) : 0<r<1, 0\le\theta\le\frac{\pi}{2}\}$, we have (ii). □

Now we return to the proof of Theorem 9.

Proof. If $\phi = \frac{2}{\pi}\arctan$, then by Proposition 3, $M_\phi(r) = \frac{2}{\pi}\ln\frac{1+r}{1-r}$, $0<r<1$. Then by subordination $F(\mathbb{U}_r)\subset\phi(\mathbb{U}_r)$ and therefore $M_F(r)\le M_\phi(r)$.

b) Let $z\in\mathbb{B}_n$, $r=|z|$ and $X=\{\lambda z/|z| : \lambda\in\mathbb{U}\}$ and define g by $g(\lambda) = f(\lambda z/|z|)$.

Since g is holomorphic in λ and $f(z) = g(|z|)$ by a) $|f(z)| = |g(|z|)| \le \frac{2}{\pi}\ln\frac{1+r}{1-r}$. □

6. APPENDIX; FURTHER RESULTS

6.1 Starlike functions

After writing the manuscript our attention have been turned to The monograph Differential Subordinations Theory and Applications [**16**] which describes for the first time in one volume the basic theory and multitude of applications in the study of differential subordinations. It seems that we can get further results using the methods developed in this book. See also [**17**].

Here we also give some additional comments about starlike functions.

If $a \neq 0$, $0 < r < |a|$ and $f : B(a,r) \to C^*$ is holomorphic, then we can write $z = re^{i\theta}$ in polar form on $B(a,r)$ and there is a continuous branch $\theta^*(r,\theta)$ of $\arg h(re^{i\theta})$ and holds formula

$$D_\theta \theta^*(r,\theta) = \operatorname{Re} I_f(z). \tag{5}$$

For fixed r we can consider $\theta^*(r,\theta)$ as a function of θ. (I1) If $I_f : B(a, r_0) \to \mathbb{P}$, then $\theta^*(r,\theta)$ increases strictly.

From (I1) it follows

(I2) If f is holomorphic on $\mathbb{U}$, $I_f : \mathbb{U} \to \mathbb{P}$ and f has a simple zero at 0, then f is injective and $f(\mathbb{U})$ is a starlike domain.

Now we write $z = re^{i\theta}$ in polar form. Suppose that $h \in P$. Then

(i1) $h(z) \neq 0$ if $z \neq 0$. If $z \neq 0$, since h is analytic then locally in the disk $B(z,|z|)$ we can write

$$\partial_\theta \arg h(re^{i\theta}) = \partial_\theta \operatorname{Im} \ln h(z) = \operatorname{Im} \partial_\theta \ln h(z) = \operatorname{Im} \frac{\partial z}{\partial \theta} \cdot \partial_z \ln h(z) = \operatorname{Re} z\frac{h'(z)}{h(z)}.$$

Let γ_r be the curve defined by $\gamma_r(\theta) = h(re^{i\theta})$, $0 \leq \theta \leq 2\pi$. Thus as z traces the circle $z = re^{i\theta}$, then

(I1) the argument of the image $h(re^{i\theta})$ is defined locally and it increases strictly.

Suppose first that h has a simple zero at 0.

Fix $r \in (0,1)$ for a moment. By the argument principle, since h has a simple zero at 0 and (i1) holds, h in $\mathbb{U}$ has only 0 at 0, and therefore

(i2) γ_r circles the origin just once.

It follows from (i2) and (I1) that γ_r is a Jordan closed curve. Hence h is injective in $B(0,r)$ and therefore in $\mathbb{U}$.

We also can use an alternative argument. The interior of the region bounded by the curve it traces is therefore starlike. If a is a point in the interior, then the number of solutions $N(a)$ of $h(z) = a$ with $|z| < r$ is given by

$$N(a) = \frac{1}{2\pi i} \int_{|z|=r} \frac{h'(z)}{h(z) - a}\, dz.$$

Since this is an integer, depends continuously on a and $N(0) = 1$, it is identically 1. So h is univalent and starlike in each disk $|z| < r$ and hence everywhere.

If $h \in P$ and $g = h(z^n)$, then $I_g = nI_h$ and $g \in P$.

6.2 Further Results

Here we give a short review of some result related to the project in connection with to Schwarz lemma mentioned in the introduction. In [**13**] the authors give simple proofs of various versions of the Schwarz lemma for real valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly HQR) mappings with the strip codomain. Along the way using the principle of subordination and the corresponding conformal mapping, depicted on the Figure 1, the authors get a simple proof of a new version of the Schwarz lemma for real valued harmonic functions (see [**13**, Theorems 4 and 5]) and [**13**, Theorem 6] related to holomorphic mappings. Using the Schwarz-Pick lemma related to distortion for harmonic mappings and the elementary properties of the hyperbolic geometry of the strip we can prove Lemma 4 in [**13**], which is a key ingredient in the proof of Theorem 7 in [**13**], and also yields optimal estimates for modulus of HQR mappings.

In [**5**] the authors establish some Schwarz type Lemmas for mappings defined on the unit disk with bounded Laplacian. Then we apply these results to obtain boundary versions of the Schwarz lemma.

In [**11**] the authors mainly consider various version of Schwarz lemma and its relatives related to harmonic and holomorphic functions including several variables. It turns out that our methods (results) unify very recent approaches. In particular our considerations include domains on which we can compute Kobayashi-Finsler norm.

Recently Maitra [**6**] has obtained some new results related the subject:

(a) Estimates of the Kobayasi metric on convex domains in $\mathbb{C}^n$

(b) Estimates of the distance $\phi(K)$ to ∂D_2, where $\phi : D_1 \to D_2$ is holomorphic, $D_1 \subset \mathbb{C}^n$, $D_2 \subset \mathbb{C}^m$ are convex domains. More precisely, let $D_1 \subset \mathbb{C}^n$, $D_2 \subset \mathbb{C}^m$ are convex domains. Fix $a \subset D_1$ and $b \subset D_1$, and let D_1 be bounded $\phi : D_1 \to D_2$ holomorphic with $\phi(a) = b$. There are constants $\alpha \geq 1$ and $C > 0$, where α depends only on D_1 and C depends only on D_1 and and a such that

$dist(w, D_2^c) \geq C dist(b, D_2^c) r^\alpha$, where $w = \phi(z)$ and $r = dist(z, D_1^c)$.

It will be of interest to researchers working in the field of geometric aspects of theory of several complex variables. It seems that author's results described here in the item (b) are related to the papers [**10, 12, 3**]; in connection see [**10**, Theorem 1.1], [**12**, Theorem 1.3] etc . It seems that Maitra's result Theorem 1.2 [**6**], described here in the item (b), is related to the Mercer's paper [**15**]; in connection with it see Proposition 2.3 and Proposition 2.6, the inequality (7), etc. in [**15**] .

Note that in the Mercer paper [**15**] in Proposition 2.3 it is assumed that the mapping is proper; but it is interesting that in [**6**, Theorem 1.2] this hypothesis can be omitted.

There is analogy the role of the Hopf lemma and the connection with vector valued Euclidean harmonic functions (It seems to me that we can get further results related to Theorem 1.2). See for example [**12**, Theorem 1.3], [**3**, Theorem 1.1] etc.

Let $\mathbb{C}$ be the complex plane and let $\mathbb{U}$ be the unit disk in $\mathbb{C}$. In [**9**] the hyperbolic densities of some strip domains and the right half plane in $\mathbb{C}$ are determined. Using these results, for real valued harmonic mappings from $\mathbb{U}$ into $(-1,1)$ and so on the author discusses sorts of Schwarz lemma and related results. For harmonic mappings f in the unit disk $\mathbb{U}$, the maximum of the absolute values of the directional derivatives of f at the origin is estimated by the diameter d of the surface $f(\mathbb{U})$. And the diameter d is estimated by the length of the boundary of the surface. For harmonic mappings f in $\mathbb{U}$ which have a continuous extension to the closure of $\mathbb{U}$, absolute values of Fourier coefficients of f at the origin, some directional derivative of f at any point $a \in \mathbb{U}$ and the derivative $f'_x(a)$ are estimated by the length of the curve $f(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. For harmonic mappings f in U which have a continuous extension to the closure of $\mathbb{U}$, the area $A(S)$ of the surface $S = f(\mathbb{U})$ is estimated by the length of the boundary curve $f(\mathbb{T})$ of the surface. If further, the mapping f is K-quasiconformal, then the Dirichlet integral of f is shown to be less than $2KA(S)$. For the harmonic mapping f, the maximum of the absolute values of the directional derivatives of f at the origin is estimated by the Dirichlet integral of f. Finally for harmonic mappings from $\mathbb{U}$ into $(-1,1)$, a bit improved form of Schwarz lemma is given.

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