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3-PATH VERTEX COVER AND DISSOCIATION NUMBER OF HEXAGONAL GRAPHS

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A subset P of vertices of a graph G is called a k-path vertex cover if every path of order k in G contains at least one vertex from P. The cardinality of a minimum k-path vertex cover is called the k-path vertex cover number of G, and is denoted by $\psi_k(G)$. It is known that the problem of finding a minimum 3-path vertex cover is NP-hard for planar graphs. In this paper we establish an upper bound for $\psi_3(G)$, where G is from an important family of planar graphs, called hexagonal graphs, arising from real world applications.

1. INTRODUCTION

For a positive integer k, a subset P of vertices of a graph G is called a *k-path vertex cover* if every path on k vertices in G contains at least one vertex from P. Brešar et al. [**3**] introduced the *k-path vertex cover number*, $\psi_k(G)$, as the cardinality of a minimum k-path vertex cover in G. Vertices of P are referred to as *protectors*, since the motivation for introducing this concept comes from designing wireless sensor network security protocols. In the graph model of such a network there exist two types of vertices representing two kinds of sensor devices, protected and unprotected, and edges represent communication channels between pairs of sensor devices. For application reasons (see, e.g, [**8, 16**]), vertices in the model have to satisfy the condition that at least one protected vertex exists on each path of order k, and one wants to minimize the cost of the network by minimizing the number of protectors. Clearly, the k-path vertex cover number of a non-connected graph is obtained as the sum of k-path vertex cover numbers of its connected components.

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The k-path vertex cover is a generalization of the *vertex cover*, which is a subset of vertices such that each edge of the graph is incident to at least one vertex in this subset. Definitions imply that $\psi_2(G)$ equals the size of a minimum vertex cover, therefore $\psi_2(G) = |V(G)| - \alpha(G)$, where $\alpha(G)$ is the well known independence number of G, i.e. the number of vertices in a maximum independent set of G. The *dissociation number*, $\mathrm{diss}(G)$, of a graph G is the number of vertices in a maximum size induced subgraph of G with vertex degree at most 1, see [**25**]. Therefore finding the minimum 3-path vertex cover is a dual problem to finding the dissociation number, i.e.

$$\psi_3(G) = |V(G)| - \mathrm{diss}(G).$$

In the seminal paper [**25**] Yannakakis proved that determining the dissociation number of a graph is NP-hard in the class of bipartite graphs, and Brešar et al. [**3**] showed that the problem of computing $\psi_k(G)$ is NP-hard for each $k \geq 2$ for general graphs, but polynomial for trees. Algorithmic aspects of the k-path vertex cover (in particular for $k = 3$) attracted attention of many researchers, see, e.g., [**1, 12, 14, 17, 21, 22**], and [**5**] where the weighted version of the k-path vertex cover problem was studied.

Early studies on mathematical properties of the 3-path vertex cover and the dissociation number include the paper of Boliac et al. [**6**] in which bipartite graphs were considered. In [**9**], Göring et al. established a lower bound for the dissociation number, expressed in terms of vertex degrees and cardinalities of closed neighborhoods. Brešar et al. [**3**] investigated upper bounds on the value of $\psi_k(G)$ and provided some exact values of this parameter. In particular, they proved that $\psi_3(G) \leq (2n + m)/6$ for every graph G on n vertices and m edges. The authors of [**2**] extended this result by presenting an upper bound in terms of the size and the order of a graph for general k. In addition, they presented a lower bound for regular graphs.

The k-path vertex cover was investigated also for special families of graphs and graph operations. In [**2**] the exact value of $\psi_3(G)$ for grids is provided, and lower and upper bounds are given for $\psi_k(G)$ if $k \geq 4$, which were later improved and shown to be tight in [**11**]. In addition, Jakovac and Taranenko derived bounds for the strong product of paths, which are tight in the case when $k = 3$. They also derived bounds of $\psi_k(G)$ for the lexicographic product of arbitrary two graphs, and obtained exact values for $k \in \{2, 3\}$. Li and Zuo [**15**] considered Cartesian product graphs where one of the factors is a path and the other a cycle, a wheel or a complete bipartite graph. Jakovac [**10**] established the bounds for $\psi_k(G)$ of rooted product graphs. Larena and Baldado Jr. [**13**] recently studied the 3-path vertex cover number of the join of graphs. In [**4**] recent and more complex view on the parameters related to k-path vertex cover problem and dissociation number is presented.

The concept of the k-path vertex cover found many real life applications, see [**8**]. One of them pertains to large real-world road networks, thus planar graphs in particular present an interesting study object. It is known that the 3-path vertex cover problem remains NP-hard even in planar graphs [**22**], however, Tu and Shi [**23**] presented an efficient polynomial time approximation scheme for graphs of bounded branchwidth. On the other hand mathematical properties of 3-path vertex cover of planar graphs is still an interesting research area. Besides partial results, mentioned above, it is known that the 3-path vertex cover number of an outerplanar graph of order n is bounded above by $\frac{n}{2}$, [**3**]. A conjecture that for any planar graph G it holds $\psi_3(G) \leq \frac{2n}{3}$ remains open, thus studying large families of planar graphs may lead to better understanding of the mentioned problem.

In this paper we establish an upper bound for an important family of planar graph, so called hexagonal graphs, which also arise from a real world application (frequency assignment problem of cellular networks[**18, 20**], see also [**7**]). The definition and properties of hexagonal graphs are discussed in the next two sections together with other necessary tools. Then, in the fourth section the upper bound for the 3-path vertex cover number of a hexagonal graph is derived. As a consequence, the lower bound for the dissociation number is established.

2. PRELIMINARIES

All graphs considered in this paper are finite, simple and undirected. Let G be a graph. For an arbitrary vertex $v \in V(G)$ we use $N_G(v)$ to denote the *open neighborhood* of v, i.e. $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *closed neighborhood* of v, $N_G[v]$, is the set $N_G[v] = \{v\} \cup N_G(v)$. For $v \in V(G)$, $d_G(v) = |N_G(v)|$ is the *vertex degree* of v, and $\delta_{V(G)}$ and $\Delta_{V(G)}$ denote the *minimal* and the *maximal*, respectively, vertex degree in G. For an edge $e = uv \in E(G)$, the *open neighborhood* of e is the set $N_G(e) = (N_G(u) \cup N_G(v)) \setminus \{u, v\}$, and $N_G[e] = N_G(u) \cup N_G(v)$ is the *closed neighborhood* of e; when the graph is clear from the context, we simply write $N(e)$ and $N[e]$. We will use the notation $G - N[e]$ for a subgraph of G induced by the vertex set $V(G) - N[e]$. We use $d_G(e)$ for the *edge degree* of e, defined as $d_G(e) = |N_G(uv)| = |N_G(u) \cup N_G(v)| - 2$, and $\delta_{E(G)}$ denotes the *minimal edge degree* in G.

Recall that a triangular lattice is comprised of three sets of parallel lines, where the position of vertices in their intersections can be described as an integer linear combination $x\overrightarrow{p} + y\overrightarrow{q}$ of vectors $\overrightarrow{p} = (1, 0)$ and $\overrightarrow{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. In this sense vertices of triangular lattices may be identified with pairs (x, y) of integers, and two vertices are adjacent in the lattice when the Euclidean distance between them is one. Note that each vertex (x, y) has six neighbors: $(x-1, y), (x-1, y+1), (x, y-1), (x+1, y), (x, y+1)$ and $(x+1, y-1)$, which for simplicity are referred to as: left, up-left, down-left, right, up-right and down-right respectively. It is sometimes convenient, see [**19**], to represent vertices of triangular lattices as ordered triples $(i, j, k) = (x, y, x + y)$, where (x, y) are the coordinates as defined above. One can note that for each vertex on a horizontal line the second coordinate, j, is constant. Similarly, vertices on each line which goes from bottom-left to up-right have the first coordinate, i, constant, and for vertices from the third line set, the coordinate k is constant, see Figure 1.

A graph G is a *hexagonal* graph if it is an induced subgraph of a triangular lattice G', that is, two vertices of G are adjacent in G if and only if they are adjacent in G'. An example of a hexagonal graph is displayed in Figure 1. Observe that in a hexagonal graph G for every $u \in V(G)$ we have $d_G(u) \leq 6$, and for every edge $e \in E(G)$ it holds $d_G(e) \leq 8$.

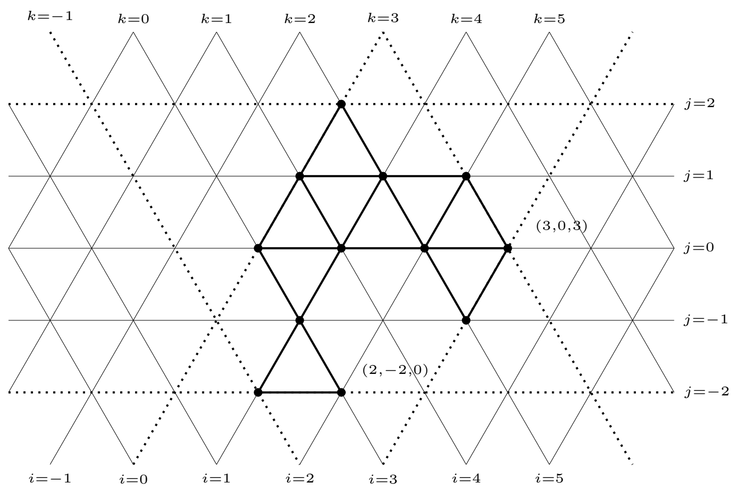


Figure 1: A hexagonal graph in triangular grid.

3. PROPERTIES OF HEXAGONAL GRAPHS

Five special hexagonal graphs will be needed in the sequel. The first two graphs, H_1 and H_2, are depicted in Figure 2. It is straightforward to see that $\psi_3(H_1) = 2$ and $\psi_3(H_2) = 3$. One can use the list in [**24**] of all graphs on at most 6 vertices to identify connected hexagonal graphs among them. Then it is easy to see that if G is a hexagonal graph of order n isomorphic to neither H_1 nor H_2, then $\psi_3(G) \leq 1$ if $n \leq 4$, and $\psi_3(G) \leq 2$, if $n \leq 6$.

In Figure 3 the graphs H_3, H_4 and H_5 are depicted, and they have $13, 18$ and 19 vertices, respectively. It can easily be seen that $\delta_{V(H_3)} = 2$, $\delta_{V(H_4)} = \delta_{V(H_5)} = 3$, and $\delta_{E(H_3)} = \delta_{E(H_4)} = \delta_{E(H_5)} = 4$. Using computer software we have checked that $\psi_3(H_3) = 6$, $\psi_3(H_4) = 9$, and $\psi_3(H_5) = 10$. In Figure 3, the black vertices in each of the graphs constitute a 3-path vertex cover of the graph.

In the next claim we collect some of the above observations, that will be referenced later on.

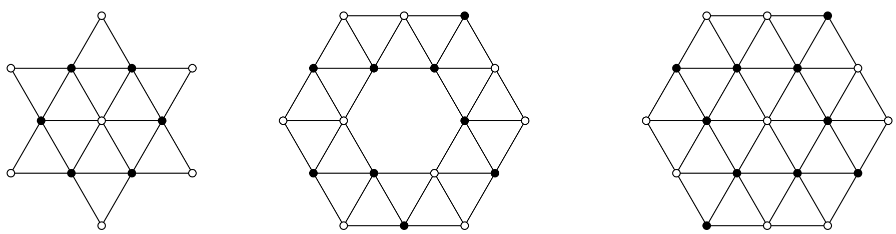
H_1 H_2

Figure 2: Graphs H_1 and H_2.

Observation 1. *The following holds.*

(a) $\psi_3(H_1) = 2$, $\psi_3(H_2) = 3$, $\psi_3(H_3) = 6$, $\psi_3(H_4) = 9$, *and* $\psi_3(H_5) = 10$.

(b) Let G *be a hexagonal graph of order* n *not isomorphic to* H_1 *nor* H_2*. If* $n \leq 2$ *then* $\psi_3(G) = 0$*, if* $n \leq 4$ *then* $\psi_3(G) \leq 1$*, and if* $n \leq 6$ *then* $\psi_3(G) \leq 2$*.*



H_3 H_4 H_5

Figure 3: Graphs H_3, H_4 and H_5.

Note that for a finite hexagonal graph G there exist $j_{\min}$ and $j_{\max}$ such that for every vertex (i, j, k) in G, according to the coordinate system we have $j_{\min} \leq j \leq j_{\max}$, see Figure 1. Lines $j = j_{\min}$ and $j = j_{\max}$ of the underlying triangular lattice will be called the *south* line, and the *north* line, respectively. Similarly, i and k are bounded: $i_{\min} \leq i \leq i_{\max}$ and $k_{\min} \leq k \leq k_{\max}$, and $i = i_{\min}$ is called the *north-west* line, $k = k_{\min}$ is called the *south-west* line, $i = i_{\max}$ the *south-east* line, and $k = k_{\max}$ the *north-east* line. For example, in Figure 1 north and north-east lines are lines $j = 2$ and $k = 3$, respectively.

We will say that vertices of G lying on the north line belong to the *first row* of G, vertices of G lying on the line $j = j_{\max} - 1$ belong to the *second row* of G, and so on. Here G and its rows were observed in the direction from the north (to the south). If we will observe G and count its rows from any of the the other 5 possible directions, we will point that out. For example, in Figure 1, $(2, -2, 0)$ lies in the first row from the south and in the second row from the south-east.

The *first vertex in the first row* of G (from the north) has the second coordinate $j_{\max}$, and the smallest possible first coordinate among all vertices from the first row. Similarly, the first vertex in the first row from the north-west can be defined as the vertex with the first coordinate $i_{\min}$ and the smallest possible k coordinate among all k coordinates of vertices in the first row of G from the north-west. In a natural way the first vertex of the first row from any other direction can be defined. For instance, in the graph from Figure 1, $(3, 0, 3)$ is the first vertex in the first row from the south-east.

By observing the first vertex in the first row (from any direction) it is clear that for each hexagonal graph G we have $\delta_{V(G)} \leq 3$ and $\delta_{E(G)} \leq 4$.

With the above notions defined we can prove the following property of hexagonal graphs.

Lemma 2. *Let G be a connected hexagonal graph of order $2 \leq n \leq 19$ which is isomorphic to neither H_3 nor H_4 nor H_5. Then $\delta_{E(G)} \leq 3$.*

Proof. Let G be a connected hexagonal graph with $2 \leq n \leq 19$ vertices and assume none of the graphs H_3, H_4 and H_5 is isomorphic to G. Since G is a hexagonal graph, $\delta_{E(G)} \leq 4$. Suppose that $\delta_{E(G)} = 4$. We will show that this assumption leads to a contradiction.

Clearly, we have $\delta_{V(G)} > 0$. If $\delta_{V(G)} = 1$, then an edge with an endvertex of degree 1 has the edge degree at most 3, a contradiction. Therefore $\delta_{V(G)} \geq 2$.

First suppose that G is a graph in which there is exactly one vertex in the first row of G from any direction. Let v be the vertex in the first row of G from the north. Then $d_G(v) = 2$ and the two neighbors of v, v_1 and v_2, lie in the second row of G. Let v_2 be the right neigbor of v_1. It holds $d_G(v_1) = d_G(v_2) = 5$, otherwise $d_G(vv_1) \leq 3$ and $d_G(vv_2) \leq 3$, respectively, which is a contradiction. Denote by G_1 the subgraph of G induced by $N_G[v_1v_2]$ and note that $|V(G_1)| = 8$. Let w_L and a be the left and the down-left, respectively, neighbor of v_1, and let w_R and b be the right and the down-right, respectively, neighbor of v_2. Note that in the open neighborhood of aw_L (resp., bw_R) there must exist at least 2 vertices in $V(G) \setminus V(G_1)$, otherwise we have $d_G(aw_L) \leq 3$ (resp., $d_G(bw_R) \leq 3$), a contradiction. Note that observing G from any direction we have a subgraph of G isomorphic to G_1. Let G_1' be the subgraph of G, isomorphic to G_1 with the corresponding vertices $v', v_1', v_2', w_L', w_R', a'$ and b', such that v' lies in the first row from the north-west. Then v_1', v_2', w_L', w_R' lie in the second row from the north-west, and a' and b' in the third row from the north-west. The assumption that G_1 and G_1' do not have common vertices leads to a contradiction, as we derive that $n \geq 8+8+4 = 20$, since $|V(G_1)| = |V(G_1')| = 8$ and $a'w_L'$ and bw_R cannot have common neighbors, but each of them has at least 2 neighbors in $V(G) \setminus (V(G_1) \cup V(G_1'))$. Thus G_1 and G_1' must have common vertices. Having in mind that there is only one vertex in the first row from any direction, we infer that either $w_R' = v$ or $w_R' = w_L$, i.e., subgraphs G_1 and G_1' have either exactly 6 or 2 common vertices. It is clear that the same property holds for two subgraphs isomorphic to G_1 from any two neighboring directions.

If any two subgraphs isomorphic to G_1 from neighboring directions have exactly 6 common vertices, then G is isomorphic to H_3, a contradiction. Therefore we may, without loss of generality, assume that G_1 and G_1' have 2 common vertices. Denote by G_1'' a subgraph isomorphic to G_1, lying in the first three rows from the north-east. Then G_1 and G_1'' have at most 6 common vertices. Therefore there exists a subgraph in G with at least $3 \cdot 8 - 2 - 6 = 16$ vertices and there are two edges ($a'w_L'$ and the corresponding edge in G_1'') which can not have common neighbors, but they need to have at least two neighbors in $V(G) \setminus (V(G_1) \cup V(G_1') \cup V(G_1''))$, thus $n \geq 20$, a contradiction.

The remaining case to consider is the case that in G there are at least two vertices in the first row from at least one direction. Without loss of generality let this direction be the north. Let v_L be the first vertex in the first row of G, and denote by v_R the last vertex in the first row of G (from the north). Clearly, it holds $d_G(v_L) \leq 3$ and $d_G(v_R) \leq 3$, thus both, v_L and v_R are of degree 2 or 3.

First suppose that $d_G(v_L) = 2$. The assumption that v_L has the right neighbor z, leads to a contradiction, since then $d_G(v_L z) \leq 3$. Thus the two neighbors of v_L, v_1 and v_2, lie in the second row of G. As before, let G_1 be a subgraph of G (with 8 vertices) induced by $N_G[v_1 v_2]$. Note that because of the symmetry analogous observations regarding v_L hold also for v_R. Thus if $d_G(v_R) = d_G(v_L) = 2$ there are two copies of G_1 in G, which can have at most 3 common vertices, therefore there are at least 13 vertices in the first three rows of G. By observing these two copies of G_1 from the north-west and north-east, respectively, one can note that at least 2 additional vertices from each direction must exist in G. Therefore there is a subgraph of G with at least 17 vertices. Now it is easy to verify that if $n \leq 19$ then $\delta_{E(G)} \leq 3$, a contradiction. If $d_G(v_R) = 3$ then v_R has the left neighbor z in the first row, and the other two neighbors of v_R lie in the second row of G. It holds $d_G(z) = 4$, otherwise $d_G(z v_R) \leq 3$, a contradiction. Note, that a subgraph of G induced by $N_G[z v_R]$ has 6 vertices, and 3 of them lie in the first row of G. Clearly, none of these vertices is the vertex v_L, and it can be verified that there are at least 14 vertices in the first three rows of G and additional 2 from two directions, hence $n \geq 18$. Again it is easy to see that if $n \leq 19$ then $\delta_{E(G)} \leq 3$, a contradiction.

Clearly, because of symmetry, the case when $d_G(v_R) = 2$ is proved analogously to the case when $d_G(v_L) = 2$, thus it remains to consider the case when $d_G(v_L) = d_G(v_R) = 3$. It is easy to see, that in this case there are at least 3 vertices in the first row and at least 4 vertices in the second row of G (from the north), otherwise $\delta_{E(G)} < 4$, a contradiction. Note, that observations for vertices in the first row from the north hold also for each of the other five possible directions. Thus, if there is more than one vertex in the first row of G from some direction, we may assume that the first and the last vertex in this row both have degree 3. Having this in mind we distinguish the following subcases.

First assume that G is a graph in which there is more than one vertex in the first row of G from every direction. Then G contains at least 3 vertices in the first row from every direction. If G contains exactly 3 vertices in the first row from every direction, then the outer face of G is isomorphic to C_{12}. It follows that

$n \leq 19$, since inside this cycle there are at most 7 vertices possible. Clearly, G must contain all four vertices in the second row from every direction, therefore H_4 is a subgraph of G. We infer that G is isomorphic to H_4, if $n = 18$, or G is isomorphic to H_5, if $n = 19$, a contradiction in either case. If G contains at least 4 vertices in the first row from some direction then the corresponding second row contains at least 5 vertices. Since the first row from every other direction must contain at least 3 vertices, and the corresponding second row at least 4, it is easy to verify that $n > 19$, a contradiction.

Finally, consider the case when in G there are at least 3 vertices in the first row from at least one direction (say from the north) and there is exactly one vertex in the first row from at least one other direction. Clearly, because of the symmetry, we may assume that there is exactly one vertex in the first row from at least one of the two neighboring directions of the north. Without loss of generality, assume that there is exactly one vertex v in the first row from the north-west, and denote by G'_1 the subgraph (with 8 vertices) induced by the closed neighborhood of the edge connecting the two neighbors of v. Let z be the right neighbor of v_L, and denote by G' the subgraph of G induced by $N_G[v_L z] \cup N_G[v_R]$. Note that vertices of G' lie in the first two rows from the north. It holds $|V(G')| \geq 7$, and it is clear, that G' and G'_1 can have at most 3 common vertices, thus there is a subgraph of G with at least $8 + 7 - 3 = 12$ vertices. By observing G from the north-east, there are two possibilities, either there is exactly one vertex in the first row (first case) or there are at least 3 vertices in the first row, and 4 vertices in the second row from the north-east (second case). In the first case there is a subgraph of G with at least 17 vertices. However, by observing this subgraph from the south-west and the south-east, respectively, one can note that at least two additional vertices from each direction must exist in G. Therefore $n \geq 21$, a contradiction. In the second case there is a subgraph of G with at least 15 vertices, and by observing this subgraph from the south-west and the south-east, respectively, at least two additional vertices from each direction must exist in G thus $n \geq 19$. Now it is easy to verify that if $n = 19$ then $\delta_{E(G)} \leq 3$, a final contradiction.

□

By Lemma 2, it is easy to see, that if G is a non-connected hexagonal graph of order $n \leq 19$ for which $\delta_{E(G)} = 4$, then one component of G is isomorphic either to H_3 or H_4 and all other components have exactly one vertex. Thus in the following corollary the structure of all hexagonal graphs of order at most 19 and with minimal edge degree equal to 4 is given.

Corollary 3. *Let G be a hexagonal graph of order $n \leq 19$ with $\delta_{E(G)} = 4$. Then G contains a connected component H isomorphic to H_3, H_4, or H_5, and $|E(G)| = |E(H)|$. Consequently, $\psi_3(G) = \psi_3(H)$.*

4. MAIN RESULTS

Let G be a graph. By finding a k-path vertex cover of a graph, the vertex set of the graph is partitioned into two sets, the set of protectors P and the set $S = V(G) - P$. Therefore, a k-path vertex cover can also be discussed as a special bicoloring $c : V(G) \to \{\text{white, black}\}$, which assigns color white to the vertices of S, and color black to the vertices of P, i.e. to protectors. Note that in the case of a 3-path vertex cover vertices of S induce a subgraph of G with maximal vertex degree at most one. If the endvertices of $e \in E(G)$ belong to S, then all vertices in its neighborhood must be protectors and are therefore colored black. Clearly, if $\Delta_{V(G)} \leq 1$ then $S = V(G)$, $P = \emptyset$ and $\psi_3(G) = 0$.

Using the following Greedy algorithm an upper bound (not necessarily tight) for $\psi_3(G)$ of an arbitrary graph G can be obtained.

Algorithm 1 : **Greedy algorithm**

Input: A graph G
Output: 3-path vertex cover P, $P \subseteq V(G)$
$P := \emptyset$
while $|E(G)| > 0$ **do**
 choose an edge e of minimum degree in G;
 $P := P \cup N(e)$;
 $G := G - N[e]$;
return P

Lemma 4. *Let G be an arbitrary graph and P be the set obtained by the Greedy algorithm. Then $\psi_3(G) \leq |P|$.*

Proof. If G is a graph with $|E(G)| = 0$, then $P = \emptyset$ and $\psi_3(G) = 0$. If G is a graph with $|E(G)| > 0$, then there exists an edge $e = uv \in E(G)$ with $d_G(e) = \delta_{E(G)}$. To the vertices u and v we assign color white, and to vertices of the open neighborhood of e color black is assigned. Then, by the algorithm, black vertices belong to P, and all colored vertices, black and white, are removed from G. The step of the algorithm is repeated on the remaining (uncolored) subgraph if it contains an edge. If (at the end of the algorithm) there is a subgraph without any edge, then we assign white color to its vertices. Thus all vertices of G are colored. Clearly, the endvertices of a chosen edge (and thus colored white) at some step cannot be connected with vertices to which white color is assigned in subsequent steps, because all neighbors of the chosen edge are colored black. It follows, that white vertices induce a subgraph with maximal vertex degree 1, thus the set of black vertices P is a 3-path vertex cover, and therefore $\psi_3(G) \leq |P|$. □

Let G be any graph with at least one edge and $e \in E(G)$. From the above proof it is clear that $\psi_3(G) \leq \psi_3(G - N[e]) + d_G(e)$. We use this inequality in the proofs of the following two results.

Proposition 5. *Let G be a hexagonal graph of order $2 \leq n \leq 19$ not isomorphic to H_1 nor H_2. Then*

$$\psi_3(G) \leq \frac{3n-4}{5}.$$

Proof. Let G be a hexagonal graph of order $2 \leq n \leq 19$. By Observation 1, it is easy to see, that the inequality from the proposition holds for the graphs H_3, H_4, and H_5, and for any hexagonal graph of order $2 \leq n \leq 6$ not isomorphic to H_1 nor H_2. Furthermore, by Corollary 3, it is clear that the inequality from the proposition also holds for any graph G with minimal edge degree 4.

Let G be a hexagonal graph of order $7 \leq n \leq 19$ with $\delta_{E(G)} \leq 3$. We assume that G has at least one edge, as $\psi_3(G) = 0 < \frac{3n-4}{5}$ clearly holds. Let $e \in E(G)$ be an edge with minimal degree in G. Then $|V(G) - N[e]| = n - d_G(e) - 2$, and $\psi_3(G) \leq \psi_3(G - N[e]) + d_G(e)$. Assume, that for a subgraph $G - N[e]$ the inequality in the proposition holds, i.e. $\psi_3(G - N[e]) \leq \frac{3|V(G)-N[e]|-4}{5}$. Since $d_G(e) \leq 3$, we derive

$$\psi_3(G) \leq \frac{3(n - d_G(e) - 2) - 4}{5} + d_G(e) = \frac{3n + 2d_G(e) - 10}{5} \leq \frac{3n-4}{5}.$$

Now we observe a subgraph $G - N[e]$. First, suppose that $G - N[e]$ has more than 6 vertices. It is clear, that if $E(G - N[e]) = \emptyset$ or $\delta_{E(G-N[e])} = 4$ then for $G - N[e]$ the inequality in the proposition holds and in these cases the proof is complete. If $G - N[e]$ has at least one edge, $|V(G) - N[e]| > 6$, and $\delta_{E(G-N[e])} \leq 3$, the above reasoning can be repeated where $G - N[e]$ is taken instead of G.

Thus it remains to consider the case when $|V(G) - N[e]| \leq 6$. Note, that $G - N[e]$ has at least 2 vertices. If $G - N[e]$ is isomorphic neither to H_1 nor H_2, then for $G - N[e]$ the inequality in the proposition holds and the proof is complete. So consider the remaining two cases, when $G - N[e]$ is isomorphic either to H_1 or H_2.

- Assume that $G - N[e]$ is isomorphic to H_1. For convenience we use the notation H_1 for this subgraph, i.e. $G - N[e] = H_1$. Let $e = uv \in E(G)$ be an edge with minimal edge degree. If $d_G(uv) = 1$, then $n = |V(G)| = 7$ and $\psi_3(G) \leq \psi_3(H_1) + 1 = 3 \leq \frac{3n-4}{5}$. If $d_G(uv) = 2$, then $n = 8$ and $\psi_3(G) \leq 4 = \frac{3n-4}{5}$.
  Now let $d_G(uv) = 3$. Clearly, vertices of H_1 are not neighbors of u nor v, thus $V(G)$ is comprised of the following 9 vertices: u,v, three neigbors of the edge uv, and four vertices of H_1. Note that G is a connected graph, otherwise a subgraph of G induced by $N[e]$ contains an edge whose edge degree is strictly less than 3, a contradiction. The structure of triangular lattice implies that neighbors (with respect to the lattice) of an edge lie on a convex polygon around that edge which is isomorphic to the cycle C_8. Similarly, neigbors of vertices of H_1 (without vertices of H_1) lie on a convex polygon, isomorphic to the cycle C_{10}. Because both cycles are convex, their common vertices must lie on the same line of the lattice. But then these common vertices can be

neighbors only of at most 2 vertices of H_1, thus there is an edge in H_1 with edge degree 2 in G, a contradiction with $\delta_{E(G)} = 3$.

- Finally, suppose that $G - N[e]$ is isomorphic to H_2. For convenience we use the notation H_2 for this subgraph, i.e. $G - N[e] = H_2$. Let $e = uv \in E(G)$ be an edge with minimal edge degree. If $d_G(uv) = 1$, then $n = 9$ and $\psi_3(G) \leq \psi_3(H_2) + 1 = 4 \leq \frac{3n-4}{5}$. If $d_G(uv) = 2$, then $n = 10$ and $\psi_3(G) \leq 5 \leq \frac{3n-4}{5}$. Now assume that $d_G(uv) = 3$. Then G contains the following 11 vertices: u, v, 3 neigbors of the edge uv, and 6 vertices of $G - N[e]$. Similarly as above, G must be connected, and because of convexity of cycles, common neighbors of the edge uv and of vertices of H_2 can only lie on a line of the underlying triangular lattice. The structure of H_2 implies that in H_2 there is either 1 or 2 vertices that have degree 2 in G. That is, there is either 1 or 2 vertices in H_2 that do not share a common neighbor with $e = uv$.
  Suppose first, that there exists exactly one vertex of H_2 of degree 2 in G, and let e' be one of the two edges incident to it. Then $N_G(e') \cap N_G(e) = \emptyset$, and it can be seen, that $G - N[e']$ is not isomorphic to H_2, thus $\psi_3(G - N[e']) \leq 2$, and therefore $\psi_3(G) \leq 5 \leq \frac{3n-4}{5}$.
  If there are exactly two vertices of H_2 of degree 2 in G, then let e' be an edge where one endvertex is a vertex of degree 2, and the other endvertex has exactly one neighbor of degree 2. It can be seen, that $G - N[e']$ has an isolated vertex, thus is not isomorphic to H_2 and as before, it follows that $\psi_3(G) \leq \frac{3n-4}{5}$.

□

Now we turn our attention to bigger graphs.

Proposition 6. *Let G be a hexagonal graph of order $n \geq 14$. Then*

$$\psi_3(G) \leq \frac{2n}{3} - 2.$$

Proof. If G is a hexagonal graph of order $14 \leq n \leq 19$ then by Proposition 5 we have that $\psi_3(G) \leq \frac{3n-4}{5}$. It can be verified, that for $14 \leq n \leq 19$ it holds $\left\lfloor \frac{3n-4}{5} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor - 2$. Clearly, $\psi_3(G)$ is an integer, therefore $\psi_3(G) \leq \frac{2n}{3} - 2$.

Let G be a hexagonal graph of order $n \geq 20$. If G does not contain an edge, then clearly $\psi_3(G) = 0 < \frac{2n}{3} - 2$. So assume that G has at least one edge. Recall that $\delta_{E(G)} \leq 4$ and let $e \in E(G)$ be an edge of minimal edge degree. Then $|V(G) - N[e]| = n - d_G(e) - 2$, and $\psi_3(G) \leq \psi_3(G - N[e]) + d_G(e)$.

Assume, that for $G - N[e]$ the inequality in the proposition holds, i.e. $\psi_3(G - N[e]) \leq \frac{2|V(G) - N[e]|}{3} - 2$. Then

$$\psi_3(G) \leq \frac{2(n - d_G(e) - 2)}{3} - 2 + d_G(e) = \frac{2n + d_G(e) - 10}{3} \leq \frac{2n-6}{3}.$$

Since $d_G(e) \leq 4$, $G - N[e]$ has at least 14 vertices, and the proof is complete by induction. □

Theorem 7. *Let G be a hexagonal graph of order $n \geq 2$ not isomorphic to H_1. Then*

$$\psi_3(G) \leq \frac{2n}{3} - 1.$$

Moreover, if $n \geq 14$ or $n \in \{9, 11, 12\}$ then

$$\psi_3(G) \leq \frac{2n}{3} - 2.$$

Proof. Let G be a hexagonal graph on at least two vertices, not isomorphic to H_1. If $2 \leq n \leq 6$, the claim holds by Observation 1. If $n \geq 14$, then the claim is true by Proposition 6.

Therefore assume that $7 \leq n \leq 13$. Then $\psi_3(G) \leq \frac{3n-4}{5}$, by Proposition 5. Note, that $\frac{3n-4}{5} \leq \frac{2n}{3} - 1$ for each $n \geq 3$. Checking all cases of $7 \leq n \leq 13$ we find that $\lfloor \frac{3n-4}{5} \rfloor = \lfloor \frac{2n}{3} \rfloor - 2$, if and only if $n \in \{9, 11, 12\}$. Clearly, $\psi_3(G)$ is an integer, thus the proof is complete. $\square$

Since for any graph G it holds $\mathrm{diss}(G) = |V(G)| - \psi_3(G)$, we have the following immediate corollary of Theorem 7.

Corollary 8. *Let G be a hexagonal graph of order $n \geq 2$, not isomorphic to H_1. Then* $\mathrm{diss}(G) \geq \frac{n}{3} + 1$. *Moreover, if $n \geq 14$ or $n \in \{9, 11, 12\}$ then* $\mathrm{diss}(G) \geq \frac{n}{3} + 2$.

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