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COUNTING SUBWORD PATTERNS IN PERMUTATIONS ARISING AS FLATTENED PARTITIONS OF SETS

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We consider various statistics on the set $\mathcal{F}_n$ consisting of the distinct permutations of length $n+1$ that arise as a flattening of some partition of the same size. In particular, we enumerate members of $\mathcal{F}_n$ according to the number of occurrences of three-letter consecutive patterns, considered more broadly in the context of r-partitions. As special cases of our results, we obtain formulas for the number of members of $\mathcal{F}_n$ avoiding a given consecutive pattern and for the total number of occurrences of a pattern over all members of $\mathcal{F}_n$.

1. INTRODUCTION

Let $\pi = \pi_1\pi_2\cdots\pi_n$ and $\tau = \tau_1\tau_2\cdots\tau_m$ be permutations of length n and m, where $n \geq m$. Then π is said to *contain* an occurrence of τ if there exists a subsequence of π that is order-isomorphic to τ. Otherwise, it is said that π *avoids* τ, with τ often being referred to as a *pattern* in this context. If in addition an occurrence of τ within π corresponds to consecutive entries $\pi_i\pi_{i+1}\cdots\pi_{i+m-1}$ for some $1 \leq i \leq n-m+1$, then π is said to contain τ as a *subword* (or *consecutive* pattern). If no set of consecutive letters of π exists that is isomorphic to τ, then π avoids τ as a subword.

For example, the permutation $\pi = 14827356 \in \mathcal{S}_8$ has two occurrences of the subword 312 (as witnessed by 827 and 735), but avoids 321 as a subword (though it contains three occurrences of 321 in the classical sense). Note that occurrences

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of a subword pattern need not consist of disjoint letters. Subwords of the form 12 and 21 are often referred to as *ascents* and *descents*, respectively. The problem of enumerating permutations (see, e.g., [**13**]) and other discrete structures (see [**11, 14**] and references contained therein) according to the number of occurrences of a given subword pattern is one that has been studied from several standpoints in both enumerative and algebraic combinatorics.

In this paper, we will be considering certain statistics on finite set partitions defined in terms of a particular representation. By a *partition* of $[n] = \{1, 2, \ldots, n\}$, we mean a collection of nonempty, mutually disjoint subsets, called *blocks*, whose union is $[n]$. Let $\mathcal{P}_n$ denote the set of all partitions of $[n]$. Recall that $|\mathcal{P}_n| = B_n$, the n-th Bell number (see entry A00110 in [**23**]). A partition $\pi = \mathcal{B}_1/\mathcal{B}_2/\cdots$ is said to be in *standard form* if the elements within each block $\mathcal{B}_i$ are written in increasing order, with the $\mathcal{B}_i$ arranged from left to right in increasing order according to the size of their smallest elements.

Let us now recall the definition of flattened partition introduced by Callan [**6**]. Suppose $\pi \in \mathcal{P}_n$ is represented in standard form. We erase the parentheses enclosing the blocks of π and consider the word that results when one combines in order from left to right the lists from the various blocks. Denote by $\text{Flatten}(\pi)$ the member of $\mathcal{S}_n$ (in the one-line notation) obtained in this manner. For example, if $\pi = \{8, 5, 1\}, \{7, 6, 3\}, \{4\}, \{2\} \in \mathcal{P}_8$, then its standard form is $\{1, 5, 8\}, \{2\}, \{3, 6, 7\}, \{4\}$, from which we get $\text{Flatten}(\pi) = 15823674 \in \mathcal{S}_8$. It is seen that any $\tau \in \mathcal{S}_n$ which arises as a flattening of some partition may be so obtained from as few as one to as many as 2^{n-1} distinct members of $\mathcal{P}_n$, depending on the particular τ.

Callan [**6**] coined the phrase "flatten" since it coincides with the command in *Mathematica* which takes lists of sets (arranged lexicographically), removes the enclosing brackets and concatenates the contents of the set into one long list. See, for example, [**24**] and also the text [**25**]. We remark that earlier, Carlitz [**8**] applied this notion of flattening to permutations whose disjoint cycles are expressed in standard form and used it to define a certain kind of inversion statistic on $\mathcal{S}_n$. Furthermore, variations on the flattening procedure, wherein the blocks of a partition are arranged using other orderings, have been used in obtaining new q-analogues of the Bell numbers such as in [**10**, Eqn. 2.9].

Callan [**6**] considered the problem of enumeration of subsets of $\mathcal{P}_n$ consisting of those π for which $\text{Flatten}(\pi)$ avoids a three-letter pattern in the classical sense. Further in this direction, in [**16**], the authors enumerated $\pi \in \mathcal{P}_n$ according to the number of subwords in $\text{Flatten}(\pi)$ and obtained avoidance results as a corollary. Alternatively, one might consider the avoidance problem not on the underlying set of partitions, but rather on the set of distinct permutations of length n that arise (not necessarily uniquely) as $\text{Flatten}(\pi)$ for some $\pi \in \mathcal{P}_n$. It was seen in [**16**] that there are B_{n-1} distinct permutations that arise, which are often referred to as *flattened partitions*. A bijective proof of this fact is provided in [**21**].

Let $\mathcal{P}'_n$ denote the set of partitions of $[n]' = \{0, 1, \ldots, n\}$ and $\mathcal{F}_n$ the set of (distinct) permutations of $[n]'$ that arise when flattening the members of $\mathcal{P}'_n$. For example, we have

$$\mathcal{P}_3' = \{0123, 0|123, 01|23, 012|3, 0|1|23, 0|12|3, 01|2|3, 0|1|2|3, 013|2, 0|13|2, 02|13, 02|1|3, 023|1, 03|12, 03|1|2\}$$

and

$$\mathcal{F}_3 = \{0123, 0132, 0213, 0231, 0312\}.$$

The reason for including 0 at this point is that, later on, we will frequently make use of the fact that the permutations in $\mathcal{F}_n$ are in one-to-one correspondence with members of $\mathcal{P}_n$ of length one less. Thus, we have $|\mathcal{F}_n| = B_n$ so that q-counting statistics on $\mathcal{F}_n$ will lead to formulas for analogues of B_n, and not of B_{n-1}, which we find here more convenient. Note that a permutation τ that can be obtained as a flattening of a set partition necessarily starts with its smallest letter and has the property that the second elements of all descents form an increasing subsequence. Conversely, by inserting delimiters at all descent locations within such a permutation τ, it is possible to obtain a partition π for which $\text{Flatten}(\pi) = \tau$.

In [**21**], the distribution of some statistics, including number of runs, are considered on the set of flattened partitions of a given length. This work is continued in [**4**], where the authors deal with further statistics on flattened partitions such as number of weak exceedances and right-left minima and introduce non-crossing flattened partitions. In [**20**], the classical avoidance problem on $\mathcal{F}_n$ is undertaken and all patterns of length three and pattern pairs are treated. Here, we consider the avoidance of three-letter subword patterns as well as the enumeration of members of $\mathcal{F}_n$ according to the number of occurrences of these patterns.

We consider this avoidance problem more generally within the framework of r-partitions, wherein the smallest r elements are required to belong to distinct blocks. We remark that we were unable to find in the literature previous avoidance results on r-partitions (flattened or otherwise) for either classical or consecutive patterns. Let us describe now the relevant weighted structures more fully. By an r-partition of $[n+r]$, it is meant a partition in which the elements of $[r]$ belong to distinct blocks, the set of which will be denoted by $\mathcal{P}_{n,r}$. Consider the statistic (marked by m) on $\mathcal{P}_{n,r}$ which records the number of non-minimal elements in the blocks not containing a member of $[r]$. Given $n, k, r \geq 0$, let $W_{m,r}(n,k)$ denote the corresponding weighted sum of r-partitions of $[n+r]$ having $k+r$ blocks in total. It is seen that $W_{m,r}(n,k)$ satisfies the recurrence

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (r+mk)W_{m,r}(n-1,k), \qquad n,k \geq 1,$$

with $W_{m,r}(0,k) = \delta_{k,0}$ for $k \geq 0$ and $W_{m,r}(n,0) = r^n$ for $n \geq 0$.

The $W_{m,r}(n,k)$ are known as the r-Whitney numbers of the second kind. See [**2, 3**], where the $W_{m,r}(n,k)$ were studied in the $r=0$ case, and [**9**] (and also [**17**]), where they are introduced for general r and connections are made with the Dowling lattice. Note that when $m=1$, the $W_{m,r}(n,k)$ reduce to the r-Stirling numbers of the second kind first considered in [**5**]. The r-Dowling polynomials (see, e.g., [**9**]) are defined as

$$D_{m,r}(n;t) = \sum_{k=0}^{n} W_{m,r}(n,k)t^k, \qquad n, r \geq 0,$$

and reduce to the r-Bell numbers $B_n^{(r)}$ (see [**18**]) when $m = t = 1$, the $r = 0$ case of which corresponds to the usual Bell numbers B_n. Note that $|\mathcal{P}_{n,r}| = B_n^{(r)}$ for all n and r. When $m = 1$, the $D_{m,r}(n;t)$ reduce to the r-Bell polynomials studied in [**19**].

In the next section, it is shown that the number of distinct permutations that arise as a flattening of some member of $\mathcal{P}_{n,r+1}$ is given by $B_n^{(r)}$. Let $\mathcal{F}_{n,r}$ denote this set of permutations. We then count the members of $\mathcal{F}_{n,r}$ according to the number of occurrences of a given subword pattern (marked by q) along with two other statistics. Taking $q = 1$ will be seen to recover $D_{m,r}(n;t)$, while taking $q = 0$ (and $m = t = 1$) will give formulas for the number of members of $\mathcal{F}_{n,r}$ avoiding a given subword. This approach is adopted in the subsequent section in conjunction with the patterns 123, 132 and 231, where the relevant exponential generating function (egf) for the distribution in each case can be found explicitly from which other results may be derived.

We remark that the patterns 213 and 312 require a different strategy that entails working instead with ordinary generating functions (ogf's) and applying the *kernel method* [**12**]. Here, we treat only the avoidance case of 213 and 312 as enumerating the members of $\mathcal{F}_n$ according to the number of occurrences of either of these patterns seems to be an intractable problem. In the case of 213, the sequence enumerating the avoiders does not appear in the OEIS. Finally, it should be noted that the subword pattern 321 is a triviality since no flattening of a partition in standard form can have two consecutive descents. Thus, all members of $\mathcal{F}_n$ or $\mathcal{F}_{n,r}$ avoid 321 as a subword.

The organization of this paper is as follows. In the next section, we enumerate the permutations arising as flattened r-partitions according to the number of occurrences of 123, 132 and 231 subwords. Indeed, we are able to find recurrences satisfied by the joint distribution for the occurrences of these three patterns, along with the egf for the joint distribution, from which one may derive results specific to each subword. In the third section, we determine the ogf for the number of members of $\mathcal{F}_n$ that avoid the 213 subword. In order to do so, we refine this number according to two auxiliary statistics which allow for one to write a system of recurrences. From these recurrences, a functional equation may be derived which, after a fortunate cancellation of terms, may be solved explicitly and yields the desired ogf. A similar method is applied in the final section where the avoidance of 312 is treated. Here, the Motzkin number M_n arises as the enumerator of the avoidance class in question and a relatively simple ogf formula may also be given in the general r-case.

Table 1 below gives for each subword pattern τ of length three the number of members of $\mathcal{F}_n$ for $n \geq 0$ that avoid τ.

τ	First terms $0 \leq n \leq 9$	Formula	Reference/Theorem
123	1,1,1,2,3,8,15,48,105,384	$(n-1)!!$	A006882/Corollary 1
132	1,1,1,2,6,17,53,205,871,3876	J_n	A097514/Corollary 3
213	1,1,2,4,10,31,108,416,1777,8289	k_n	No Seq/Theorem 5
231	1,1,2,4,10,26,76,232,764,2620	I_n	A000085/Corollary 3
312	1,1,2,4,9,21,51,127,323,835	M_n	A001006/Theorem 7
321	1,1,2,5,15,52,203,877,4140,21147	B_n	A000110/Trivial

Table 1: Number of τ-avoiding flattened partitions of length $n+1$.

2. OCCURRENCES OF SUBWORDS IN FLATTENED PARTITIONS

Given $n, r \geq 0$, let $\mathcal{P}_{n,r}$ denote the set of partitions of $[n+r]$ wherein the elements $1, \ldots, r$ are required to lie in distinct blocks and let $\mathcal{P}'_{n,r}$ be the set of partitions of $[n+r]'$ wherein $0, 1, \ldots, r$ belong to distinct blocks. Let $\mathcal{F}_{n,r}$ denote the set of distinct permutations of $[n+r]'$, expressed in the one-line notation, that arise as a flattening of some member of $\mathcal{P}'_{n,r}$. When $r = 0$, the sets $\mathcal{P}_{n,r}$ and $\mathcal{F}_{n,r}$ will often be denoted simply by $\mathcal{P}_n$ and $\mathcal{F}_n$.

In [**21**], the following bijection f was described between $\mathcal{F}_n$ and $\mathcal{P}_n$. Given $\pi \in \mathcal{F}_n$, insert a vertical bar directly to the right of each right-left minimum. Then the blocks of $f(\pi) \in \mathcal{P}_n$ are obtained by considering the set of elements between each pair of consecutive bars, with the element 0 being discarded from the resulting partition. It is seen that this mapping extends to a bijection between $\mathcal{F}_{n,r}$ and $\mathcal{P}_{n,r}$ for all $r \geq 0$, which we will again denote by f. Note that any elements between j and $j+1$ within $\pi \in \mathcal{F}_{n,r}$ for $0 \leq j \leq r-1$ correspond to the non-minimal elements in the block of $f(\pi)$ that contains $j+1$. For example, if $\pi = 0416238597 \in \mathcal{F}_{7,2}$, then inserting bars as described gives $0|41|62|3|85|97|$ and hence

$$f(\pi) = \{1,4\}, \{2,6\}, \{3\}, \{5,8\}, \{7,9\} \in \mathcal{P}_{7,2}.$$

A block containing an element of $[r]$ within a member of $\mathcal{P}_{n,r}$ will be referred to as *special*, with all other blocks being *non-special*. This terminology will also be applied to the respective elements lying within a particular kind of block. Let $\mathcal{F}_{n,k,r}$ denote the subset of $\mathcal{F}_{n,r}$ consisting of those π for which $f(\pi)$ contains exactly k non-special blocks. In terms of a statistic defined directly in terms of π, the set $\mathcal{F}_{n,k,r}$ is seen to consist of those $\pi \in \mathcal{F}_{n,r}$ in which there are exactly k right-left minima occurring to the right of the letter r. Let $\sigma(\pi)$ be the number of non-minimal elements in all the non-special blocks of $f(\pi)$ (i.e., the total number of non-special elements minus the number of non-special blocks). Alternatively, we have that $\sigma(\pi)$ gives the number of positions to the right of r within π that do not correspond to right-left minima.

Given $\pi \in \mathcal{F}_{n,r}$ and a subword pattern ρ, let $\mu_\rho(\pi)$ denote the number of occurrences of ρ within π. For a pattern ρ, define the distribution polynomial

$$W_{m,r}^{(\rho)}(n,k;q) = \sum_{\pi\in\mathcal{F}_{n,k,r}} m^{\sigma(\pi)}q^{\mu_\rho(\pi)}.$$

We have for all ρ that $W_{m,r}^{(\rho)}(n,k;1) = W_{m,r}(n,k)$. For $n, r \geq 0$, define the further distribution

$$D_{m,r}^{(\rho)}(n;q,t) = \sum_{k=0}^{n} W_{m,r}^{(\rho)}(n,k;q)t^k;$$

note that $D_{m,r}^{(\rho)}(n;1,t) = D_{m,r}(n;t)$, the r-Dowling polynomial of order n.

It is possible to treat the patterns 123, 132 and 231 simultaneously and here we consider the joint distribution

$$W_{m,r}(n,k;a,b,c) = \sum_{\pi\in\mathcal{F}_{n,r,k}} a^{\mu_{123}(\pi)}b^{\mu_{132}(\pi)}c^{\mu_{231}(\pi)}m^{\sigma(\pi)}.$$

Define $D_{m,r}(n;a,b,c,t) = \sum_{k=0}^{n} W_{m,r}(n,k;a,b,c)t^k$, which will be denoted by $u_r(n)$. Note that each of the individual distributions $D_{m,r}^{(\rho)}(n;q,t)$ for $\rho = 123, 132$, 231 can be obtained from $u_r(n)$ by setting one of $\{a,b,c\}$ equal to q and the other two to 1. Each such distribution will be seen to be distinct and hence furnishes a different q-generalization of the r-Dowling polynomials. See, e.g., [**15**] for other generalizations of $W_{m,r}(n,k)$ and $D_{m,r}(n;t)$.

The sequence $u_r(n)$ is given recursively as follows.

Lemma 1. *If $n \geq 1$, then*

$$u_r(n) = au_{r-1}(n) + nbu_{r-1}(n-1) + c\sum_{i=2}^{n}\binom{n}{i}a^{i-1}u_{r-1}(n-i), \qquad r \geq 1, \tag{1}$$

and

$$\begin{aligned} u_0(n) = {} & a^{n-1}t^n + bm\sum_{i=0}^{n-2}(n-i-1)a^it^{i+1}u_0(n-i-2)\\ & + c\sum_{i=0}^{n-3}\sum_{j=2}^{n-i-1}\binom{n-i-1}{j}a^{i+j-1}m^jt^{i+1}u_0(n-i-j-1), \end{aligned} \tag{2}$$

with $u_r(0) = a^{r-1}$ for $r \geq 1$ and $u_0(0) = 1$.

Proof. The initial conditions when $n = 0$ follow from the definitions since $\mathcal{F}_{0,r}$ consists of the single permutation $01\cdots r$, so we may assume $n \geq 1$. If $r \geq 1$, then the weight of the members of $\mathcal{F}_{n,r}$ in which 1 directly follows 0 is given by $au_{r-1}(n)$ since $n, r \geq 1$ implies 0 must be involved in an occurrence of 123. Otherwise, there exists at least one element of $I = [r+1, r+n]$ between 0 and 1. If within $\pi \in \mathcal{F}_{n,r}$, there exist $i \geq 1$ such elements, then there are $i-1$ occurrences of 123 involving letters to the left of 1 as they must occur in ascending order. Furthermore, neither the last nor the next-to-last of these letters can start a 123. On the other hand, an occurrence of 132 is introduced at the beginning if $i=1$, whereas there is an extra occurrence of 231 starting with the penultimate letter prior to 1 if $i \geq 2$. Thus, the weight of such π is given by $nbu_{r-1}(n-1)$ if $i=1$ and by $\binom{n}{i}a^{i-1}cu_{r-1}(n-i)$ if $i \geq 2$ and considering all possible i implies (1).

For (2), first note that the permutation $\pi = 01\cdots n$ has weight $a^{n-1}t^n$, so assume $\pi \in \mathcal{F}_n$ has at least one descent. By a descent bottom, we mean the smaller (i.e., the second) element within a descent. Suppose that π has leftmost descent bottom $i+1$, where $0 \leq i \leq n-2$. Then we have $\pi = 01\cdots i\alpha(i+1)\beta$, where α is non-empty and starts with an element of $[i+2,n]$ and β is possibly empty. Denote the length of α by j. Then there are $\binom{n-i-1}{j}$ possibilities for α whose elements are increasing, with $u_0(n-i-j-1)$ accounting for the contribution of the section $(i+1)\beta$ towards the weight. Further, each member of $[i]$ within π corresponds to a (singleton) block of $f(\pi) \in \mathcal{P}_n$ and hence contributes a factor of t towards the weight, as does the section α as a whole, which accounts for t^{i+1}. Since members of α correspond to non-minimal elements in their (non-special) block within $f(\pi)$, a further factor of m^j is introduced, with a^{i+j-1} accounting for the $i+j-1$ occurrences of 123 within $01\cdots i\alpha$ in addition to those within $(i+1)\beta$. Finally, a factor of b or c is introduced depending on whether $j=1$ or $j \geq 2$. Considering all possible i and j yields the two sum formulas in (2) and completes the proof. □

We now determine a formula for the egf of $u_r(n)$. Define

$$U_r(x) = \sum_{n\geq 0} u_r(n)\frac{x^n}{n!}$$

for all $r \geq 0$. Then (1) above can be written as

$$U_r(x) = \left(a + (b-c)x + c\frac{e^{ax}-1}{a}\right)U_{r-1}(x), \qquad r \geq 2, \tag{3}$$

$$U_1(x) = \left(a + (b-c)x + c\frac{e^{ax}-1}{a}\right)U_0(x) + 1 - a. \tag{4}$$

By two applications of (2) and subtraction, we have for $n \geq 2$,

$$u_0(n) - atu_0(n-1) = bmt(n-1)u_0(n-2) + ct\sum_{j=2}^{n-1}\binom{n-1}{j}a^{j-1}m^j u_0(n-j-1),$$

with $u_0(0)=1$ and $u_0(1)=t$.

Multiplying both sides of the last equation by $x^{n-1}/(n-1)!$, and summing over $n \geq 2$, we get

$$\frac{\partial}{\partial x}U_0(x) = t(a+bmx)U_0(x) + \frac{ct}{a}(e^{amx}-1-amx)U_0(x) + (1-a)t,$$

with $U_0(0)=1$. Solving explicitly this linear first order differential equation gives

$$U_0(x)=e^{atx+\frac{(b-c)mt}{2}x^2+\frac{ct}{a^2m}(e^{amx}-1-amx)} \cdot\left(1+(1-a)t\int_0^x e^{-atz-\frac{(b-c)mt}{2}z^2-\frac{ct}{a^2m}(e^{amz}-1-amz)}dz\right). \tag{5}$$

Hence, from (3) and (4), we obtain the following result.

Theorem 1. *For all* $r\geq 1$*,*

$$U_r(x)=\left(a+(b-c)x+\frac{c}{a}(e^{ax}-1)\right)^{r-1} \cdot\left(\left(a+(b-c)x+\frac{c}{a}(e^{ax}-1)\right)U_0(x)+1-a\right), \tag{6}$$

where $U_r(x)=\sum_{n\geq0}D_{m,r}(n;a,b,c,t)\frac{x^n}{n!}$ *and* $U_0(x)$ *is given by* (5)*.*

2.1 Occurrences of the pattern 123

In this subsection, we study further the distribution of the subword pattern 123 on members of $\mathcal{F}_{n,r}$. Let $a_r(n)=D_{m,r}(n;q,1,1,t)=D_{m,r}^{(123)}(n;q,t)$ and $A_r(x)=\sum_{n\geq0}a_r(n)\frac{x^n}{n!}$. Setting $a=q$ and $b=c=1$ in (6) implies for $r\geq1$,

$$\sum_{n\geq0}D_{m,r}^{(123)}(n;q,t)\frac{x^n}{n!} =\left(q+\frac{1}{q}(e^{qx}-1)\right)^{r-1}\left(\left(q+\frac{1}{q}(e^{qx}-1)\right)A_0(x)+1-q\right), \tag{7}$$

where $A_0(x)=\sum_{n\geq0}D_{m,0}^{(123)}(n;q,t)\frac{x^n}{n!}$ is given by

$$A_0(x)=e^{t(q-1/q)x+\frac{t}{mq^2}(e^{mqx}-1)}\left(1+t(1-q)\int_0^x e^{-t(q-1/q)z-\frac{t}{mq^2}(e^{mqz}-1)}dz\right). \tag{8}$$

Clearly,

$$\sum_{n\geq0}D_{m,r}^{(123)}(n;1,t)\frac{x^n}{n!}=\exp\left(rx+t\frac{e^{mx}-1}{m}\right),$$

which agrees with the egf of $D_{m,r}(n;t)$. We now consider the avoidance case of the 123 subword. Taking $a=0$ and all other parameters to be unity in (1) and (2) implies $a_r(n)=na_{r-1}(n-1)$ if $n,r\geq1$ in this case, with $a_0(n)=(n-1)a_0(n-2)$ if $n\geq2$. Thus, we get the following result, where $n!!=n(n-2)\cdots2$ if n is even and $n(n-2)\cdots1$ if n is odd, with $0!!=(-1)!!=1$.

Corollary 1. *The number of members of* $\mathcal{F}_{n,r}$ *that avoid* 123 *(as a subword) is given by*

$$n(n-1)\cdots(n-r+1)(n-r-1)!!,\qquad n\geq r\geq0,$$

and by $(r-1)!$ *if* $n=r-1$ *and* $r\geq1$ *and zero otherwise.*

It is possible to explain directly the prior formula as follows. Suppose $\pi \in \mathcal{F}_{n,r}$ is 123-avoiding, where $n \geq r \geq 0$. Then one may verify that $f(\pi)$ must consist exclusively of doubletons in this case, with only the last block possibly being a singleton. Let us assume $r \geq 1$, as a similar argument will be seen to apply in the $r = 0$ case. Note that there are n possibilities for the larger element in the block of $f(\pi)$ containing 1 and, subsequently, $n-i$ possibilities concerning the block containing $i+1$ for $1 \leq i \leq r-1$. The remaining $n-r$ elements of I are then paired according to a perfect matching if $n-r$ is even or to an involution with only a single 1-block occurring at the end if $n-r$ is odd. This yields, in either case, $(n-r-1)!!$ possible ways in which to arrange the remaining $n-r$ elements of I. On the other hand, if $n < r$, then there can be 123-avoiders in $\mathcal{F}_{n,r}$ only when $r \geq 1$ and $n = r-1$, in which case the final letter must be r with a single member of I separating j and $j+1$ for $0 \leq j \leq r-2$. □

We now substitute $q = 0$ in (7), which will require taking a limit. To do so, first observe that $A_0(x)$ may be rewritten as

$$A_0(x) = e^{qtx+\frac{t}{mq^2}(e^{mqx}-1-mqx)} + t(1-q)\int_0^x e^{qtz-\frac{t}{mq^2}e^{mqx}\left(e^{-mqz}-1+mqze^{-mqx}\right)}dz,$$

which can be obtained by substituting z with $x-z$ in the integral in (8). Taking the limit as $q \to 0$ in the last formula, noting

$$\lim_{q\to 0}\left(\frac{e^{mqx}-1-mqx}{mq^2}\right) = \frac{mx^2}{2}$$

and

$$\lim_{q\to 0}\left(\frac{e^{-mqz}-1+mqze^{-mqx}}{mq^2}\right) = \frac{m(z^2-2xz)}{2},$$

and again replacing z by $x-z$ in the integral implies

$$A_0(x)\mid_{q=0} = e^{\frac{mtx^2}{2}}\left(1+t\int_0^x e^{-\frac{mtz^2}{2}}dz\right). \tag{9}$$

Thus, by (7), we have

$$\sum_{n\geq 0} D_{m,r}^{(123)}(n;0,t)\frac{x^n}{n!} = x^{r-1} + x^r e^{\frac{mtx^2}{2}}\left(1+t\int_0^x e^{-\frac{mtz^2}{2}}dz\right), \qquad r \geq 1. \tag{10}$$

Note that extracting the coefficient of $x^n/n!$ when $m = t = 1$ in (9) or (10) and $n-r$ is even is seen to recover the formula from the corresponding case of Corollary 1 above. When $n-r$ is odd, however, it leads to a derivation of the following binomial formula for the quotient of double factorials.

Corollary 2. *If $a \geq 0$, then*

$$\sum_{j=0}^{a}\frac{(-1)^j}{2j+1}\binom{a}{j} = \frac{(2a)!!}{(2a+1)!!}. \tag{11}$$

Proof. By (9), we have if $m = t = 1$ and $q = 0$,

$$
\begin{aligned}
&[x^{2a+1}/(2a+1)!]A_0(x) = [x^{2a+1}/(2a+1)!]\left(e^{x^2/2}\int_0^x e^{-z^2/2}dz\right)\\
&= [x^{2a+1}/(2a+1)!]\left(\sum_{n\geq 0}(2n-1)!!\frac{x^{2n}}{(2n)!}\cdot\sum_{n\geq 0}(-1)^n(2n-1)!!\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{j=0}^{a}(-1)^{a-j}\binom{2a+1}{2j}(2j-1)!!(2a-2j-1)!!\\
&= \frac{1}{2^a}\sum_{j=0}^{a}(-1)^{a-j}\binom{2a+1}{2j}(2j)^{\underline{j}}(2a-2j)^{\underline{a-j}}\\
&= \frac{1}{2^a}\sum_{j=0}^{a}(-1)^{a-j}\frac{(2a+1)!}{j!(a-j)!(2a-2j+1)} = \frac{(2a+1)!}{2^a a!}\sum_{j=0}^{a}\frac{(-1)^{a-j}}{2a-2j+1}\binom{a}{j}\\
&= (2a+1)!!\sum_{j=0}^{a}\frac{(-1)^j}{2j+1}\binom{a}{j},
\end{aligned}
$$

where we have used the fact $2^k(2k-1)!! = (2k)^{\underline{k}} = 2k(2k-1)\cdots(k+1)$ in the fourth equality. On the other hand, by Corollary 1, we have that this same coefficient of $A_0(x)$ is given by $(2a)!!$, which implies the desired formula. $\square$

Remark: Note that the right side of (11) gives $I(a)$, where $I(a) = \int_0^{\pi/2}\sin^{2a+1}(x)dx$ plays an instrumental role in the derivation of the Wallis product formula for π.

We now find formulas for the number of occurrences of 123 within all members of $\mathcal{F}_{n,r}$, where we must differentiate the case $r = 0$ from the others.

Case $r = 0$. Taking the derivative with respect to q in (8), and setting $q = 1$, yields

$$
\begin{aligned}
&\frac{\partial}{\partial q}\sum_{n\geq 0}D_{m,0}^{(123)}(n;q,t)\frac{x^n}{n!}\ |_{q=1}\\
&= \frac{\partial}{\partial q}\left(e^{t\left(q-\frac{1}{q}\right)x+\frac{t}{mq^2}(e^{mqx}-1)}\right.\\
&\qquad\left.+\,t(1-q)e^{t\left(q-\frac{1}{q}\right)x+\frac{t}{mq^2}(e^{mqx}-1)}\int_0^x e^{-t(q-1/q)z-\frac{t}{mq^2}(e^{mqz}-1)}dz\right)_{q=1}
\end{aligned}
$$

$$
(12)\quad = e^{\frac{t}{m}(e^{mx}-1)}\left(2tx-\frac{2t}{m}(e^{mx}-1)+txe^{mx}\right)-te^{\frac{t}{m}(e^{mx}-1)}\int_0^x e^{-\frac{t}{m}(e^{mz}-1)}dz.
$$

When $m = t = 1$ in (12), we get

$$
\frac{\partial}{\partial q}\sum_{n\geq 0}D_{1,0}^{(123)}(n;q,1)\frac{x^n}{n!}\ |_{q=1} = e^{e^x-1}(2x+xe^x-2e^x+2)-e^{e^x}\int_0^x e^{-e^z}dz.
$$

Using the well-known fact $\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{e^x-1}$, we then have

$$\begin{aligned}\frac{\partial}{\partial q} D_{1,0}^{(123)}(n;q,1)\mid_{q=1} &= 2n![x^{n-1}]e^{e^x-1} + n![x^{n-1}]\frac{d}{dx}e^{e^x-1} - 2n![x^n]\frac{d}{dx}e^{e^x-1} \\ &\quad + 2n![x^n]e^{e^x-1} - n![x^n]\left(e^{e^x}\int_0^x e^{-e^z}dz\right) \\ &= 2nB_{n-1} + (n+2)B_n - 2B_{n+1} - n![x^n]\left(e^{e^x}\int_0^x e^{-e^z}dz\right).\end{aligned}$$

Let B_n^* denote the complementary Bell number defined by

$$e^{1-e^x} = \sum_{n\geq 0} B_n^* \frac{x^n}{n!};$$

see, e.g., [**22**] and A000587 in [**23**]. Then from the preceding we get the following result.

Theorem 2. *The total number of occurrences of* 123 *in all members of* $\mathcal{F}_n$ *is given by*

$$2nB_{n-1} + (n+2)B_n - 2B_{n+1} - \sum_{i=0}^{n-1}\binom{n}{i}B_i B_{n-i-1}^*.$$

It is possible to estimate the average number of 123 occurrences as follows. First recall the asymptotic estimate

$$B_{n+h} = B_n \frac{(n+h)!}{n!\alpha^h}\left(1 + O\left(\frac{\log n}{n}\right)\right), \tag{13}$$

where $h = O(\log n)$ and $\alpha = \log n - \log\log n + O(\log\log n/\log n)$ is the positive root of $\alpha e^\alpha = n+1$ (see [**7**] for an even stronger estimate that includes further terms in the asymptotic expansion). By a general result from [**1**], we obtain

$$\frac{\partial}{\partial q} D_{1,0}^{(123)}(n;q,1)\mid_{q=1} = B_n\left(2n\frac{B_{n-1}}{B_n} + n + 2 - 2\frac{B_{n+1}}{B_n} - G + O(e^{-\kappa n/log^2 n})\right),$$

where $G = \int_0^\infty e^{1-e^x}dx = 0.5963473623\cdots$ and κ is a positive constant. This implies by (13),

$$\frac{\partial}{\partial q} D_{1,0}^{(123)}(n;q,1)\mid_{q=1} = B_n\left(2n\frac{\alpha}{n} + n + 2 - 2\frac{n+1}{\alpha} - G + O\left(\frac{\log n}{n}\right)\right),$$

from which we conclude the following result.

Theorem 3. *The average number of occurrences of* 123 *in all members of* $\mathcal{F}_n$ *is asymptotic to*

$$n + 2 + 2(\log n - \log\log n) - \frac{2(n+1)}{\log n - \log\log n} - G.$$

Case $r \geq 1$. Taking the derivative with respect to q in (7), and setting $q = 1$, yields for $r \geq 1$,

$$\frac{\partial}{\partial q}\sum_{n\geq 0} D_{m,r}^{(123)}(n;q,t)\frac{x^n}{n!}\;|_{q=1}$$
$$= \frac{\partial}{\partial q}\left((1-q)\left(q+\frac{1}{q}(e^{qx}-1)\right)^{r-1} + \left(q+\frac{1}{q}(e^{qx}-1)\right)^{r} A_0(x)\right)_{q=1},$$

which, by (8) and (12), leads to

$$\frac{\partial}{\partial q}\sum_{n\geq 0} D_{m,r}^{(123)}(n;q,t)\frac{x^n}{n!}\;|_{q=1} = -e^{(r-1)x} + re^{(r-1)x}(2-e^x+xe^x)e^{\frac{t}{m}(e^{mx}-1)}$$
$$+ e^{rx+\frac{t}{m}(e^{mx}-1)}\left(2tx - \frac{2t}{m}(e^{mx}-1) + txe^{mx} - t\int_0^x e^{-\frac{t}{m}(e^{mz}-1)}dz\right).$$

In particular, for $m = t = 1$, we get

$$\frac{\partial}{\partial q}\sum_{n\geq 0} D_{1,r}^{(123)}(n;q,1)\frac{x^n}{n!}\;|_{q=1} = -e^{(r-1)x} + re^{(r-1)x}(2-e^x+xe^x)e^{e^x-1}$$
$$+ (2x - 2(e^x-1) + xe^x)e^{e^x+rx-1}$$
$$- e^{e^x+rx-1}\int_0^x e^{-e^z+1}dz.$$

By induction on r, we have

$$e^{e^x+rx-1} = \sum_{j=0}^{r} s(r,j)\frac{d^j}{dx^j}(e^{e^x-1}), \tag{14}$$

where $s(r,j)$ is the Stirling number of first kind. Hence,

$$\frac{\partial}{\partial q}\sum_{n\geq 0} D_{1,r}^{(123)}(n;q,1)\frac{x^n}{n!}\;|_{q=1} = -e^{(r-1)x} + 2r\sum_{j=0}^{r-1} s(r-1,j)\frac{d^j}{dx^j}(e^{e^x-1})$$
$$+ (rx+2x-r+2)\sum_{j=0}^{r} s(r,j)\frac{d^j}{dx^j}(e^{e^x-1})$$
$$+ (x-2)\sum_{j=0}^{r+1} s(r+1,j)\frac{d^j}{dx^j}(e^{e^x-1})$$
$$- e^{e^x+rx-1}\int_0^x e^{-e^z+1}dz.$$

This implies

$$
\begin{aligned}
&\frac{\partial}{\partial q} D_{1,r}^{(123)}(n;q,1) \mid_{q=1} \\
&= -(r-1)^n + 2r\sum_{j=0}^{r-1} s(r-1,j)B_{n+j} + n(r+2)\sum_{j=0}^{r} s(r,j)B_{n+j-1} \\
&\quad - (r-2)\sum_{j=0}^{r} s(r,j)B_{n+j} + n\sum_{j=0}^{r+1} s(r+1,j)B_{n+j-1} \\
&\quad - 2\sum_{j=0}^{r+1} s(r+1,j)B_{n+j} - n![x^n]\left(e^{e^x+rx-1}\int_0^x e^{-e^z+1}dz\right).
\end{aligned} \tag{15}
$$

By induction on r, one can show the identity

$$
\begin{aligned}
e^{e^x+rx-1}\int_0^x e^{-e^z+1}dz &= \sum_{j=1}^{r} s(r,j)\frac{d^j}{dx^j}\left(e^{e^x-1}\int_0^x e^{-e^z+1}dz\right) \\
&\quad + \sum_{j=0}^{r-1}(-1)^{r-j}(r-j-1)!e^{jx},
\end{aligned}
$$

which implies

$$
\begin{aligned}
n![x^n]\left(e^{e^x+rx-1}\int_0^x e^{-e^z+1}dz\right) &= \sum_{j=1}^{r}\sum_{i=0}^{n+j-1}\binom{n+j}{i}s(r,j)B_iB^*_{n+j-i-1} \\
&\quad + \sum_{j=0}^{r-1}(-1)^{r-j}(r-j-1)!j^n.
\end{aligned}
$$

Thus, by (15), we get the following result.

Theorem 4. *The total number of occurrences of* 123 *in all members of* $\mathcal{F}_{n,r}$ *for* $r \geq 1$ *is given by*

$$
\begin{aligned}
&2r\sum_{j=0}^{r-1} s(r-1,j)B_{n+j} + n(r+2)\sum_{j=0}^{r} s(r,j)B_{n+j-1} - (r-2)\sum_{j=0}^{r} s(r,j)B_{n+j} \\
&+ n\sum_{j=0}^{r+1} s(r+1,j)B_{n+j-1} - 2\sum_{j=0}^{r+1} s(r+1,j)B_{n+j} - \sum_{j=0}^{r-2}(-1)^{r-j}(r-j-1)!j^n \\
&- \sum_{j=1}^{r}\sum_{i=0}^{n+j-1}\binom{n+j}{i}s(r,j)B_iB^*_{n+j-i-1}.
\end{aligned}
$$

Remarks: Let $T_n^{(r)}$ denote the quantity for the total in Theorem 4. Then $T_n^{(r)}/B_n^{(r)}$ gives the average number of occurrences of 123 in all members of $\mathcal{F}_{n,r}$. Note the identity $B_n^{(r)} = \sum_{j=0}^{r} s(n,j)B_{n+j}$ for all $n, r \geq 0$, which follows from comparing coefficients in (14) (or by taking $x = 1, r = 0$ in [**19**, Eqn. 11]). Further, we have by [**1**],

$$n![x^n]\left(e^{e^x+rx-1}\int_0^x e^{-e^z+1}dz\right) = G\sum_{j=1}^{r} s(r,j)B_{n+j} + \sum_{j=0}^{r-1}(-1)^{r-j}(r-j-1)!j^n + B_nO(e^{-\kappa n/log^2 n}),$$

where G is as above and κ is a positive constant. Thus, for a fixed r, we have $T_n^{(r)}/B_n^{(r)} = \left(T_n^{(r)}/B_n\right)\cdot\left(B_n^{(r)}/B_n\right)^{-1}$, where both $T_n^{(r)}/B_n$ and $B_n^{(r)}/B_n$ may be estimated for large n by (13). This then yields an asymptotic estimate for the average number of occurrences of 123 in $\mathcal{F}_{n,r}$ for any $r > 0$. Using the fact $(n+j)!/n! \approx n^j$ for large n and fixed j and considering dominant terms, it is possible to simplify this estimate, the details of which we leave to the reader.

2.2 Occurrences of 132 and 231

Letting $a = 1$, $b = p$ and $c = q$ in Theorem 1 gives the following egf for the joint distribution of 132 and 231 on $\mathcal{F}_{n,r}$:

$$\sum_{n\geq 0} D_{m,r}(n;1,p,q,t)\frac{x^n}{n!} = (1-q+(p-q)x+qe^x)^r e^{t\left(\frac{q}{m}(e^{mx}-1)+(1-q)x+\frac{(p-q)m}{2}x^2\right)}. \tag{16}$$

Taking $p = 1$ or $q = 1$ (and replacing p by q thereafter if the latter) gives the respective egf's for $D_{m,r}^{(231)}(n;q,t)$ and $D_{m,r}^{(132)}(n;q,t)$.

We now consider the avoidance case. Let $\mathcal{I}_{n,r}$ denote the subset of $\mathcal{P}_{n,r}$ whose members only have blocks of size one or two (such partitions could rightfully be termed *r-involutions* of $[n+r]$) and $\mathcal{J}_{n,r}$ the subset of $\mathcal{P}_{n,r}$ whose members contain no blocks of size two. Define $I_n^{(r)} = |\mathcal{I}_{n,r}|$ and $J_n^{(r)} = |\mathcal{J}_{n,r}|$, which are seen to be analogues of the r-Bell numbers. Note that

$$\sum_{n\geq 0} I_n^{(r)}\frac{x^n}{n!} = (1+x)^r e^{x+\frac{x^2}{2}} \quad \text{and} \quad \sum_{n\geq 0} J_n^{(r)}\frac{x^n}{n!} = (e^x - x)^r e^{e^x-1-\frac{x^2}{2}}.$$

Taking $(p,q) = (0,1), (1,0)$ in (16) yields respectively

$$\sum_{n\geq 0} D_{m,r}^{(132)}(n;0,t)\frac{x^n}{n!} = (e^x - x)^r e^{\frac{t}{m}(e^{mx}-1)-\frac{mt}{2}x^2}$$

and

$$\sum_{n\geq 0} D_{m,r}^{(231)}(n;0,t)\frac{x^n}{n!} = (1+x)^r e^{tx+\frac{mt}{2}x^2}.$$

Let $\mathcal{F}_{n,r}(\rho)$ denote the subset of $\mathcal{F}_{n,r}$ avoiding the subword ρ, with the $r=0$ case being denoted by $\mathcal{F}_n(\rho)$. Letting $m=t=1$ in the last two formulas yields immediately the following corollary which may also be realized directly using the bijection f.

Corollary 3. *For $n,r\geq 0$, we have $|\mathcal{F}_{n,r}(132)|=J_n^{(r)}$ and $|\mathcal{F}_{n,r}(231)|=I_n^{(r)}$.*

Remark: Let $I_n=I_n^{(0)}$ and $J_n=J_n^{(0)}$ for $n\geq 0$. Then I_n and J_n enumerate the involutions of $[n]$ and the partitions of $[n]$ without doubletons and occur as A000085 and A097514 in [**23**], respectively.

We next consider the sum of the μ_{132} and μ_{231} statistic values taken over all members of $\mathcal{F}_{n,r}$. Note that by (16), we have

$$\begin{aligned}\frac{\partial}{\partial q}\left(\sum_{n\geq 0}D_{m,r}^{(132)}(n;q,t)\frac{x^n}{n!}\right)_{q=1} &= \frac{\partial}{\partial q}\left(((q-1)x+e^x)^r e^{\frac{t}{m}(e^{mx}-1)+\frac{(q-1)mt}{2}x^2}\right)_{q=1}\\ &= \left(rxe^{-x}+\frac{mt}{2}x^2\right)e^{rx+\frac{t}{m}(e^{mx}-1)} \qquad (17)\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial q}\left(\sum_{n\geq 0}D_{m,r}^{(231)}(n;q,t)\frac{x^n}{n!}\right)_{q=1}\\ &= \frac{\partial}{\partial q}\left(((1-q)(1+x)+qe^x)^r\, e^{\frac{qt}{m}(e^{mx}-1)+\frac{t(1-q)}{2}(2x+mx^2)}\right)_{q=1}\\ &= \left(r(e^x-1-x)+\frac{te^x}{m}\left(e^{mx}-1-mx-\frac{m^2}{2}x^2\right)\right)e^{(r-1)x+\frac{t}{m}(e^{mx}-1)}. \qquad (18)\end{aligned}$$

Let $B_n^{(r)}=0$ if n or r is negative. Setting $m=t=1$ in (17) and (18), and extracting the coefficient of $\frac{x^n}{n!}$, implies the following result.

Corollary 4. *If $n,r\geq 0$, then the total number of occurrences of 132 and of 231 in $\mathcal{F}_{n,r}$ is given respectively by*

$$nrB_{n-1}^{(r-1)}+\binom{n}{2}B_{n-2}^{(r)}$$

and

$$(r-1)B_n^{(r)}+B_n^{(r+1)}-rB_n^{(r-1)}-nB_{n-1}^{(r)}-nrB_{n-1}^{(r-1)}-\binom{n}{2}B_{n-2}^{(r)}.$$

Remarks: Let $\mathcal{F}_{n,r}^{*}$ denote the set of colored members $\pi\in\mathcal{F}_{n,r}$ wherein each letter contributing towards the $\sigma(\pi)$ value is colored in one of m ways and each letter of π corresponding to the smallest element in some non-special block of $f(\pi)$ is assigned one of t colors. Let $d_r(n) = D_{m,r}(n;t)$. Recall that $\sum_{n\geq 0} d_r(n)\frac{x^n}{n!} = \exp\left(rx + t\frac{e^{mx}-1}{m}\right)$. Then finding the coefficient of $\frac{x^n}{n!}$ in (17) and (18) implies that the total number of occurrences of 132 and of 231 in $\mathcal{F}^*_{n,r}$ is given respectively by

$$nrd_{r-1}(n-1) + mt\binom{n}{2}d_r(n-2)$$

and

$$rd_r(n) - rd_{r-1}(n) - ntd_r(n-1) - nrd_{r-1}(n-1) - mt\binom{n}{2}d_r(n-2) + t\sum_{i=1}^{n}\binom{n}{i}m^{i-1}d_r(n-i).$$

These formulas reduce to those for $\mathcal{F}_{n,r}$ given in Corollary 4 when $m = t = 1$, the second via the identity $B_n^{(r+1)} = \sum_{i=0}^{n}\binom{n}{i}B_{n-i}^{(r)}$. Note that other formulas above pertaining to $\mathcal{F}_{n,r}$, such as Corollary 3, may be comparably generalized to $\mathcal{F}^*_{n,r}$. A combinatorial proof of Corollary 4, which can be extended to $\mathcal{F}^*_{n,r}$, is provided below.

From the egf (16), one may deduce results involving joint behavior such as the total number of occurrences of one pattern within the members of another avoidance class. For example, we have

$$\frac{\partial}{\partial p}\left(\sum_{n\geq 0} D_{m,r}(n;1,p,0,t)\frac{x^n}{n!}\right)_{p=1} = \frac{\partial}{\partial p}\left((1+px)^r e^{tx+\frac{mpt}{2}x^2}\right)_{p=1}$$

$$= \left(rx + \frac{mt}{2}x^2(1+x)\right)(1+x)^{r-1}e^{tx+\frac{mt}{2}x^2} \tag{19}$$

and

$$\frac{\partial}{\partial q}\left(\sum_{n\geq 0} D_{m,r}(n;1,0,q,t)\frac{x^n}{n!}\right)_{q=1}$$

$$= \frac{\partial}{\partial q}\left((1-q-qx+qe^x)^r e^{qt\left(\frac{e^{mx}-1}{m}\right)+(1-q)tx-\frac{mqt}{2}x^2}\right)_{q=1}$$

$$= \left(r(e^x-1-x) + \frac{t}{m}(e^x-x)\left(e^{mx}-1-mx-\frac{m^2}{2}x^2\right)\right) \cdot (e^x-x)^{r-1}e^{\frac{t}{m}(e^{mx}-1)-\frac{mt}{2}x^2}. \tag{20}$$

Setting $m = t = 1$ in (19) and (20) implies the following result.

Corollary 5. *If $n, r \geq 0$, then the total number of occurrences of 132 in all members of $\mathcal{F}_{n,r}(231)$ is given by*

$$nrI_{n-1}^{(r-1)} + \binom{n}{2}I_{n-2}^{(r)},$$

and the number of occurrences of 231 *in* $\mathcal{F}_{n,r}(132)$ *is given by*

$$J_n^{(r+1)} + (r-1)J_n^{(r)} - rJ_n^{(r-1)} - \binom{n}{2}J_{n-2}^{(r)}.$$

Remarks: Let $\mathcal{F}^*_{n,r}(\rho)$ denote the subset of $\mathcal{F}^*_{n,r}$ whose members avoid ρ. Define $I_{m,r}(n;t)$ and $J_{m,r}(n;t)$ as the Whitney number analogues of $I_n^{(r)}$ and $J_n^{(r)}$ obtained by restricting the joint distribution of the statistics marked by m and t above to the respective subsets $\mathcal{I}_{n,r}$ and $\mathcal{J}_{n,r}$ of $\mathcal{P}_{n,r}$. Note that $I_{m,r}(n;t)$ and $J_{m,r}(n;t)$ are synonymous with $D_{m,r}^{(231)}(n;0,t)$ and $D_{m,r}^{(132)}(n;0,t)$, respectively, by Corollary 3. Then extracting coefficients of $\frac{x^n}{n!}$ in (19) and (20) yields the analogous totals on $\mathcal{F}^*_{n,r}(231)$ and $\mathcal{F}^*_{n,r}(132)$ given by

$$nrI_{m,r-1}(n-1;t) + mt\binom{n}{2}I_{m,r}(n-2;t)$$

and

$$rJ_{m,r}(n;t) - rJ_{m,r-1}(n;t) + t\sum_{i=3}^{n}\binom{n}{i}m^{i-1}J_{m,r}(n-i;t).$$

Note that these totals reduce to those in Corollary 5 when $m = t = 1$, the second via the identity $J_n^{(r+1)} = J_n^{(r)} + \sum_{i=2}^{n}\binom{n}{i}J_{n-i}^{(r)}$.

We conclude this section with combinatorial proofs of Corollaries 4 and 5. Note that these arguments may be extended to provide explanations for the comparable totals given above on $\mathcal{F}^*_{n,r}$.

Combinatorial proof of Corollary 4.

We assume $r \geq 1$ in our arguments as only slight adjustments will be required to handle the $r = 0$ case as well. For the first formula, we count, equivalently by the bijection f, the doubleton blocks in all members of $\mathcal{P}_{n,r}$. Note that this is the same as counting "marked" members of $\mathcal{P}_{n,r}$ where a doubleton block is marked. Enumerating separately the special and non-special blocks, it is seen that there are $nrB_{n-1}^{(r-1)}$ doubletons of the former kind and $\binom{n}{2}B_{n-2}^{(r)}$ of the latter, which implies the first formula. For the second, we count, equivalently by f, the blocks of cardinality at least three within members of $\mathcal{P}_{n,r}$. To do so, it is easier to subtract from the total number of blocks in $\mathcal{P}_{n,r}$ the combined number of singleton and doubleton blocks. There are clearly $rB_n^{(r)}$ special blocks in $\mathcal{P}_{n,r}$. To count the non-special blocks, first consider forming members of $\mathcal{P}_{n,r+1}$ wherein the block containing $r+1$ is a non-singleton, for which there are $B_n^{(r+1)} - B_n^{(r)}$ possibilities. Then remove $r+1$ and designate the resulting (non-special) block. This implies that there are $B_n^{(r+1)} - B_n^{(r)}$ non-special blocks in $\mathcal{P}_{n,r}$ and hence $B_n^{(r+1)} + (r-1)B_n^{(r)}$ blocks altogether. From this, we subtract the total number of singletons and doubletons in $\mathcal{P}_{n,r}$. Since there are $rB_n^{(r-1)} + nB_{n-1}^{(r)}$ singletons and $nrB_{n-1}^{(r-1)} + \binom{n}{2}B_{n-2}^{(r)}$ doubletons in all in $\mathcal{P}_{n,r}$, the second formula follows by subtraction. $\square$

Combinatorial proof of Corollary 5.

By the bijection f, the subsets $\mathcal{F}_{n,r}(231)$ and $\mathcal{F}_{n,r}(132)$ of $\mathcal{F}_{n,r}$ correspond respectively to the subsets $\mathcal{I}_{n,r}$ and $\mathcal{J}_{n,r}$ of $\mathcal{P}_{n,r}$. So we count, equivalently, all doubletons occurring within members of $\mathcal{I}_{n,r}$ and blocks of cardinality at least three within members of $\mathcal{J}_{n,r}$. For the former, note that there are $nrI_{n-1}^{(r-1)} + \binom{n}{2}I_{n-2}^{(r)}$ doubletons in $\mathcal{I}_{n,r}$, which yields the first formula. For the latter, we must count equivalently all non-singleton blocks in $\mathcal{J}_{n,r}$ as the doubletons are excluded. To count these blocks in $\mathcal{J}_{n,r}$, we proceed analogously as before and form members of $\mathcal{J}_{n,r+1}$ wherein $\{r+1\}$ does not occur, of which there are $J_n^{(r+1)} - J_n^{(r)}$ possibilities. However, if one of the formed members of $\mathcal{J}_{n,r+1}$ had exactly three elements in its block containing $r+1$, then removal of $r+1$ results in a doubleton which is not allowed. Hence, we must exclude these $\binom{n}{2}J_{n-2}^{(r)}$ possibilities. Further, no non-special singleton block can arise from the removal of $r+1$ as described since no doubletons occur in members of $\mathcal{J}_{n,r+1}$. Thus, there are $J_n^{(r+1)} - J_n^{(r)} - \binom{n}{2}J_{n-2}^{(r)}$ non-special non-singleton blocks in $\mathcal{J}_{n,r}$. Adding to this the $r\left(J_n^{(r)} - J_n^{(r-1)}\right)$ special non-singleton blocks in $\mathcal{J}_{n,r}$ yields the second formula and completes the proof. □

3. AVOIDING THE SUBWORD PATTERN 213

In this section, we consider the distribution of the subword pattern 213 on $\mathcal{F}_{n,r}$, focusing on the avoidance case. We first treat the $r = 0$ case and enumerate members of $\mathcal{F}_n(213)$. To do so, it is more convenient to enumerate the subset of $\mathcal{P}_n$ corresponding to $\mathcal{F}_n(213)$ under f. Let $\mathcal{K}_n$ denote this subset of $\mathcal{P}_n$, i.e., $\mathcal{K}_n = f(\mathcal{F}_n(213))$. Then one may verify that $\mathcal{K}_n$ consists of those partitions such that when expressed in standard form no non-singleton block with greatest element say x is directly followed by a singleton $\{y\}$ or by another non-singleton with second smallest element y where $x < y$ in either case. Note, for instance, that the partitions $\{1,5\},\{2,3,4\},\{6\},\{7,8\}$ and $\{1,3,5\},\{2,7,8\},\{4,6\}$ both fail to belong to $\mathcal{K}_8$, the former since the non-singleton $\{2,3,4\}$ is directly followed by $\{6\}$ with $4 < 6$ and the latter since $\{1,3,5\}$ is followed by $\{2,7,8\}$ with $5 < 7$. Let $k_n = |\mathcal{K}_n|$ for $n \geq 0$. We have, for example, when $n = 4$,

$$\mathcal{F}_4(213) = \{02341, 01342, 01243, 01234, 02413, 01423, 03124, 03412, 04132, 04123\}$$

and

$$\mathcal{K}_4 = \{1234, 1|234, 1|2|34, 1|2|3|4, 124|3, 1|24|3, 13|2|4, 134|2, 14|23, 14|2|3\}.$$

where the i-th element of the latter set corresponds under f to the respective element of the former for all i.

Let $k_n = |\mathcal{K}_n|$ for $n \geq 0$. One may verify, for example, that $k_0 = k_1 = 1$, $k_2 = 2$, $k_3 = 4$ and $k_4 = 10$. In order to determine k_n, we refine it as follows. Let

$\mathcal{K}_{n,i}$ denote the subset of $\mathcal{K}_n$ in whose members the first block is a non-singleton with second smallest element i and let $k_{n,i} = |\mathcal{K}_{n,i}|$. Further, let $\mathcal{K}_{n,i,j}$ be the subset of $\mathcal{K}_{n,i}$ wherein the greatest element in the first block is j for $2 \le i \le j \le n$ and let $k_{n,i,j} = |\mathcal{K}_{n,i,j}|$. Note that $i = j$ occurs precisely when the first block is a doubleton. Put $k_{n,i} = 0$ if $i < 2$ or $i > n$. By the definitions, we have

$$k_n = k_{n-1} + \sum_{i=2}^{n} k_{n,i} = k_{n-1} + \sum_{i=2}^{n}\sum_{j=i}^{n} k_{n,i,j}, \qquad n \ge 2,$$

with $k_0 = k_1 = 1$.

The $k_{n,i,j}$ are determined recursively as follows.

Lemma 2. *We have*

$$k_{n,i,n} = \sum_{a=0}^{n-i-1}\binom{n-i-1}{a}k_{n-a-3}, \qquad 2 \le i \le n-1, \tag{21}$$

$$k_{n,i,i} = k_{n-2} - \sum_{a=i-1}^{n-2} k_{n-2,a}, \qquad 3 \le i \le n-1, \tag{22}$$

$$k_{n,2,j} = \sum_{a=0}^{j-4}\binom{j-3}{a}\left(k_{n-a-3} - \sum_{b=j-a-2}^{n-a-3} k_{n-a-3,b}\right), \ 4 \le j \le n-1, \tag{23}$$

and

(24)

$$k_{n,i,j} = \sum_{a=0}^{j-i-1}\binom{j-i-1}{a}\left(k_{n-a-3} - \sum_{b=j-a-2}^{n-a-3} k_{n-a-3,b}\right), \ 4 \le i+1 \le j \le n-1,$$

with $k_{n,n,n} = k_{n-2}$ *and* $k_{n,2,2} = \delta_{n,2}$ *if* $n \ge 2$ *and* $k_{n,2,3} = 0$ *if* $n \ge 4$.

Proof. The boundary conditions for $k_{n,n,n}$, $k_{n,2,2}$ and $k_{n,2,3}$ may be verified from the definitions, upon noting that the first block in each case is given by $\{1, n\}$, $\{1,2\}$ and $\{1, 2, 3\}$, respectively. To show (21), let a denote the number of elements in $[i+1, n-1]$ in the first block of $\pi \in \mathcal{K}_{n,i,n}$ where $2 \le i \le n-1$. Since the largest element in the first block of π is n in this case, no restriction is imposed on the subsequent block and hence the remaining members of $[n]$ may be arranged in k_{n-a-3} ways, with $\binom{n-i-1}{a}$ possibilities for the elements in the first block. Considering all possible i then gives (21). For (22), first note that within members of $\mathcal{K}_{n,i,i}$ where $3 \le i \le n-1$, the first block is the doubleton $\{1, i\}$, whereas the second is either the singleton $\{2\}$ or a non-singleton with second smallest element belonging to $[i-1]$. Thus, we must exclude from our count of possible partitions of $[n] - \{1, i\}$ those in which the first block is a non-singleton with second smallest element belonging to $[i+1,n]$. It is seen that such partitions number $\sum_{a=i-1}^{n-2} k_{n-2,a}$, which implies (22), by subtraction.

To show (23), suppose $\pi \in \mathcal{K}_{n,2,j}$ where $4 \le j \le n-1$. Note that the first block of π cannot comprise all of the elements of $[j]$ in this case, for otherwise $j < n$ implies that either the second block is the singleton $\{j+1\}$ or a non-singleton with second smallest element $> j+1$, neither of which is permissible. Let S denote the subset of $[3, j-1]$ lying in the first block of π and let $a = |S|$, where $0 \le a \le j-4$. Then the $n-a-3$ elements of $[3,n] - S - \{j\}$ may be arranged according to any $\lambda \in \mathcal{K}_{n-a-3}$ except for those in which the first block of λ is a non-singleton with second element $b \ge j-a-2$ (to see this, note that $\ell \in [j+1,n]$ in the original partition π corresponds to $\ell - a - 3$ in λ). Then for each of the $\binom{j-3}{a}$ possibilities for S, there are $k_{n-a-3} - \sum_{b=j-a-2}^{n-a-3} k_{n-a-3,b}$ possibilities for λ and considering all a gives (23). A similar proof applies to (24) except now the restriction that the first block cannot contain all elements between its second smallest and its largest elements is no longer seen to apply. $\square$

The preceding system of recurrences may be rewritten more conveniently in terms of $k_{n,i}$ as follows.

Lemma 3. *We have*

$$k_{n,n} = k_{n-2}, \qquad n \ge 2, \tag{25}$$

$$k_{n,2} = k_{n-1} - k_{n-2}, \qquad n \ge 3, \tag{26}$$

and for $3 \le i \le n-1$,

$$k_{n,i} = k_{n-2} + \sum_{a=0}^{n-i-1} \binom{n-i}{a+1} k_{n-a-3} - \sum_{a=0}^{n-i-1} \sum_{b=i-1}^{n-a-2} \binom{a+b-i+1}{a} k_{n-a-2,b}. \tag{27}$$

Proof. Formula (25) follows from the definitions. For (26), observe that removal of the element 1 from $\pi \in \mathcal{K}_{n,2}$ where $n \ge 3$ defines a bijection with members of $\mathcal{K}_{n-1}$ whose first block is not a singleton (as $n \ge 3$ implies the first block of π cannot be a doubleton). To show (27), we proceed as follows. Combining the fact

$$k_{n,i} = k_{n,i,i} + k_{n,i,n} + \sum_{j=i+1}^{n-1} k_{n,i,j}, \qquad 2 \le i \le n-1,$$

with the formulas (21), (22) and (24), we obtain for $3 \le i \le n-1$,

$$\begin{aligned} k_{n,i} = k_{n-2} &- \sum_{a=i-1}^{n-2} k_{n-2,a} + \sum_{a=0}^{n-i-1} \binom{n-i-1}{a} k_{n-a-3} \\ &+ \sum_{j=i+1}^{n-1} \sum_{a=0}^{j-i-1} \binom{j-i-1}{a} k_{n-a-3} - \sum_{j=i+1}^{n-1} \sum_{a=0}^{j-i-1} \sum_{b=j-a-2}^{n-a-3} \binom{j-i-1}{a} k_{n-a-3,b}. \end{aligned}$$

After reordering some of the sums, we have

$$\begin{aligned}k_{n,i} &= k_{n-2} - \sum_{a=i-1}^{n-2} k_{n-2,a} + \sum_{a=0}^{n-i-1} \binom{n-i-1}{a} k_{n-a-3} \\ &+ \sum_{a=0}^{n-i-2} \sum_{j=a+i+1}^{n-1} \binom{j-i-1}{a} k_{n-a-3} \\ &- \sum_{a=0}^{n-i-2} \sum_{b=i-1}^{n-a-3} \sum_{j=a+i+1}^{a+b+2} \binom{j-i-1}{a} k_{n-a-3,b}.\end{aligned}$$

Hence,

$$\begin{aligned}k_{n,i} &= k_{n-2} - \sum_{a=i-1}^{n-2} k_{n-2,a} + \sum_{a=0}^{n-i-1} \binom{n-i-1}{a} k_{n-a-3} \\ &+ \sum_{a=0}^{n-i-2} \binom{n-i-1}{a+1} k_{n-a-3} - \sum_{a=0}^{n-i-2} \sum_{b=i-1}^{n-a-3} \binom{a+b-i+2}{a+1} k_{n-a-3,b} \\ &= k_{n-2} + \sum_{a=0}^{n-i-1} \binom{n-i}{a+1} k_{n-a-3} - \sum_{a=0}^{n-i-1} \sum_{b=i-1}^{n-a-2} \binom{a+b-i+1}{a} k_{n-a-2,b},\end{aligned}$$

as required. □

Define $k(x) = \sum_{n\geq 1} k_n x^n$ and $k(x,v) = \sum_{n\geq 2} \left(\sum_{i=2}^{n} k_{n,i} v^{i-2}\right) x^n$. Then from the fact $k_n = k_{n-1} + \sum_{i=2}^{n} k_{n,i}$ for $n \geq 2$, we get

$$k(x) = \frac{x + k(x,1)}{1-x}. \tag{28}$$

Multiplying both sides of (27) by $x^n v^{i-2}$, summing over $n \geq 4$ and $3 \leq i \leq n-1$ making use of (25)–(26), and noting the initial values $k_0 = k_1 = 1$, we obtain

$$\begin{aligned}\sum_{n\geq 4} \sum_{i=3}^{n-1} k_{n,i} x^n v^{i-2} &= k(x,v) - x(1-x)k(x) - x^2 k(xv), \\ \sum_{n\geq 4} \sum_{i=3}^{n-1} k_{n-2} x^n v^{i-2} &= \frac{x^2(vk(x) - k(xv))}{1-v},\end{aligned}$$

$$\sum_{n\geq 4}\sum_{i=3}^{n-1}\sum_{a=0}^{n-i-1}\binom{n-i}{a+1}k_{n-a-3}x^n v^{i-2} = \sum_{i\geq 3}\sum_{n\geq i}k_{n-2}x^n v^{i-2}\left(\frac{1}{(1-x)^{n-i+1}}-1\right)$$
$$= \sum_{n\geq 3}\sum_{i=3}^{n}k_{n-2}x^n v^{i-2}\left(\frac{1}{(1-x)^{n-i+1}}-1\right)$$
$$= \frac{x^2v(k(x/(1-x))-k(xv))}{1-(1-x)v} - \frac{x^2v(k(x)-k(xv))}{1-v},$$
$$\sum_{n\geq 4}\sum_{i=3}^{n-1}\sum_{a=0}^{n-i-1}\sum_{b=i-1}^{n-a-2}\binom{a+b-i+1}{a}k_{n-a-2,b}x^n v^{i-2}$$
$$= \sum_{i\geq 3}\sum_{n\geq 0}\sum_{b=i-1}^{n+i-1}\sum_{a\geq 0}\binom{a+b-i+1}{a}k_{n+i-1,b}x^{n+a+i+1}v^{i-2}$$
$$= \sum_{i\geq 3}\sum_{n\geq 0}\sum_{b=i-1}^{n+i-1}k_{n+i-1,b}\frac{x^{n+i+1}v^{i-2}}{(1-x)^{b-i+2}} = \sum_{i\geq 3}\sum_{n\geq i}\sum_{b=i-1}^{n-1}k_{n-1,b}\frac{x^{n+1}v^{i-2}}{(1-x)^{b-i+2}}$$
$$= \sum_{n\geq 3}\sum_{b=2}^{n-1}k_{n-1,b}\frac{x^{n+1}}{(1-x)^{b+2}}\frac{(1-x)^3v-(1-x)^{b+2}v^b}{1-(1-x)v}$$
$$= \frac{x^2v(k(x,1/(1-x))-(1-x)vk(x,v))}{(1-x)(1-(1-x)v)}.$$

From the preceding calculations and (28), we obtain the following result.

Lemma 4. *We have*

$$\left(1-\frac{x^2v^2}{1-(1-x)v}\right)k(x,v)$$
$$= xk(x,1) + \frac{x^2v}{1-(1-x)v}\left(\frac{1-x}{1-2x}k(\frac{x}{1-x},1) - \frac{1}{1-x}k(x,\frac{x}{1-x})\right)$$
$$- \frac{x^2v}{(1-xv)(1-(1-x)v)}k(xv,1) + \frac{x^2(2x^2v-2x+1)}{(1-2x)(1-xv)}.$$

Lemma 4 with $v=1$ gives

$$\frac{1-2x+2x^2}{1-x}k(x,1)$$
$$= \frac{x(1-x)}{1-2x}k(\frac{x}{1-x},1) - \frac{x}{1-x}k(x,\frac{x}{1-x}) + \frac{x^2(2x^2-2x+1)}{(1-2x)(1-x)},$$

which implies

$$\frac{x(1-x)}{1-2x}k(\frac{x}{1-x},1) - \frac{x}{1-x}k(x,\frac{x}{1-x})$$
$$= \frac{1-2x+2x^2}{1-x}k(x,1) - \frac{x^2(2x^2-2x+1)}{(1-2x)(1-x)}.$$

Hence, by Lemma 4, we have

$$(1-(1-x)v-x^2v^2)k(x,v) = \frac{(x^2v-x+1)x}{1-x}k(x,1)-\frac{x^2v}{1-xv}k(xv,1)+\frac{(1-v)(x^2v-x+1)x^2}{(1-x)(1-xv)}. \tag{29}$$

Let $\rho(x)=\frac{x-1+\sqrt{1-2x+5x^2}}{2x}$ and define $\rho_0(x)=x$ and $\rho_m(x)=\rho(\rho_{m-1})$ for $m\geq 1$. Applying the kernel method, taking $v=\frac{\rho(x)}{x}$ in the preceding functional equation and using $x\rho^2(x)=x+(x-1)\rho(x)$ to simplify the resulting expression for $k(x,1)$ leads to

$$k(x,1)=\left(1-\frac{1-2x}{x}\rho(x)\right)k(\rho(x),1)+x\rho(x).$$

Iteration of this last equation yields

$$k(x,1)=\sum_{d\geq 0}\rho_d(x)\rho_{d+1}(x)\prod_{j=0}^{d-1}\left(1-(1-2\rho_j(x))\frac{\rho_{j+1}(x)}{\rho_j(x)}\right).$$

Hence, by (28), we get the following result.

Theorem 5. *The generating function* $\sum_{n\geq 1}|\mathcal{F}_n(213)|x^n$ *is given by*

$$k(x)=\frac{x+\sum_{d\geq 0}\rho_d(x)\rho_{d+1}(x)\prod_{j=0}^{d-1}\left(1-(1-2\rho_j(x))\frac{\rho_{j+1}(x)}{\rho_j(x)}\right)}{1-x} = x+2x^2+4x^3+10x^4+31x^5+108x^6+416x^7+1777x^8+8289x^9+\cdots. \tag{30}$$

Concerning the avoidance of 213 in the r-case for $r\geq 1$, we enumerate the subset of $\mathcal{P}_{n,r}$ that is equivalent under f to $\mathcal{F}_{n,r}(213)$. Let $\mathcal{K}_n^{(r)}$ be the subset of $\mathcal{P}_{n,r}$ whose members when expressed in standard form are such that no non-singleton block with greatest element x is directly followed by a singleton $\{y\}$ or by another non-singleton with second smallest element y where $x<y$ in either case. This restriction is to apply equally to both special and non-special blocks. Given $1\leq i\leq n$, let $\mathcal{K}_{n,i}^{(r)}\subseteq\mathcal{K}_n^{(r)}$ comprise those members in which the first (special) block is a non-singleton with second smallest element $r+i$. Let $\mathcal{K}_{n,i,j}^{(r)}\subseteq\mathcal{K}_{n,i}^{(r)}$ consist of those partitions in which the largest element in the first block is $r+j$ where $1\leq i\leq j\leq n$. Define $k_n^{(r)}=|\mathcal{K}_n^{(r)}|$, $k_{n,i}^{(r)}=|\mathcal{K}_{n,i}^{(r)}|$ and $k_{n,i,j}^{(r)}=|\mathcal{K}_{n,i,j}^{(r)}|$.

Considering the contents of the block containing 1 yields

$$k_n^{(r)}=k_n^{(r-1)}+\sum_{i=1}^n k_{n,i}^{(r)}=k_n^{(r-1)}+\sum_{i=1}^n\sum_{j=i}^n k_{n,i,j}^{(r)},\qquad n,r\geq 1,$$

with $k_n^{(0)}=k_n$. If $r=1$, then we have $k_{n,i,j}^{(1)}=k_{n+1,i+1,j+1}$ for all i and j, which implies $k_{n,i}^{(1)}=k_{n+1,i+1}$ and $k_n^{(1)}=k_{n+1}$. Let $k_0^{(r)}=1$ for all $r\geq 0$. By a proof similar to that given for Lemma 2 above, we have the following recurrence for the array $k_{n,i,j}^{(r)}$ when $r \geq 2$.

Lemma 5. *If $r \geq 2$, then*

$$k_{n,i,j}^{(r)} = \sum_{a=0}^{j-i-1} \binom{j-i-1}{a} \left(k_{n-a-2}^{(r-1)} - \sum_{b=j-a-1}^{n-a-2} k_{n-a-2,b}^{(r-1)} \right), \qquad 1 \leq i < j \leq n-1, \tag{31}$$

$$k_{n,i,n}^{(r)} = \sum_{a=0}^{n-i-1} \binom{n-i-1}{a} k_{n-a-2}^{(r-1)}, \qquad 1 \leq i \leq n-1, \tag{32}$$

and

$$k_{n,i,i}^{(r)} = k_{n-1}^{(r-1)} - \sum_{b=i}^{n-1} k_{n-1,b}^{(r-1)}, \qquad 1 \leq i \leq n-1, \tag{33}$$

with $k_{n,n,n}^{(r)} = k_{n-1}^{(r-1)}$ for $n \geq 1$.

Define $K_r(x) = \sum_{n\geq 1} k_n^{(r)} x^n$ and $K_r(x,v) = \sum_{n\geq 1} \left(\sum_{i=1}^n k_{n,i}^{(r)} v^{i-1} \right) x^n$ for $r \geq 1$, with $K_0(x) = k(x)$ and $K_0(x,v) = k(x,v)$. Note that $k(x,v)$ is determined by the formula above for $k(x,1)$ and (29). Then we have

$$K_1(x) = \frac{1}{x}(K_0(x) - x), \quad K_1(x,v) = \frac{1}{x} K_0(x,v),$$

and $K_r(x) = K_{r-1}(x) + K_r(x,1)$ for $r \geq 1$. Upon proceeding as before in the $r = 0$ case, we get from Lemma 5 the functional recurrence relation

$$\begin{aligned} K_r(x,v) = {} & \frac{x}{(1-x)(1-(1-x)v)} \left(K_{r-1}(\frac{x}{1-x}) - K_{r-1}(x, \frac{1}{1-x}) \right) \\ & - \frac{xv}{1-(1-x)v} (K_{r-1}(xv) - K_{r-1}(x,v)) + \frac{x}{1-x}, \qquad r \geq 2. \end{aligned}$$

Define $K(x,y) = \sum_{r\geq 1} K_r(x) y^r$ and $K(x,y,v) = \sum_{r\geq 1} K_r(x,v) y^r$. Then we have

$$K(x,y,1) = (1-y)K(x,y) - yK_0(x) \tag{34}$$

and

$$\begin{aligned} & \left(1 - \frac{xyv}{1-(1-x)v}\right) K(x,y,v) \\ & = \frac{y}{x} K_0(x,v) - \frac{xyv}{1-(1-x)v} K(xv,y) + \frac{xy^2}{(1-x)(1-y)} \\ & \quad + \frac{xy}{(1-x)(1-(1-x)v)} \left(K(\frac{x}{1-x}, y) - K(x,y,\frac{1}{1-x}) \right). \end{aligned} \tag{35}$$

Substituting $v=1$ in (35), and noting (34), we obtain

$$\begin{aligned}&\frac{y}{1-x}\left(K(\frac{x}{1-x},y)-K(x,y,\frac{1}{1-x})\right)\\&=(1-y+y^2)K(x,y)-\frac{y(1-xy)}{x}K_0(x)+\frac{y(1-x-y)}{(1-x)(1-y)},\end{aligned}$$

where we have also used the fact $K_0(x,1)=(1-x)K_0(x)-x$. Hence, formula (35) gives

$$\begin{aligned}&\left(1-\frac{xyv}{1-(1-x)v}\right)K(x,y,v)\\&=\frac{y}{x}K_0(x,v)-\frac{xyv}{1-(1-x)v}K(xv,y)+\frac{xy^2}{(1-x)(1-y)}\\&\quad+\frac{x}{1-(1-x)v}\left((1-y+y^2)K(x,y)-\frac{y(1-xy)}{x}K_0(x)+\frac{y(1-x-y)}{(1-x)(1-y)}\right).\end{aligned}$$

By taking $v=\frac{1}{1-x+xy}$ in this last equation, we obtain

$$\begin{aligned}K(x,y)=&\frac{y}{(1-y+y^2)(1-x+xy)}K\left(\frac{x}{1-x+xy},y\right)\\&-\frac{y^2}{x(1-y+y^2)(1-x+xy)}K_0\left(x,\frac{1}{1-x+xy}\right)+\frac{y(1-xy)}{x(1-y+y^2)}K_0(x)\\&-\frac{y(1-x)}{(1-y+y^2)(1-x+xy)}.\end{aligned}$$

If we write $K(x,y)=\frac{y}{(1-y+y^2)(1-x+xy)}K\left(\frac{x}{1-x+xy},y\right)+\rho(x,y)$, then iteration (where we assume $|y|<1$) leads to the following result.

Theorem 6. *We have*

$$K(x,y)=\sum_{j\geq0}\frac{y^j}{(1-jx+jxy)(1-y+y^2)^j}\rho\left(\frac{x}{1-jx+jxy},y\right),$$

where

$$\begin{aligned}&\rho(x,y)\\&=\frac{y}{x(1-y+y^2)}\left((1-xy)K_0(x)-\frac{y}{1-x+xy}K_0(x,\frac{1}{1-x+xy})-\frac{x(1-x)}{1-x+xy}\right)\end{aligned}$$

and $K_0(x,y)$ and $K_0(x)$ are given by (29) and (30), respectively.

4. AVOIDING THE PATTERN 312

We first consider the $r = 0$ case of the avoidance. Again, it is more convenient to enumerate the corresponding set partitions under f. Let $\mathcal{L}_n = f(\mathcal{F}_n(312))$. It is seen that $\mathcal{L}_n$ consists of those members of $\mathcal{P}_n$ for which no non-singleton block with largest element x is directly followed by a singleton $\{y\}$ or by another non-singleton with second smallest element y such that $x > y$ in either case. For example, when $n = 4$, we have

$$\mathcal{F}_4(312) = \{02341, 01342, 02143, 02314, 01243, 01324, 02134, 01234, 03142\}$$

and

$$\mathcal{L}_4 = \{1234, 1|234, 12|34, 123|4, 1|2|34, 1|23|4, 12|3|4, 1|2|3|4, 13|24\},$$

where the i-th element of the latter corresponds under f to the respective element of the former.

Let $\ell_n = |\mathcal{L}_n|$ for $n \geq 0$. For example, one may verify $\ell_0 = \ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 4$ and $\ell_4 = 9$. To determine ℓ_n, we refine it analogously as before. Let $\mathcal{L}_{n,i}$ consist of those $\pi \in \mathcal{L}_n$ in which the first block is a non-singleton with second smallest element i for $2 \leq i \leq n$, and let $\mathcal{L}_{n,i,j}$ consist of the $\pi \in \mathcal{L}_{n,i}$ in which the first block has greatest element j where $2 \leq i \leq j \leq n$. Let $\ell_{n,i} = |\mathcal{L}_{n,i}|$ and $\ell_{n,i,j} = |\mathcal{L}_{n,i,j}|$. Note that

$$\ell_n = \ell_{n-1} + \sum_{i=2}^{n} \ell_{n,i} = \ell_{n-1} + \sum_{i=2}^{n}\sum_{j=i}^{n} \ell_{n,i,j}, \qquad n \geq 2,$$

with $\ell_0 = \ell_1 = 1$.

The $\ell_{n,i,j}$ are given recursively as follows.

Lemma 6. *We have*

$$\ell_{n,i,i} = \sum_{a=i-1}^{n-2} \ell_{n-2,a}, \qquad 3 \leq i \leq n-1, \tag{36}$$

$$\ell_{n,2,j} = \ell_{n-j} + \sum_{a=0}^{j-4}\sum_{b=j-a-2}^{n-a-3} \binom{j-3}{a} \ell_{n-a-3,b}, \qquad 4 \leq j \leq n-1, \tag{37}$$

and

$$\ell_{n,i,j} = \sum_{a=0}^{j-i-1}\sum_{b=j-a-2}^{n-a-3} \binom{j-i-1}{a} \ell_{n-a-3,b}, \qquad 4 \leq i+1 \leq j \leq n-1, \tag{38}$$

with $\ell_{n,i,n} = \delta_{i,2}$ *for* $2 \leq i \leq n$, $\ell_{n,2,2} = \ell_{n-2}$ *if* $n \geq 2$ *and* $\ell_{n,2,3} = \ell_{n-3}$ *if* $n \geq 3$.

Proof. The formulas for $\ell_{n,2,2}$ and $\ell_{n,2,3}$ are easily seen to hold as the initial block in either case imposes no restriction on subsequent blocks. That $\ell_{n,i,n} = \delta_{i,2}$ follows from the fact that $\mathcal{L}_{n,i,n}$ consists of the single-block partition of $[n]$ if $i = 2$ and is empty otherwise. To show (36), let $\pi \in \mathcal{L}_{n,i,i}$ where $3 \le i \le n-1$. Note that $i \ge 3$ implies that if the second block of π were a singleton, then it would have to be $\{2\}$, which is not allowed since $\{2\}$ cannot directly follow $\{1, i\}$. Thus, the second block must be a non-singleton with second smallest element at least $i+1$. This implies that there are $\sum_{a=i-1}^{n-2} \ell_{n-2,a}$ possibilities for the remaining blocks of π, which yields (36).

To show (37), suppose $\pi \in \mathcal{L}_{n,2,j}$ where $4 \le j \le n-1$. Let S denote the subset of $[3, j-1]$ whose elements lie in the initial block of π and let $a = |S|$. If $a = j-3$, then the first block of π is precisely the set $[j]$ and thus has no effect on subsequent blocks, which yields ℓ_{n-j} possibilities. So assume $0 \le a \le j-4$. Then for each of the $\binom{j-3}{a}$ possible S, there are $\sum_{b=j-a-2}^{n-a-3} \ell_{n-a-3,b}$ possibilities concerning the remaining blocks of π since the second block must be a non-singleton with second smallest element at least $j+1$. Formula (37) then follows from considering all possible a. The proof of (38) is similar except that now the case when the first block of π equals $\{1\} \cup [i, j]$ is not to be differentiated from the other cases when $i \ge 3$ as it was above when $i = 2$. □

By Lemma 6, we obtain the following.

Lemma 7. *For all* $3 \le i \le n-1$,

$$\ell_{n,i} = \sum_{a=i-1}^{n-2} \ell_{n-2,a} + \sum_{j=i+1}^{n-1} \sum_{a=0}^{j-i-1} \sum_{b=j-a-2}^{n-a-3} \binom{j-i-1}{a} \ell_{n-a-3,b}, \tag{39}$$

with $\ell_{n,2} = \ell_{n-1}$ *if* $n \ge 2$ *and* $\ell_{n,n} = 0$ *if* $n \ge 3$.

Define $\ell(x) = \sum_{n\ge1} \ell_n x^n$, $\ell_n(v) = \sum_{i=2}^{n} \ell_{n,i} v^{i-2}$ and $\ell(x, v) = \sum_{n\ge2} \ell_n(v) x^n$. By the fact $\ell_n = \ell_{n-1} + \sum_{i=2}^{n} \ell_{n,i}$ for $n \ge 2$, we have

$$\ell(x) = \frac{x + \ell(x, 1)}{1 - x}. \tag{40}$$

Multiplying each part of equation (39) by $x^n v^{i-2}$, and summing over $n \ge 4$ and $3 \le i \le n-1$, we get

$$\sum_{n\ge4} \sum_{i=3}^{n-1} \ell_{n,i} x^n v^{i-2} = \sum_{n\ge4} \sum_{i=2}^{n} \ell_{n,i} x^n v^{i-2} - \sum_{n\ge4} \ell_{n-1} x^n$$
$$= \ell(x, v) - x^2 - 2x^3 - x(\ell(x) - x - 2x^2) = \ell(x, v) - x\ell(x),$$

$$\sum_{n\ge4} \sum_{i=3}^{n-1} \sum_{a=i-1}^{n-2} \ell_{n-2,a} x^n v^{i-2} = \sum_{n\ge4} \sum_{a=2}^{n-2} \sum_{i=3}^{a+1} \ell_{n-2,a} x^n v^{i-2}$$
$$= \frac{x^2 v}{1-v} \sum_{n\ge4} (\ell_{n-2}(1) - v\ell_{n-2}(v)) x^{n-2} = \frac{x^2 v}{1-v} (\ell(x, 1) - v\ell(x, v)),$$

and

$$\begin{aligned}
&\sum_{n\geq4}\sum_{i=3}^{n-1}\sum_{j=i+1}^{n-1}\sum_{a=0}^{j-i-1}\sum_{b=j-a-2}^{n-a-3}\binom{j-i-1}{a}\ell_{n-a-3,b}x^nv^{i-2}\\
&=\sum_{n\geq5}\sum_{i=3}^{n-2}\sum_{a=0}^{n-i-2}\sum_{b=i-1}^{n-a-3}\sum_{j=i+a+1}^{a+b+2}\binom{j-i-1}{a}\ell_{n-a-3,b}x^nv^{i-2}\\
&=\sum_{n\geq5}\sum_{i=3}^{n-2}\sum_{a=0}^{n-i-2}\sum_{b=i-1}^{n-a-3}\binom{a+b-i+2}{a+1}\ell_{n-a-3,b}x^nv^{i-2}\\
&=\sum_{i\geq3}\sum_{a\geq0}\sum_{n\geq i+2}\sum_{b=i-1}^{n-3}\binom{a+b-i+2}{a+1}\ell_{n-3,b}x^{n+a}v^{i-2}\\
&=\sum_{i\geq3}\sum_{n\geq i+2}\sum_{b=i-1}^{n-3}\left(\frac{1}{(1-x)^{b-i+2}}-1\right)\ell_{n-3,b}x^{n-1}v^{i-2}\\
&=\sum_{n\geq5}\sum_{b=2}^{n-3}\sum_{i=3}^{b+1}\left(\frac{1}{(1-x)^{b-i+2}}-1\right)\ell_{n-3,b}x^{n-1}v^{i-2}\\
&=\sum_{n\geq5}\sum_{b=2}^{n-3}\left(\frac{-v^b}{1-(1-x)v}+\frac{v^b}{1-v}+\frac{v}{(1-x)^{b-1}(1-(1-x)v)}-\frac{v}{1-v}\right)\ell_{n-3,b}x^{n-1}\\
&=x^2\sum_{n\geq5}\left(\frac{-v^2\ell_{n-3}(v)}{1-(1-x)v}+\frac{v^2\ell_{n-3}(v)}{1-v}+\frac{v\ell_{n-3}(\frac{1}{1-x})}{(1-x)(1-(1-x)v)}-\frac{v\ell_{n-3}(1)}{1-v}\right)x^{n-3}\\
&=\frac{-x^2v^2\ell(x,v)}{1-(1-x)v}+\frac{x^2v^2\ell(x,v)}{1-v}+\frac{x^2v\ell(x,\frac{1}{1-x})}{(1-x)(1-(1-x)v)}-\frac{x^2v\ell(x,1)}{1-v}.
\end{aligned}$$

By collecting the appropriate terms from the preceding calculations and (40), we have

$$\begin{aligned}
\ell(x,v)&=\frac{x}{1-x}(x+\ell(x,1))-\frac{x^2v^2}{1-(1-x)v}\ell(x,v)\\
&\quad+\frac{x^2v}{(1-x)(1-(1-x)v)}\ell(x,\frac{1}{1-x}).
\end{aligned}\tag{41}$$

Setting $v=1$ in (41), and solving for $\ell(x,\frac{1}{1-x})$, yields

$$\ell(x,\frac{1}{1-x})=\frac{1-x-x^2}{x}\ell(x,1)-x.$$

Hence, (41) may be rewritten as

$$\left(1+\frac{x^2v^2}{1-(1-x)v}\right)\ell(x,v) = \frac{x(1-x^2v)}{(1-x)(1-(1-x)v)}\ell(x,1)+\frac{x^2(1-v)}{(1-x)(1-(1-x)v)}. \tag{42}$$

Applying the kernel method, and taking $v=v_0=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ in (42), we obtain

$$\ell(x,1)=\frac{x(v_0-1)}{1-x^2v_0}=(1-x)v_0-1,$$

which, by (40), implies

$$\ell(x)=v_0-1.$$

Note that $v_0=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}=\sum_{n\geq 0}M_nx^n$, where M_n denotes the n-th Motzkin number. Thus we may conclude the following.

Theorem 7. *The number of members of $\mathcal{F}_n$ that avoid the* 312 *subword is given by M_n for all $n\geq 0$.*

Concerning the avoidance in the general r-case, we enumerate the equivalent subset of $\mathcal{P}_{n,r}$ under f defined as follows. Let $\mathcal{L}_n^{(r)}$ denote the subset of $\mathcal{P}_{n,r}$ whose members meet the same requirements concerning adjacent blocks as the members of $\mathcal{L}_n$ above. Given $1\leq i\leq n$, let $\mathcal{L}_{n,i}^{(r)}\subseteq\mathcal{L}_n^{(r)}$ comprise those members in which the first (special) block is a non-singleton with second smallest element $r+i$. Let the further subset $\mathcal{L}_{n,i,j}^{(r)}\subseteq\mathcal{L}_{n,i}^{(r)}$ consist of those partitions in which the largest element in the first block is $r+j$ for $1\leq i\leq j\leq n$. Let $\ell_n^{(r)}=|\mathcal{L}_n^{(r)}|$, $\ell_{n,i}^{(r)}=|\mathcal{L}_{n,i}^{(r)}|$ and $\ell_{n,i,j}^{(r)}=|\mathcal{L}_{n,i,j}^{(r)}|$.

Considering whether or not the block containing the (special) element 1 is a singleton yields

$$\ell_n^{(r)}=\ell_n^{(r-1)}+\sum_{i=1}^{n}\ell_{n,i}^{(r)}=\ell_n^{(r-1)}+\sum_{i=1}^{n}\sum_{j=i}^{n}\ell_{n,i,j}^{(r)},\qquad n,r\geq 1,$$

with $\ell_n^{(0)}=\ell_n$. Further, it is seen that $\ell_{n,i,j}^{(1)}=\ell_{n+1,i+1,j+1}$ for all i and j, which implies $\ell_{n,i}^{(1)}=\ell_{n+1,i+1}$ and $\ell_n^{(1)}=\ell_{n+1}$. Put $\ell_0^{(r)}=1$ for all $r\geq 0$. By reasoning comparable to that used in the proof of Lemma 6 above, we have the following recurrence for the array $\ell_{n,i,j}^{(r)}$ when $r\geq 2$.

Lemma 8. *If $r\geq 2$, then*

$$\ell_{n,i,j}^{(r)}=\sum_{a=0}^{j-i-1}\sum_{b=j-a-1}^{n-a-2}\binom{j-i-1}{a}\ell_{n-a-2,b}^{(r-1)},\qquad 1\leq i<j\leq n-1, \tag{43}$$

and

$$\ell^{(r)}_{n,i,i} = \sum_{a=i}^{n-1} \ell^{(r-1)}_{n-1,a}, \qquad 1 \le i \le n-1, \tag{44}$$

with $\ell^{(r)}_{n,i,n} = 0$ *for* $1 \le i \le n$.

From these recurrences, it is possible to determine the ogf of $\ell^{(r)}_n$ for a given r as follows. Define $\ell_r(x) = \sum_{n\ge1} \ell^{(r)}_n x^n$ and $\ell_r(x,v) = \sum_{n\ge1}\left(\sum_{i=1}^n \ell^{(r)}_{n,i} v^{i-1}\right) x^n$. Then we have immediately

$$\ell_1(x) = \frac{1}{x}(\ell(x) - x), \quad \ell_1(x,v) = \frac{1}{x}\ell(x,v), \tag{45}$$

with

$$\ell_r(x) = \ell_{r-1}(x) + \ell_r(x,1), \qquad r \ge 1. \tag{46}$$

Proceeding as before using Lemma 8, we obtain

$$\ell_r(x,v) = \frac{x}{(1-x)(1-(1-x)v)}\ell_{r-1}(x,1/(1-x)) - \frac{xv}{1-(1-x)v}\ell_{r-1}(x,v), \tag{47}$$

for all $r \ge 2$. Define $\ell(x,y,v) = \sum_{r\ge1} \ell_r(x,v)y^r$. Multiplying both sides of (47) by y^r and summing over $r \ge 2$ yields

$$\ell(x,y,v) = \ell_1(x,v)y + \frac{xy}{(1-x)(1-(1-x)v)}\ell(x,y,1/(1-x)) - \frac{xyv}{1-(1-x)v}\ell(x,y,v),$$

which is equivalent to

$$\begin{aligned}\left(1+\frac{xyv}{1-(1-x)v}\right)\ell(x,y,v) &= \ell_1(x,v)y \\ &\quad + \frac{xy}{(1-x)(1-(1-x)v)}\ell(x,y,1/(1-x)).\end{aligned} \tag{48}$$

By taking $v = \frac{1}{1-x-xy}$, we obtain

$$\ell(x,y,1/(1-x)) = \frac{y(1-x)}{1-x-xy}\ell_1\left(x, \frac{1}{1-x-xy}\right),$$

which, by (45) and (42), implies

$$\begin{aligned}&\ell(x,y,1/(1-x)) \\ &= \frac{y(1-x)((1+x)(2x^2y+x^2-xy-3x+1)+(x^2+xy+x-1)\sqrt{1-2x-3x^2})}{2x^3(xy^2+xy+x-y)}.\end{aligned}$$

Hence, by (48), one can state the following result.

Theorem 8. *We have*

$$\ell(x,y,v)=\frac{y((1+x)(x^2yv-x^2v-xy-2x+1)+(x^2yv+x^2v+xy-1)\sqrt{1-2x-3x^2})}{2x(x^2v^2+xv-v+1)(xy^2+xy+x-y)}.$$

Extracting the coefficient of y^r in $\ell(x,y,1)$, we have

$$\ell_r(x,1)=\frac{2^{r+1}x^r}{(1-x+\sqrt{1-2x-3x^2})^{r+1}}.$$

Then (46) implies the relation

$$\ell_r(x)=\ell_1(x)+\sum_{j=2}^{r}\ell_j(x,1),$$

which leads to the following formula for $\ell_r(x)$.

Theorem 9. *If $r\geq 1$, then the generating function $\ell_r(x)=\sum_{n\geq 1}|\mathcal{F}_n^{(r)}(312)|x^n$ is given by*

$$\begin{aligned}\ell_r(x)=&\frac{8x^2\left(1-\frac{2^{r-1}x^{r-1}}{(1-x+\sqrt{1-2x-3x^2})^{r-1}}\right)}{(1-3x+\sqrt{1-2x-3x^2})(1-x+\sqrt{1-2x-3x^2})^2}\\&+\frac{1-x-2x^2-\sqrt{1-2x-3x^2}}{2x^3}-1.\end{aligned}$$

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