Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **16** (2022), 178–188.
https://doi.org/10.2298/AADM210419030P

COMMUTATIVE VON NEUMANN REGULAR RINGS ARE 1-GRÖBNER

Zoran Z. Petrović and Maja Roslavcev[*]

Let R be a commutative von Neumann regular ring. We show that every finitely generated ideal I in the ring of polynomials $R[X]$ has a strong Gröbner basis. We prove this result using only the defining property of a von Neumann regular ring.

1. INTRODUCTION

When considering the Gröbner bases for ideals in polynomial rings over a ring R, most of the theory and applications is developed for R — a field or a Noetherian commutative domain (e.g., for R a PID or a Dedekind domain); see [**1**], [**2**]. Of interest are also the cases when R is not a Noetherian ring, for example, for R a valuation domain (see [**7**]). Here we deal with a specific kind of ring that is commutative and a non-domain. A von Neumann regular ring R is a ring in which for every $a \in R$ there exists $b \in R$ such that $a = aba$. This type of rings was originally introduced in [**8**] and as a basic reference, one can use [**4**]. Generally, R does not need to be commutative, as the stated definition suggests, but here we make that extra assumption. For more information about von Neumann regular condition in commutative rings, see [**3**].

According to [**7**], a ring R is 1-Gröbner if for an ideal $I \triangleleft R[X]$ that is finitely generated, there is a Gröbner basis for I. In [**6**], the authors prove that a valuation ring of dimension zero is 1-Gröbner. A von Neumann regular ring is another kind of zero-dimensional ring and here we prove that it is also 1-Gröbner, if it is commutative.

[*]Corresponding author. M.Roslavcev
2020 Mathematics Subject Classification. 13P10, 13B25, 13F99.
Keywords and Phrases. Von Neumann Regular Ring, Gröbner Bases.

Thus, we introduce a new class of non-Noetherian rings which are 1-Gröbner. Note that Boolean rings are von Neumann regular rings, so all the results of this paper hold for Boolean rings too. Also, here we actually prove that there is a strong Gröbner basis for every finitely generated ideal in $R[X]$, for a commutative von Neumann regular ring R. Throughout the remaining part of the paper, all rings R are assumed to be commutative and von Neumann regular, unless otherwise stated.

The plan of the paper is as follows. In the second section, we recall several basic results concerning von Neumann regular rings and then proceed to prove a couple of lemmas which allow us to prove Theorem 8 as follows: In a finitely generated ideal I of $R[X]$, there is a polynomial whose leading coefficient divides all coefficients of any a polynomial of I. In the third section, we give the basic definitions concerning the Gröbner bases and present the important Lemma 9. This lemma enables us to prove the main result of the paper: every finitely generated ideal in $R[X]$ has a strong Gröbner basis. Importantly, the proofs of the lemmas and theorems in the paper, in addition to establish the existence of certain objects, are presented in such a way, that they allow us to eventually calculate the Gröbner basis for a given ideal. At the end of the paper, we present an example which illustrates the stated proofs and methods. Namely, we give an ideal in the ring $\prod_{k=1}^{\infty} \mathbb{F}_3$, where $\mathbb{F}_3$ is a field of order 3, and present all the steps for finding its Gröbner basis.

2. VON NEUMANN REGULAR RINGS

We begin this section with some preliminary results. In the following definition, we only suppose that the ring is unital.

Definition 1. *A ring R is called a von Neumann regular ring if for every element $a \in R$ there exists an element $b \in R$ so that $a = aba$.*

Because we deal with commutative rings, this condition is equivalent to the existence of $b \in R$ such that $a = a^2 b$, i.e., $a(1 - ab) = 0$. Obviously, a commutative von Neumann regular ring with no non-zero zero divisors is a field.

The following lemma gives us a useful representation of an element in a von Neumann regular ring. This result is a well known fact, but we present it here with proof for the sake of completeness (cf. Theorem 2.2 in [**3**]).

Lemma 1. *For every element $a \in R$ there is an invertible element $u \in R$ and an idempotent $e \in R$ such that $a = ue$.*

Proof. Let $b \in R$ be such that $a = aba$. We have that

$$(a + 1 - ab)ab = a^2 b + ab - a^2 b^2 = a + ab - ab = a.$$

Let us set $u = a + 1 - ab$ and $e = ab$. Since $e^2 = a^2 b^2 = ab = e$, then e is an idempotent. Also, $(a + 1 - ab)(ab^2 + 1 - ab) = 1$, so u is invertible. □

Another useful fact that will be used here throughout the text is the following.

Remark 2. *If we suppose that R is any commutative ring, then the condition $e \mid a$, for an idempotent e, is equivalent to $a = ea$. Namely, if $e \mid a$, then $a = eb$, for $b \in R$. So, $a = eb = eeb = ea$. The other direction is trivial.*

Obviously, in this kind of rings we basically deal with idempotents. So, concerning Remark 2 and Lemma 1, we may notice the following important fact. If e and s are idempotents, then $1-(1-e)(1-s)$ is an idempotent which divides both e and s. Namely, $1-e$ and $1-s$ are idempotents as well and so is their product $(1-e)(1-s)$. Then $1-(1-e)(1-s)$ is an idempotent too. Also, $(1-(1-e)(1-s))e = e$, and the same holds when multiplying with s. This short analysis allows us to deduce the following theorem, which is known (cf. Theorem 1.1 in [**4**] or Theorem 4.23 in [**5**]), but we present it in detail because its proof is the basis for Lemmas 5 and 7.

Theorem 3. *Any finitely generated ideal J in R is generated by an idempotent.*

Proof. Let $J = \langle a_1, \dots, a_n \rangle$. According to Lemma 1, there are invertibles $u_1, \dots, u_n$ and idempotents $e_1, \dots, e_n$ such that $a_i = u_i e_i$ for $1 \le i \le n$. Obviously, $J = \langle e_1, \dots, e_n \rangle$. Then

$$J = \langle 1 - \prod_{i=1}^{n} (1-e_i) \rangle.$$

Namely,

$$e_j \cdot (1 - \prod_{i=1}^{n}(1-e_i)) = e_j - e_j(1-e_j)\prod_{i \neq j}(1-e_i) = e_j,$$

for each $j \in \{1, \dots, n\}$. So, according to Remark 2, the element $1 - \prod_{i=1}^{n}(1-e_i)$ divides each generator of J and then

$$\langle e_1, \dots, e_n \rangle \subseteq \langle 1 - \prod_{i=1}^{n}(1-e_i) \rangle.$$

We also have that

$$1 - \prod_{i=1}^{n}(1-e_i) = \sum_{i=1}^{n} e_i - \sum_{i \neq j} e_i e_j + \cdots + (-1)^{n-1} \prod_{i=1}^{n} e_i,$$

and therefore, $1 - \prod_{i=1}^{n}(1-e_i) \in \langle e_1, \dots, e_n \rangle$. Since $1-e_1, \dots, 1-e_n$ are idempotents, so is their product $\prod_{i=1}^{n}(1-e_i)$ and then $1 - \prod_{i=1}^{n}(1-e_i)$ is also an idempotent. □

This theorem could also be easily proved by mathematical induction. That is, if $1 - \prod_{i=1}^{k}(1-e_i)$ is an idempotent that divides $e_1, \dots, e_k$ and e_{k+1} is a new idempotent, then

$$1 - (1 - (1 - \prod_{i=1}^{k}(1-e_i)))(1-e_{k+1}) = 1 - \prod_{i=1}^{k}(1-e_i) \cdot (1-e_{k+1}) = 1 - \prod_{i=1}^{k+1}(1-e_i)$$

is an idempotent that divides $e_1, \dots, e_{k+1}$.

In the following lemma we only suppose that the ring is commutative. The lemma is known (see, e.g., 2.3(3), p.8, in [**9**]) and very easy to prove, so we present it without proof. It concerns the question of the annihilator of an idempotent element.

Lemma 4. *Let $e \in R$ be an idempotent. Then* $\operatorname{Ann}(e) = \langle 1 - e \rangle$.

Let us introduce some notions regarding the ring of polynomials $R[X]$ that will be needed in this paper. If $f = a_n X^n + \cdots + a_1 X + a_0 \in R[X]$, where $a_n \neq 0$, then the leading term of f is $\mathrm{LT}(f) = a_n X^n$ and the leading coefficient of f is $\mathrm{LC}(f) = a_n$. Also, for a non-zero ideal $I \triangleleft R[X]$, $\mathcal{LT}(I) = \{\mathrm{LT}(f) \mid f \in I \setminus \{0\}\}$ and $\mathrm{LT}(I) = \langle \mathcal{LT}(I) \rangle$. We also define the content of a polynomial and the only assumption on the ring here is that it is commutative.

Definition 2. *Let $f \in R[X]$ be such that $f(X) = a_n X^n + \cdots + a_1 X + a_0$. Then the content $c(f)$ of the polynomial f is the ideal $\langle a_n, \dots, a_0 \rangle \subseteq R$.*

We present here several results which enable us to prove that a finitely generated ideal in the ring of polynomials contains a polynomial whose leading coefficient divides all coefficients of any a polynomial of I. We begin with a lemma.

Lemma 5. *Let $p(X) \in R[X]$ is such that* $\deg(p) = n$. *Then, there is a polynomial $r \in \langle p(X) \rangle$ such that* $\deg(r) = n$ *and* $\langle \mathrm{LC}(r) \rangle = c(p)$.

Proof. Without loss of generality we can assume that the leading coefficient of p is an idempotent. So, let

$$p(X) = e_n X^n + u_{n-1} e_{n-1} X^{n-1} + u_{n-2} e_{n-2} X^{n-2} + \cdots + u_1 e_1 X + u_0 e_0,$$

where $u_{n-1}, \dots, u_0$ are invertible and $e_n, \dots, e_0$ are idempotents. Let us first define the sequence of elements $e^{(k)}$ recursively:

$$\begin{aligned} e^{(0)} &= e_n \\ e^{(k)} &= e^{(k-1)} + (1 - e^{(k-1)}) e_{n-k}, \end{aligned}$$

for $k \in \{1, \dots, n\}$. As we discussed earlier, these elements are idempotents and $e^{(k)}$ divides $e_n, \dots, e_{n-k}$ for $0 \leq k \leq n$. According to the presented proof of Theorem 3, $c(p) = \langle e^{(n)} \rangle$.

Now we can define the polynomial $r^* \in R[X]$ such that

$$r^*(X) = 1 + u_{n-1}^{-1}(1 - e^{(0)})X + u_{n-2}^{-1}(1 - e^{(1)})X^2 + \cdots + u_0^{-1}(1 - e^{(n-1)})X^n.$$

Since $e_s(1-e^{(k)}) = 0$ for each $s \geq n-k$, we have that $\deg(p \cdot r^*) = n$. Actually,

$$\begin{aligned}
\mathrm{LC}(pr^*) &= e_n + (1-e^{(0)})e_{n-1} + \cdots + (1-e^{(n-2)})e_1 + (1-e^{(n-1)})e_0 \\
&= e^{(1)} + (1-e^{(1)})e_{n-2} + \cdots + (1-e^{(n-2)})e_1 + (1-e^{(n-1)})e_0 \\
&= \cdots \\
&= e^{(n-1)} + (1-e^{(n-1)})e_0 \\
&= e^{(n)}.
\end{aligned}$$

We set $r = pr^*$ and this concludes the proof. □

Remark 6. *In Lemma 5, if one of the coefficients of the polynomial p is invertible, that is, if some idempotent e_k equals 1, the given construction produces a monic polynomial r. In the sequence of elements $e^{(s)}$, the elements $e^{(n-k)}, \ldots, e^{(n)}$ would all be equal to 1.*

Beside the previous result, we can also construct the polynomial whose leading coefficient divides all the coefficients of two given polynomials p and q and its degree equals $\max\{\deg(p), \deg(q)\}$.

Lemma 7. *Let $p(X), q(X) \in R[X]$ are such that $\deg(p) = n$ and $\deg(q) = m \leq n$. Then, there is a polynomial $r \in \langle p(X), q(X)\rangle$ such that $\deg(r) = n$ and $\langle \mathrm{LC}(r)\rangle = c(p) + c(q)$.*

Proof. Due to Lemma 5, there is $r_1 \in \langle p(X)\rangle$ such that $\deg(r_1) = n$ and $\langle \mathrm{LC}(r_1)\rangle = c(p)$. We can suppose that $\mathrm{LC}(r_1) = e$, where e is an idempotent. Let

$$q(X) = e_m X^m + u_{m-1}e_{m-1}X^{m-1} + u_{m-2}e_{m-2}X^{m-2} + \cdots + u_1 e_1 X + u_0 e_0,$$

where $u_{m-1}, \ldots, u_0$ are invertible and $e_m, \ldots, e_0$ are idempotents. (We also suppose here that $\mathrm{LC}(q) = e_m$ is an idempotent.) As in the previous lemma, we define the sequence of elements $e^{(k)}$:

$$\begin{aligned}
e^{(0)} &= e \\
e^{(1)} &= e^{(0)} + (1-e^{(0)})e_m \\
e^{(k)} &= e^{(k-1)} + (1-e^{(k-1)})e_{m-k+1},
\end{aligned}$$

for $k \in \{2, \ldots, m+1\}$. These elements are idempotents and $e^{(k)}$ divides $e, e_m, \ldots, e_{m-k+1}$, for $1 \leq k \leq m+1$. Obviously, $e^{(m+1)}$ is an element which divides e (that is, all the coefficients of p) and also all the coefficients of q. So, $\langle e^{(m+1)}\rangle = c(p) + c(q)$. Let $r^* \in R[X]$ be such that

$$\begin{aligned}
r^*(X) = (1-e^{(0)})X^{n-m} &+ u_{m-1}^{-1}(1-e^{(1)})X^{n-m+1} + u_{m-2}^{-1}(1-e^{(2)})X^{n-m+2} + \\
&\cdots + u_1^{-1}(1-e^{(m-1)})X^{n-1} + u_0^{-1}(1-e^{(m)})X^n.
\end{aligned}$$

Let us observe the polynomial $r_1 + qr^*$. Considering the fact that $e_s(1 - e^{(k)}) = 0$ for each $s \geq m - k + 1$, we have that

$$\begin{aligned}\mathrm{LC}(r_1 + qr^*) &= e + (1 - e^{(0)})e_m + \cdots + (1 - e^{(m-1)})e_1 + (1 - e^{(m)})e_0 \\ &= e^{(1)} + (1 - e^{(1)})e_{m-1} + \cdots + (1 - e^{(m-1)})e_1 + (1 - e^{(m)})e_0 \\ &= \cdots \\ &= e^{(m)} + (1 - e^{(m)})e_0 \\ &= e^{(m+1)}.\end{aligned}$$

Let $r = r_1 + qr^*$. Then $r \in \langle p(X)\rangle + \langle q(X)\rangle = \langle p(X), q(X)\rangle$. Since $\deg(r) = n$ and $\mathrm{LC}(r) = e^{(m+1)}$, we finally have that $\langle \mathrm{LC}(r)\rangle = c(p) + c(q)$. □

Now we prove the main result of this section.

Theorem 8. *Let R be a commutative von Neumann regular ring and $I = \langle f_1, \ldots, f_m\rangle$ be an ideal in $R[X]$. Then, there is a polynomial $f \in I$ such that $\deg(f)$ equals $max\{\deg(f_1), \ldots, \deg(f_m)\}$ and $\langle \mathrm{LC}(f)\rangle = c(f_1) + \cdots + c(f_m)$.*

Proof. We start with the generator of I which is of the highest degree. Then by the consecutive application of the previous lemma we get the element of I such that its leading coefficient divides all the coefficients of all the generators of I. □

3. GRÖBNER BASES

Let us recall the definition of (strong) Gröbner bases for an ideal in $R[X]$ (see, e.g., [**1**]), where R is only assumed to be a commutative ring.

Definition 3. *Let I be a non-zero ideal in $R[X]$. A subset $G = \{g_1, \ldots, g_m\}$ of I is a Gröbner basis for I if $\mathrm{LT}(I) = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_m)\rangle$.*

Definition 4. *Let I be a non-zero ideal in $R[X]$. A subset $G = \{g_1, \ldots, g_m\}$ of I is a strong Gröbner basis for I if for any $f \in I \setminus \{0\}$, there exists $g_i \in G$ such that $\mathrm{LT}(g_i) \mid \mathrm{LT}(f)$.*

The aim here is to prove that every finitely generated ideal I in $R[X]$ has a Gröbner basis. Together with such an ideal, we can observe certain R-submodules, which are assigned to I. These modules consist of all the polynomials of that ideal which are of a bounded degree. Namely, let $R[X]_k$ be the submodule of $R[X]$ generated by $1, X, X^2, \ldots, X^k$ and I_k be the submodule $I \cap R[X]_k$ of $R[X]$. These submodules I_k also can be seen as R-modules, as we observe them in the next lemma.

Lemma 9. *Let $I = \langle f_1, \ldots, f_m\rangle$ be a finitely generated ideal in the ring $R[X]$. Then, the R-modules $I_0, I_1, \ldots I_{n-1}$, where $n = max\{\deg(f_1), \ldots, \deg(f_m)\}$, are finitely generated.*

Proof. According to Theorem 8 there is a polynomial $f \in I$ such that $\mathrm{LC}(f)$ divides all the coefficients of any a polynomial of I and so that $\deg(f) = n$. Then of course $I = \langle f, f_1, \dots, f_m \rangle$.

Let us suppose that $\mathrm{LC}(f) = e$, for an idempotent e. First, we show that I_{n-1} is a finitely generated R-module. Actually, the set of elements

$$\rho(X^i f_j, f), \qquad 0 \le i \le n-1, \qquad 1 \le j \le m,$$

where $\rho(g, h)$ stands for the remainder in the division of g by h, forms the set of generators for this R-module. These remainders here are well-defined, since all the coefficients of $f_1, \dots, f_m$ are divisible by e.

Let $h \in I_{n-1}$. Since $h \in I$, we have that

$$h = rf + r_1 f_1 + \cdots + r_m f_m,$$

for $r, r_1, \dots, r_m \in R[X]$ and we can suppose that $\deg(r_1), \dots, \deg(r_m) < n$. If not, we could write $f_j = ep_j$, for $1 \le j \le m$, and then

$$h = rf + r_1 f_1 + \cdots + r_m f_m = rf + er_1 p_1 + \cdots + er_m p_m.$$

Then $er_1, \dots, er_m$ can be divided by f and let

$$er_j = q_j f + t_j, \qquad \deg(t_j) < n,\ 1 \le j \le m.$$

Also, all the coefficients of t_j are divisible by e. If $t_j = es_j$, for $1 \le j \le m$, then

$$\begin{aligned} h &= rf + (q_1 f + t_1)p_1 + \cdots + (q_m f + t_m)p_m \\ &= rf + (q_1 f + es_1)p_1 + \cdots + (q_m f + es_m)p_m \\ &= (r + q_1 p_1 + \cdots + q_m p_m)f + s_1 ep_1 + \cdots + s_m ep_m \\ &= (r + q_1 p_1 + \cdots + q_m p_m)f + s_1 f_1 + \cdots + s_m f_m. \end{aligned}$$

Obviously, $\deg(s_j) (= \deg(t_j))$ is less than n.

So, let

$$r_j = \sum_{i=0}^{k_j} \alpha_i^{(j)} X^i, \qquad k_j < n, \qquad 1 \le j \le m,$$

$$X^i f_j = f s_i^{(j)} + \rho(X^i f_j, f), \qquad 0 \le i \le k_j, \qquad 1 \le j \le m.$$

It follows that

$$h = f\left(r + \sum_{j=1}^{m}\sum_{i=0}^{k_j} \alpha_i^{(j)} s_i^{(j)}\right) + \sum_{j=1}^{m}\sum_{i=0}^{k_j} \alpha_i^{(j)} \rho(X^i f_j, f).$$

So, the polynomial h is of the form $fp + q$, $p \in R[X]$ and q being the R-linear combination of the polynomials $\rho(X^i f_j, f)$, $0 \le i \le k_j$, $1 \le j \le m$. Since $h \in I_{n-1}$, then $\deg(h) < n$.

If $p = 0$, then h is an R-linear combination of the remainders $\rho(X^i f_j, f)$. If not, let $p = \beta_l X^l + \cdots + \beta_0$, where $\beta_l \neq 0$. We can see the product fp as

$$fp = f \cdot \beta_l X^l + f \cdot \beta_{l-1} X^{l-1} + \cdots + f \cdot \beta_0.$$

It is necessary that the coefficient of fp with X^{n+l} is equal to zero. So, firstly, we must have $e\beta_l = 0$. Due to Lemma 4, $\mathrm{Ann}(e) = \langle 1-e \rangle$. It follows that $\beta_l = a(1-e)$, for $a \in R$. Since each coefficient of f is divisible by e, we have that $f \cdot \beta_l X^l = 0$. In the same way, since now $e\beta_{l-1}$ must be zero, it follows that β_{l-1} is divisible by $1-e$ and then $f \cdot \beta_{l-1} X^{l-1} = 0$. By repeating this method we get that $f \cdot p = 0$.

So we have that h is an R-linear combination of the remainders $\rho(X^i f_j, f)$ anyway. From the fact that $\rho(X^i f_j, f) = X^i f_j - f s_i^{(j)}$ it follows that all the remainders $\rho(X^i f_j, f)$, for $0 \leq i \leq k_j$ and $1 \leq j \leq m$, belong to I, and consequently to I_{n-1}. Therefore, these polynomials form the generating set for the R-module I_{n-1}.

Let us move on to prove that the rest of the modules $I_{n-2}, \ldots, I_0$ are finitely generated. Let $\pi_k : R[X]_k \to R$ be the homomorphisms such that $\pi_k(p)$ is the coefficient of the monomial X^k in the polynomial p. If $I_{n-1} = \{0\}$, then $I_0 = \cdots = I_{n-2} = \{0\}$. If $I_{n-1} \neq \{0\}$, then the image $\pi_{n-1}(I_{n-1})$ is also a finitely generated submodule of R, that is, a finitely generated ideal. According to Theorem 3, this image must be a principal ideal $\pi_{n-1}(I_{n-1}) = \langle c_{n-1} \rangle$ and c_{n-1} is a non-zero idempotent. Let $g_{n-1} \in I_{n-1}$ be the polynomial such that $\mathrm{LT}(g_{n-1}) = c_{n-1} X^{n-1}$.

Let us prove that I_{n-2} is a finitely generated R-module also by explicitly finding its generating set. If $h \in I_{n-2}$, then $h \in I_{n-1}$ and

$$h = \sum r_{ij} \rho(X^i f_j, f), \qquad r_{ij} \in R,\ 1 \leq j \leq m,\ 0 \leq i \leq n-1.$$

Since all the remainders $\rho(X^i f_j, f)$ are in I_{n-1}, we can divide them by g_{n-1}. So, $\rho(X^i f_j, f) = a_{ij} g_{n-1} + f_{ij}$. Note that it may happen that $a_{ij} = 0$. We now have that

$$h = \sum r_{ij}(a_{ij} g_{n-1} + f_{ij}) = \left(\sum r_{ij} a_{ij}\right) g_{n-1} + \sum r_{ij} f_{ij}.$$

Since $h \in I_{n-2}$, $\sum r_{ij} a_{ij}$ must belong to $\mathrm{Ann}(c_{n-1})$. According to Lemma 4, $\mathrm{Ann}(c_{n-1}) = \langle 1 - c_{n-1} \rangle$. It follows that each element in I_{n-2} is an R-linear combination of $(1 - c_{n-1}) g_{n-1}$ and $f_{ij} = \rho(\rho(X^i f_j, f), g_{n-1})$, where $1 \leq j \leq m$ and $0 \leq i \leq n-1$.

We repeat this procedure in order to prove that I_{n-3} is a finitely generated R-module. As before, if $I_{n-2} = \{0\}$, then $I_0 = \cdots = I_{n-3} = \{0\}$. If $I_{n-2} \neq \{0\}$, then $\pi_{n-2}(I_{n-2})$ is a finitely generated submodule of R, so, there is an idempotent $c_{n-2} \in R$ such that $\pi_{n-2}(I_{n-2}) = \langle c_{n-2} \rangle$. Let $g_{n-2} \in I_{n-2}$ be the polynomial such that $\mathrm{LT}(g_{n-2}) = c_{n-2} X^{n-2}$. The generating set for I_{n-3} consists of the remainders in the division of the generators for I_{n-2} by g_{n-2} together with $(1 - c_{n-2}) g_{n-2}$. We can continue in the same manner to prove that all these modules are finitely generated. □

Now, thanks to the previous lemma, we can prove the main theorem.

Theorem 10. *For a commutative von Neumann regular ring R, each finitely generated ideal of $R[X]$ contains a strong Gröbner basis.*

Proof. Let $I = \langle f_1, \ldots, f_m \rangle$. According to Theorem 8, there is a polynomial $f \in I$ such that $\deg(f) = max\{\deg(f_1), \ldots, \deg(f_m)\}(= n)$ and $\mathrm{LC}(f)$ divides all the coefficients of all the polynomials in I. Next, by Lemma 9, $I_{n-1}, \ldots, I_0$ are finitely generated R-modules. As in the proof of this lemma, let $\pi_k : R[X]_k \to R$ be the homomorphisms such that $\pi_k(p)$ is the coefficient of the monomial X^k in the polynomial p. It follows that the images $\pi_k(I_k)$ are principal ideals in R. Let $\pi_k(I_k) = \langle c_k \rangle$ and let $g_k \in I_k$ be such that $\mathrm{LT}(g_k) = c_k X^k$. Then a strong Gröbner basis G for I is given by $G = \{f, g_i \mid i \in \{0, \ldots, n-1\}, g_i \neq 0\}$. Namely, suppose that $p \in I \setminus \{0\}$. If $\deg(p) \geq n$, then $\mathrm{LT}(f) \mid \mathrm{LT}(p)$. If $\deg(p) = k < n$, then $p \in I_k$; so, $c_k X^k \mid \mathrm{LT}(p)$. This concludes the proof. $\square$

Among the examples of commutative von Neumann regular rings, we have direct products of fields. Here we give the following example.

Example 11. Let $R = \prod_{k=1}^{\infty} \mathbb{F}_3$, where $\mathbb{F}_3$ is a field of order 3. Let $f_1 = \alpha_2 X^2 + \alpha_1 X + \alpha_0$ and $f_2 = \beta_2 X^2 + \beta_0$, where

$$\begin{aligned} \alpha_2 &= (1,2,1,0,0,0,0,0,\ldots) & \beta_2 &= (1,0,1,0,1,0,1,0,\ldots) \\ \alpha_1 &= (1,1,1,1,0,0,0,0,\ldots) & \beta_0 &= (2,0,0,0,2,0,0,0,2,\ldots) \\ \alpha_0 &= (0,0,0,1,2,0,0,0,\ldots). \end{aligned}$$

Let us find the Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$. Since α_2 is not an idempotent, we can multiply f_1 by $(1,2,1,1,\ldots)$ to get

$$\tilde{f}_1 = (1,1,1,0,0,\ldots)X^2 + (1,2,1,1,0,0,\ldots)X + (0,0,0,1,2,0,\ldots),$$

and also $I = \langle \tilde{f}_1, f_2 \rangle$. We follow the steps in Lemma 5 to get the polynomial r^* and then the polynomial

$$r_1 = \tilde{f}_1 \cdot r^* = (1,1,1,1,1,0,0,\ldots)X^2 + (1,2,1,2,2,0,0,\ldots)X + (0,0,0,0,1,2,0,\ldots)$$

is such that $\mathrm{LC}(r_1)$ divides all the coefficients of $\tilde{f}_1$. We proceed to the application of Lemma 7 and form the polynomial r^{**} such that the leading coefficient of $r_1 + f_2 r^{**}$ divides all the coefficients of $\tilde{f}_1$ and f_2. Let $f = r_1 + f_2 r^{**}$. It equals

$$\begin{aligned} f = {} & (1,1,1,1,1,0,1,0,1,0,1,\ldots)X^2 + (1,2,1,2,2,0,0,\ldots)X + \\ & + (0,0,0,1,2,0,0,0,2,0,0,0,2,\ldots). \end{aligned}$$

Now we have that $I = \langle f, \tilde{f}_1, f_2 \rangle$. As in Theorem 10, we find the generators for the R-modules I_1 and I_0. We have that

$$\begin{aligned} \rho(\tilde{f}_1, f) &= (0,0,0,1,0,\ldots)X + (0,0,0,1,2,0,\ldots) \\ \rho(X\tilde{f}_1, f) &= (0,0,0,2,2,0,\ldots)X + (0,0,0,2,0,\ldots) \\ \rho(f_2, f) &= (2,0,2,0,1,0,\ldots)X + (2,0,0,\ldots) \\ \rho(Xf_2, f) &= (0,0,1,0,1,0,\ldots)X + (0,0,0,0,1,0,\ldots). \end{aligned}$$

Since $\pi_1(I_1) = \langle(1,0,1,1,1,0,0,\dots)\rangle$, we take

$$\begin{aligned} g_1 &= (2,1,2,1,1,1,\dots)\cdot\rho(f_2,f)+\rho(\tilde{f}_1,f)\\ &= (1,0,1,1,1,0,\dots)X+(1,0,0,1,2,0,\dots). \end{aligned}$$

Next, we calculate

$$\operatorname{Ann}((1,0,1,1,1,0,0,\dots)) = \langle(0,1,0,0,0,1,1,\dots)\rangle,$$

and see that $(0,1,0,0,0,1,1,\dots)\cdot g_1 = 0$. We also calculate

$$\begin{aligned} \rho(\rho(\tilde{f}_1,f),g_1) &= (0,0,0,0,2,0,\dots)\\ \rho(\rho(X\tilde{f}_1,f),g_1) &= (0,0,0,0,2,0,\dots)\\ \rho(\rho(f_2,f),g_1) &= (0,0,0,0,1,0,\dots)\\ \rho(\rho(Xf_2,f),g_1) &= (0,0,0,0,2,0,\dots). \end{aligned}$$

Now, we get that $I_0 = \pi(I_0) = \langle(0,0,0,0,1,0,\dots)\rangle$. So, $g_0 = (0,0,0,0,1,0,\dots)$. According to Theorem 10, the Gröbner basis for I is given by $\{f, g_1, g_0\}$. $\Diamond$

Acknowledgments. The authors would like to thank the Ministry of Education, Science and Technological Development of Republic of Serbia (Grant No: 451-03-9/2021-14/200104) for financial support. Also, we very much appreciate the valuable comments and suggestions of the anonymous referees.

REFERENCES

1. W. W. Adams, P. Loustaunau: *An Introduction to Gröbner Bases.* Graduate Studies in Mathematics, Vol. 3, AMS, 1994.
2. W. W. Adams, P. Loustaunau: *Gröbner bases and primary decomposition in polynomial rings in one variable over Dedekind domains.* J. Pure Appl. Algebra, **121** (1997), no. 1, 1–15
3. D. F. Anderson, A. Badawi: *Von Neumann regular and related elements in commutative rings.* Algebra Colloq., **19** (Spec 1) (2012), 1017–1040.
4. K. R. Goodearl: *Von Neumann regular rings.* Monographs and Studies in Mathematics, No. 4, Pitman, London, San Francisco, Melbourne, 1979.
5. T. Y. Lam: *A first course in noncommutative rings.* Springer-Verlag, New York, 1991.
6. D. Li, J. Liu, L. Zheng: *A zero-dimensional valuation ring is 1-Gröbner.* J. Algebra, **484** (2017), 334–343.
7. H. Lombardi, P. Schuster, I. Yengui: *The Gröbner Ring Conjecture in One Variable.* Math. Z., **270** (2012), no. 3–4, 1181–1185.
8. J. von Neumann: *Continuous geometry.* Princeton Mathematical Series, No. 25, Princeton University Press, Princeton, N.J., 1960.

9. R. Wisbauer: *Foundations of Module and Ring Theory.* Gordon and Breach Science Publishers, Reading, 1991.

Zoran Z. Petrović
Faculty of Mathematics,
University of Belgrade,
Studentski trg 16, Belgrade, Serbia,
E-mail: *zoranp@matf.bg.ac.rs*

(Received 19. 04. 2021.)
(Revised 24. 09. 2021.)

Maja Roslavcev
Faculty of Mathematics,
University of Belgrade,
Studentski trg 16, Belgrade, Serbia,
E-mail: *roslavcev@matf.bg.ac.rs*