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FIXED POINT EQUATIONS WITH ABSTRACT VOLTERRA OPERATORS ON SPACES OF FUNCTIONS OF SEVERAL VARIABLES

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Several notions of abstract Volterra operators on spaces of functions of one variable are well known. In this paper, we introduce various notions of abstract Volterra operators in spaces of functions of several variables. Some fixed point equations with such abstract Volterra operators are also studied. The basic ingredient in the theory of step by step contraction is the notion of G-contraction. The relevance of step by step contraction principle is illustrated by applications in the theory of Darboux-Ionescu problem.

1. INTRODUCTION

Several notions of abstract Volterra operators on spaces of functions of one variables are well-known (see [**4**], [**31**], [**25**], [**24**], [**3**], [**1**], [**13**], [**8**], [**12**], ...). For the study of Volterra operators in spaces of summable functions of several variables see [**28**] and the references therein. In this paper, we introduce various notions of abstract Volterra operators in spaces of functions of several variables. Some fixed point equations with such operators are also studied. The basic ingredient in the theory of step by step contraction is the notion of G-contraction ([**21**], [**27**]). The relevance of step by step contraction principle is illustrated by applications in the theory of Darboux-Ionescu problem ([**9**], [**23**], [**10**], [**2**], [**5**], [**6**], ...). Using a new variant of the fibre contraction principle (see [**16**]), the convergence of successive approximations is also studied.

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The outline of this paper is as follows:

2. PRELIMINARIES

In this section we recall some notions and results which are important for the development of our main results. For details and related concepts see [**20**], [**19**], [**27**] (for the theory of Picard and weakly Picard operators), [**21**], [**25**] (for the notion of G-contraction and main results involving this concept), [**29**], [**24**], [**25**], [**19**], [**27**], [**16**] (for Fibre Contraction Principle and its generalizations).

2.1 Weakly Picard operators

Let $(X, \to)$ be an L-space, where X is a nonempty set and $\to$ is a convergence structure, in the sense of Fréchet, defined on X. Usual examples of L-spaces are:
i) metric spaces (X, d), where $\to := \stackrel{d}{\to}$;
ii) topological spaces (X, τ), where $\to := \stackrel{\tau}{\to}$;
iii) normed spaces $(X, \|\cdot\|)$, where $\to := \stackrel{\|\cdot\|}{\to}$ or $\to := \rightharpoonup$;
and many others.

If $(X, \to)$ is an L-space and $f : X \to X$ is a given operator, then we denote by F_f the fixed point set of f, i.e., $F_f := \{x \in X : x = f(x)\}$. In the same framework $f : X \to X$ is said to be weakly Picard operator (briefly WPO) if, for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ of Picard iterates converges in $(X, \to)$ and the limit (which may depend on x) is a fixed point of f. If $f : X \to X$ is a WPO, then we can introduce the operator $f^\infty : X \to X$, by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$. It is obvious that $f^\infty(X) = F_f$, which shows that f^∞ is a set retraction of X on F_f.

If f is WPO with a unique fixed point (say $x^* \in X$), then f is called a Picard operator (briefly PO). In this case, $f^\infty(X) = \{x^*\}$.

2.2 G-contractions

Let (X, d) be a metric space and $G \subset X \times X$. An operator $f : X \to X$ is called a G-contraction with coefficient α if there exists $\alpha \in]0,1[$ such that

$$d(f(x), f(y)) \leq \alpha d(x,y), \text{ for every } (x,y) \in G.$$

For example, if $G = X \times X$ we obtain the classical notion of contraction with coefficient α (also called α-contraction), while when $G = Graph(f) := \{(x, f(x)) : x \in X\}$, then f is a graphic contraction with coefficient α. For other examples of G-contractions see [**21**] and [**27**]. For the multivalued case see [**26**].

2.3 Fibre contraction theorem

Let us recall first the following theorem, see [**17**], [**18**] and [**29**].

Theorem 1. *(***Fiber Contraction Principle.***) Let $(X_0, \to)$ be an L-space. For $m \geq 1$ we consider (X_i, d_i), $i \in \{1, \cdots, m\}$ be metric spaces. For $i \in \{0, 1, \cdots, m\}$ let us consider $f_i : X_0 \times X_1 \times \cdots \times X_i \to X_i$. We suppose that:*

(1) *for each $i \in \{1, 2, \cdots, m\}$ the metric spaces (X_i, d_i) are complete;*

(2) *f_0 is a WPO;*

(3) *for each $i \in \{1, 2, \cdots, m\}$ the operators $f_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i$ are α_i-contractions;*

(4) *for each $i \in \{1, 2, \cdots, m\}$ the operators f_i are continuous.*

Then, the operator $f = (f_0, f_1, \cdots, f_m) : \prod_{i=0}^{m} X_i \to \prod_{i=0}^{m} X_i$, defined by,

$$f(x_0, \ldots, x_m) := (f_0(x_0), f_1(x_0, x_1), \ldots, f_m(x_0, \ldots, x_m))$$

is a WPO. Moreover, when f_0 is a PO, then f is also a PO.

For our main results, we need an extension of the above theorem. Let us introduce first the framework of our result.

Let (X_i, d_i) (for $i \in \{1, ..., m\}$, where $m \geq 2$) be metric spaces and $U_1 \subset X_1 \times X_2$, $U_2 \subset U_1 \times X_3, \ldots, U_{m-1} \subset U_{m-2} \times X_m$, be nonempty subsets.

We also consider the following sets.

$$U_{1x} := \{x_2 \in X_2 \mid (x, x_2) \in U_1\}, \text{ for } x \in X_1,$$

$$U_{2x} := \{x_3 \in X_3 \mid (x, x_3) \in U_2\}, \text{ for } x \in U_1,$$

$$\vdots$$

$$U_{m-1x} := \{x_m \in X_m \mid (x, x_m) \in U_{m-1}\}, \text{ for } x \in U_{m-2}.$$

We suppose that $U_{1x}, x \in X_1; U_{2x}, x \in U_1; \ldots, U_{m-1x}, x \in U_{m-2}$ are nonempty.

If $T_1 : X_1 \to X_1$, $T_2 : U_1 \to X_2, \ldots, T_m : U_{m-1} \to X_m$, then we consider the operator

$$T : U_{m-1} \to X_1 \times X_2 \times \ldots \times X_m,$$

defined by

$$T(x_1, \ldots, x_m) := (T_1(x_1), T_2(x_1, x_2), \ldots, T_m(x_1, x_2, \ldots, x_m)).$$

The following variant of the Fibre Contraction Principle is proved in [**16**].

Theorem 2. *Let $m \in \mathbb{N}, m \geq 2$. We suppose that:*

(1) *the pairs (X_i, d_i) are complete metric spaces, for each $i \in \{2, ..., m\}$;*

(2) *for $i \in \{1, ..., m-1\}$, the sets U_i are closed;*

(3) $(T_1, T_2, \ldots, T_{i+1})(U_i) \subset U_i$, $i \in \{1, ..., m-1\}$;

(4) *T_1 is a WPO;*

(5) *there exist $L_i > 0$ and $0 < l_i < 1$, $i \in \{1, ..., m-1\}$ such that*

$$d_{i+1}(T_{i+1}(x, y,), T_{i+1}(\widetilde{x}, \widetilde{y})) \leq L_i \widetilde{d_i}(x, \widetilde{x}) + l_i d_{i+1}(y, \widetilde{y}),$$

for all $(x, y), (\widetilde{x}, \widetilde{y}) \in U_i$, $i \in \{1, ..., m-1\}$, where $\widetilde{d_i}$ is a metric induced by $d_1, \ldots, d_i$ on $X_1 \times \ldots \times X_i$, defined by $\widetilde{d_i} := \max\{d_1, \cdots, d_i\}$.

Then T is WPO. If T_1 is PO, then T is a PO too.

3. ABSTRACT VOLTERRA OPERATORS ON SPACES OF FUNCTIONS OF SEVERAL VARIABLES: NOTIONS AND EXAMPLES

Let X be an abstract set and, for $\lambda \in \Lambda$, let $R_\lambda \subset X \times X$ be a family of equivalence relations on X. An operator $V : X \to X$ is said to be an abstract Volterra operator with respect to (w.r.t) the family $\{R_\lambda\}_{\lambda \in \Lambda}$ if the following implications hold (see [**30**] and [**31**]):

$$\lambda \in \Lambda,\ (x, y) \in R_\lambda \Rightarrow (V(x), V(y)) \in R_\lambda.$$

Example 3. If $a, b \in \mathbb{R}^p$, $a_i < b_i$, for $i = \{1, \cdots, p\}$, we denote $[a, b] := \prod_{i=1}^{p} [a_i, b_i]$.

If Y is an arbitrary nonempty set, we consider the set

$$X := \mathbb{M}([a, b], Y) := \{u : [a, b] \to Y \mid u \text{ is a mapping}\},$$

$\Lambda := [a, b]$ and $R_\lambda := \{(u, v) \in X \times X \mid u\big|_{[a,\lambda]} = v\big|_{[a,\lambda]}\}$, for $\lambda \in [a, b]$.

By definition, an abstract Volterra operator w.r.t. the above family of equivalence relations $\{R_\lambda\}_{\lambda \in \Lambda}$,

$$V : \mathbb{M}([a, b], Y) \to \mathbb{M}([a, b], Y),$$

is a forward abstract Volterra operator.

For example, in the case $p = 2$, $Y = \mathbb{R}$, the operator

$$V : C([a_1, b_1] \times [a_2, b_2]) \to C([a_1, b_1] \times [a_2, b_2])$$

defined by

$$V(u)(x_1, x_2) := \int_{a_1}^{x_1} \int_{a_2}^{x_2} K(x_1, x_2, t_1, t_2, u(t_1, t_2))dt_1 dt_2,$$

where $K \in C(([a_1, b_1] \times [a_2, b_2])^2 \times \mathbb{R})$ is a forward Volterra operator.

Remark 4. In the notations of the above example, if we consider the operator V_1 defined by

$$V_1(u)(x_1, x_2) := \int_{b_1}^{x_1} \int_{b_2}^{x_2} K(x_1, x_2, t_1, t_2, v(t_1, t_2))dt_1 dt_2,$$

then V_1 is not a forward Volterra operator.

Example 5. Let us consider in Example 3,

$$\widetilde{R}_\lambda := \{(u, v) \in X \times X \mid u\big|_{[\lambda,b]} = v\big|_{[\lambda,b]}\},\ \lambda \in [a, b].$$

Then, by definition, $V : X \to X$ is called a backward abstract Volterra operator if it is an abstract Volterra operator w.r.t. the family equivalence relations $\{\widetilde{R}_\lambda\}_{\lambda\in[a,b]}$. For example, the operator V_1 from the above remark is a backward Volterra operator.

Example 6. Let us take, in Example 3, another family of equivalence relations. For $c \in \mathbb{R}^p$ such that $a_i < c_i < b_i$, for $i = \{1, \cdots, p\}$, we take $\Lambda := \{c\}$ and

$$R_c := \{(u, v) \in X \times X \mid u\big|_{[a,c]} = v\big|_{[a,c]}\}.$$

Then, by definition, an abstract Volterra operator w.r.t. the family of equivalent relations R_c is a forward Volterra operator w.r.t. the point c.

If we take $\Lambda := \{c\}$ and $\widetilde{R}_c := \{(u, v) \in X \times X \mid u\big|_{[c,b]} = v\big|_{[c,b]}\}$, then an abstract Volterra operator w.r.t. $\widetilde{R}_c$ is, by definition, a backward Volterra w.r.t. the point c.

Let us give now several such examples (see also [**15**]).

Let $[a, b] = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$. Let us consider $H \in C([a, b] \times [a, b] \times \mathbb{R})$ and $K \in C([a, b] \times [a, b] \times \mathbb{R})$. Then we define the operators $V, V_1 : X \to X$ by

$$V(u)(x_1, x_2) := \int_{a_1}^{c_1} \int_{a_2}^{c_2} H(x, t, u(t))dt_1 dt_2 + \int_{a_1}^{x_1} \int_{a_2}^{x_2} K(x, t, u(t))dt_1 dt_2$$

and

$$V_1(u)(x_1, x_2) = \int_{b_1}^{c_1} \int_{b_2}^{c_2} H(x, t, u(t))dt_1 dt_2 + \int_{b_1}^{x_1} \int_{b_2}^{x_2} K(x, t, u(t))dt_1 dt_2.$$

Example 7. For $f \in C([0,b_1] \times [0,b_2] \times \mathbb{R})$ we consider the classical Darboux problem

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = f(x_1, x_2, u(x_1, x_2)),\ x_1 \in [0,b_1],\ x_2 \in [0,b_2],$$

$$u(x_1, 0) = 0 \text{ for } x_1 \in [0,b_1] \text{ and } u(0, x_2) = 0 \text{ for } x_2 \in [0,b_2].$$

This problem is equivalent with the following integral equation

$$u(x_1,x_2) = \int_0^{x_1}\int_0^{x_2} f(s_1,s_2,u(s_1,s_2))ds_1 ds_2,\ (x_1,x_2) \in [0,b_1] \times [0,b_2].$$

Let $V : C([0,b_1] \times [0,b_2]) \to C([0,b_1] \times [0,b_2])$ be defined by

$$V(u)(x_1,x_2) := \int_0^{x_1}\int_0^{x_2} f(s_1,s_2,u(s_1,s_2))ds_1 ds_2.$$

It is clear that V is a forward Volterra operator.

Example 8. Now we consider the following Darboux problem:

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = f(x_1, x_2, u(x_1, x_2)),\ (x_1, x_2) \in [0,b_1] \times [0,b_2],$$

$$u(x_1, b_2) = 0,\ u(b_1, x_2) = 0,\ (x_1,x_2) \in [0,b_1] \times [0,b_2].$$

This problem is equivalent with the following integral equation

$$u(x_1,x_2) = \int_{b_1}^{x_1}\int_{b_2}^{x_2} f(s_1,s_2,u(s_1,s_2))ds_1 ds_2,\ (x_1,x_2) \in [0,b_1] \times [0,b_2].$$

In this case the operator $V_1 : C([0,b_1] \times [0,b_2]) \to C([0,b_1] \times [0,b_2])$ defined by the second part of this integral equation, i.e.,

$$V_1(u)(x_1,x_2) := \int_{b_1}^{x_1}\int_{b_2}^{x_2} f(s_1,s_2,u(s_1,s_2))ds_1 ds_2$$

is a backward Volterra operator.

For some examples related to the case $p = 1$ see e.g. [**8**].

4. FIXED POINT EQUATIONS WITH ABSTRACT VOLTERRA OPERATORS ON SPACES OF FUNCTIONS OF SEVERAL VARIABLES: STEP BY STEP CONTRACTIONS

Let $a, b \in \mathbb{R}^p$ and $[a,b] := \prod_{i=1}^{p}[a_i, b_i]$, where $a_i < b_i$, for $i = \{1, \cdots, p\}$. For $m \in \mathbb{N}$, $m \geq 2$ we denote

$$t^0 := a, t^k := t^0 + \frac{k}{m}(b-a),\ \text{for } k \in \{1, \cdots, m\}.$$

Let $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ be a forward abstract Volterra operator, where $\mathbb{B}$ is a Banach space. Then, V induces the following operators:

$$V_1 : C([a,t^1],\mathbb{B}) \to C([a,t^1],\mathbb{B}),\ V_1(u)(x) := V(\tilde{u})(x), x \in [a,t^1],$$

where $\tilde{u} \in C([a,b],\mathbb{B})$ is such that $\tilde{u}_{|_{[a,t^1]}} = u$;

$\vdots$

$$V_k : C([a,t^k],\mathbb{B}) \to C([a,t^k],\mathbb{B}),\ V_k(u)(x) := V(\tilde{u})(x), x \in [a,t^k],$$

where $\tilde{u} \in C([a,b],\mathbb{B})$ is such that $\tilde{u}_{|_{[a,t^k]}} = u$;

$\vdots$

$$V_{m-1} : C([a,t^{m-1}],\mathbb{B}) \to C([a,t^{m-1}],\mathbb{B}),\ V_{m-1}(u)(x) := V(\tilde{u})(x), x \in [a,t^{m-1}],$$

where $\tilde{u} \in C([a,b],\mathbb{B})$ is such that $\tilde{u}_{|_{[a,t^{m-1}]}} = u$;

We also need the following notations:

$$G_k := \{(u,v)\ : u,v \in C([a,t^{k+1}],\mathbb{B}), u_{|_{[a,t^k]}} = v_{|_{[a,t^k]}}\}$$

and, for $u_k \in C([a,t^k],\mathbb{B})$, with $k \in \{1,2,\cdots,m-1\}$

$$X_{u_k} := \{u \in C([a,t^{k+1}],\mathbb{B})\ : u_{|_{[a,t^k]}} = u_k\}.$$

In the above mentioned context and notations, we establish our first main result.

Theorem 9. (Principle of forward step by step contraction) *We suppose:*

(1) V is a forward Volterra operator;

(2) V_1 is a contraction;

(3) V_k is a G_{k-1}-contraction, for each $k \in \{2,3,\cdots,m\}$.

Then, the following conclusions hold:

(i) $F_V = \{x^*\}$;

(ii) $x^*_{|_{[a,t^1]}} = V_1^\infty(x)$, for every $x \in C([a,t^1],\mathbb{B})$;

$x^*_{|_{[a,t^2]}} = V_2^\infty(x)$, for every $x \in X_{x^*_{|_{[a,t^1]}}}$;

$\vdots$

$x^*_{|_{[a,t^{m-1}]}} = V_{m-1}^\infty(x)$, for every $x \in X_{x^*_{|_{[a,t^{m-2}]}}}$;

(iii) $x^* = V^\infty(x)$, for every $x \in X_{x^*_{|_{[a,t^{m-1}]}}}$.

Proof. In the proof all the spaces are endowed with the maximum norm. Since V_1 is a contraction, we get that V_1 is a Picard operator and $F_{V_1} = \{x_1^*\}$, with

$x_1^* \in C([a,t^1],\mathbb{B})$. Since V_2 is a G_1-contraction, we get that $V_2|_{X_{x_1^*}} : X_{x_1^*} \to X_{x_1^*}$ is a contraction. Thus, $F_{V_2} = \{x_2^*\}$, with $x_2^*|_{[a,t^1]} = x_1^*$. Moreover, $V_2|_{X_{x_1^*}} : X_{x_1^*} \to X_{x_1^*}$ is a Picard operator. By this approach, step by step, we get the conclusion of the theorem. □

Let $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ be a backward Volterra operator and $m \in \mathbb{N}$, with $m \geq 2$. We denote

$$t_0 := b, t_1 := a + \frac{m-1}{m}(b-a), \cdots, t_k := a + \frac{m-k}{m}(b-a),\ 1 \leq k \leq m.$$

We also denote $V_k : C([t_k,b],\mathbb{B}) \to C([t_k,b],\mathbb{B})$ an operator induced by V and defined by $V_k(x) = V(\tilde{x_k})$, where $\tilde{x_k} \in C([a,b],\mathbb{B})$ is such that $\tilde{x}_k|_{[t_k,b]} = x_k$, for every $k \in \{1, \cdots, m-1\}$.

We consider the following sets

$$G_k := \{(x,y) : x, y \in C([t_{k+1},b],\mathbb{B}), x|_{[t_k,b]} = y|_{[t_k,b]}\},$$

and, for $x_k \in C([t_k,b],\mathbb{B})$, the sets

$$X_{x_k} := \{y \in C([t_{k+1},b],\mathbb{B}) : y|_{[t_k,b]} = x_k\},\ \text{for } k \in \{1, \cdots, m-1\}.$$

As in the case of forward Volterra operators, we have the following result.

Theorem 10. (Principle of backward step by step contraction) *We suppose that:*

(1) V is a backward Volterra operator;
(2) V_1 is a contraction;
(3) V_k is a G_{k-1}-contraction, for each $k \in \{2, \cdots, m-1\}$;
(4) V is a G_{m-1}-contraction.

Then, we have the following conclusions:

(i) $F_V = \{x^*\}$;
(ii) $x^*|_{[t_1,b]} = V_1^\infty(x)$, for every $x \in C([t_1,b],\mathbb{B})$
$x^*|_{[t_2,b]} = V_2^\infty(x)$, for every $x \in X_{x^*|_{[t_1,b]}}$
$\vdots$
$x^*|_{[t_{m-1},b]} = V_{m-1}^\infty(x)$, for every $x \in X_{x^*|_{[t_{m-2},b]}}$
(iii) $x^* = V^\infty(x)$, for every $x \in X_{x^*|_{[t_{m-1},b]}}$.

By the above two results (Theorem 9 and Theorem 10), the following problem rises:

Open Problem. Let V be as in Theorem 9 or as in Theorem 10. Under which additional conditions, V is a Picard operator ?

5. AN APPLICATION TO DARBOUX-IONESCU PROBLEM

In this section, we consider the following Darboux-Ionescu problem:

$$(1)\quad \begin{cases} \dfrac{\partial^2 u}{\partial x_1 \partial x_2} = f(x_1, x_2, A(u)(x_1, x_2)),\ (x_1, x_2) \in I := [0, b_1] \times [0, b_2], \\ u(0, x_2) = 0,\ u(x_1, 0) = 0,\ (x_1, x_2) \in I, \end{cases}$$

where $b_1, b_2 \in \mathbb{R}_+^*$, $\mathbb{B}$ is a Banach space, $f \in C(I \times \mathbb{B}, \mathbb{B})$ and $A : C(I, \mathbb{B}) \to C(I, \mathbb{B})$ is a forward Volterra operator. For related notions and results, see [**9**], [**1**], [**5**], [**2**], [**6**], [**10**], [**11**], ...

We are searching for classical solutions of this problem, i.e., $u \in C^1(I, \mathbb{B})$ with $\dfrac{\partial^2 u}{\partial x_1 \partial x_2} \in C(I, \mathbb{B})$.

This problem is equivalent with the following integral equation

$$u(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(s_1, s_2, A(u)(s_1, s_2)) ds_1 ds_2,\ (x_1, x_2) \in I.$$

If denote

$$V(u)(x_1, x_2) := \int_0^{x_1} \int_0^{x_2} f(s_1, s_2, A(u)(s_1, s_2)) ds_1 ds_2,\ (x_1, x_2) \in I,$$

then we notice that $V : C(I, \mathbb{B}) \to C(I, \mathbb{B})$ is a forward Volterra operator.

We impose the following assumptions for the data of this problem:

(i) $f(x_1, x_2, \cdot)$ is L_f-Lipschitz, i.e., there exists $L_f > 0$ such that

$$\|f(x_1, x_2, \eta_1) - f(x_1, x_2, \eta_2)\| \le L_f \|\eta_1 - \eta_2\|,\ \text{for every } (x_1, x_2) \in I \text{ and } \eta_1, \eta_2 \in \mathbb{B};$$

(ii) there exists $L_A > 0$ such that

$$\|A(u)(x_1, x_2) - A(v)(x_1, x_2)\| \le L_A \max\{\|u(s_1, s_2) - v(s_1, s_2)\| : s_1 \in [0, x_1], s_2 \in [0, x_2]\},$$

for $0 \le x_1 \le b_1$ and $0 \le x_2 \le b_2$.

Some examples of operator A are given below.

Example 11. 1) $A(u)(x_1, x_2) := u(x_1, x_2)$;
2) $A(u)(x_1, x_2) := u(g_1(x_1, x_2), g_2(x_1, x_2))$, where $g_1 \in C(I, [0, b_1])$, $g_2 \in C(I, [0, b_2])$, with $g_1(x_1, x_2) \le x_1, g_2(x_1, x_2) \le x_2$, for each $(x_1, x_2) \in I$;
3) $A(u)(x_1, x_2) := \max\{\|u(\tau_1, \tau_2)\| \ : 0 \le \tau_1 \le x_1, 0 \le \tau_2 \le x_2\}$;
4) $A(u)(x_1, x_2) := \int_0^{x_1} \int_0^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2)) ds_1 ds_2$, where $H \in C(I^2 \times \mathbb{B}, \mathbb{B})$ has the property that $H(x, s, \cdot) : \mathbb{B} \to \mathbb{B}$ is L_H-Lipschitz, i.e., there exists $L_H > 0$ such that

$$\|H(x, s, \eta_1) - H(x, s, \eta_2)\| \le L_H \|\eta_1 - \eta_2\|,\ \text{for every } (x, s) \in I \text{ and } \eta_1, \eta_2 \in \mathbb{B}.$$

Under the notations introduced in Section 4, we have the following result for the problem (1).

Theorem 12. *Assume that the conditions (i) and (ii) hold. Let* $m \in \mathbb{N}$, $m \geq 2$ *such that* $\frac{2L_f L_A b_1 b_2}{m} < 1$. *Then, the following conclusions hold:*

(i) the problem (1) has a unique solution u^*, *i.e.,* $F_V = \{u^*\}$;

(ii) $u^*_{|_{[0,t_1]}} = V_1^\infty(u)$, *for every* $u \in C([0,t_1],\mathbb{B})$;

$u^*_{|_{[0,t_2]}} = V_2^\infty(u)$, *for every* $u \in X_{u^*_{|_{[0,t_1]}}}$;

$\vdots$

$u^*_{|_{[0,t_{m-1}]}} = V_{m-1}^\infty(u)$, *for every* $u \in X_{u^*_{|_{[0,t_{m-2}]}}}$;

(iii) $u^* = V^\infty(u)$, *for every* $u \in X_{u^*_{|_{[0,t_{m-1}]}}}$.

Proof. We notice that the operator V and the operators $V_1, V_2, \cdots, V_{m-1}$ induced by V satisfy the assumptions from Theorem 9. □

We also consider, in the final part of this section, the following variant of Darboux-Ionescu problem:

$$(2) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial x_1 \partial x_2} = f(x_1, x_2, A(u)(x_1, x_2),\ (x_1, x_2) \in I := [0, b_1] \times [0, b_2], \\ u(b_1, x_2) = 0,\ u(x_1, b_2) = 0,\ (x_1, x_2) \in I. \end{cases}$$

Then, the above problem is equivalent with the integral equation

$$u(x_1, x_2) = \int_{b_1}^{x_1} \int_{b_2}^{x_2} f(s_1, s_2, A(u)(s_1, s_2)) ds_1 ds_2,\ (x_1, x_2) \in I.$$

If denote

$$V(u)(x_1, x_2) := \int_{b_1}^{x_1} \int_{b_2}^{x_2} f(s_1, s_2, A(u)(s_1, s_2)) ds_1 ds_2,\ (x_1, x_2) \in I,$$

then $V : C(I,\mathbb{B}) \to C(I,\mathbb{B})$ is a backward Volterra operator.

Remark 13. By Theorem 10 we get a similar result for the problem (2).

Remark 14. In the case of the operator V from Theorem 12 the open problem from Section 4 has a positive answer, i.e., in the conditions of Theorem 12, the operator V is Picard. This fact follows from Theorem 2. In order to prove this assertion, we need some notations and definitions.

$$I_k := [0, t^k] := [0, \frac{k}{m} b] := [0, \frac{kb_1}{m}] \times [0, \frac{kb_2}{m}],\ \text{for } k \in \{1, \cdots, m\}, m \in \mathbb{N}, m \geq 2;$$

$$D_1 := I_1, \cdots D_k := \overline{I_k \setminus I_{k-1}}, k \in \{2, \cdots, m\},\ \text{for } m \in \mathbb{N}, m \geq 2;$$

$$\Gamma_k := \left(\{(t_1^k, 0)\} \times [0, t_2^k]\right) \cup \left([0, t_1^k] \times \{(0, t_2^k)\}\right),\ \text{for } k \in \{1, \cdots, m-1\}, m \in \mathbb{N}, m \geq 2;$$

$$X_k := C\left(D_k, \mathbb{B}\right), \text{ with max-norm, for } k \in \{1, \cdots, m\}, m \in \mathbb{N}, m \geq 2;$$

$$U_1 := \{(u_1, u_2) \in X_1 \times X_2 \mid u_{1|_{\Gamma_1}} = u_{2|_{\Gamma_1}}\},$$

$$\vdots$$

$$U_{m-1} := \{(u_1, \cdots u_m) \in U_{m-1} \times X_m \mid u_{m-1|_{\Gamma_{m-1}}} = u_{m|_{\Gamma_{m-1}}}\}, m \in \mathbb{N}, m \geq 2.$$

We also introduce, for $k \in \{1, \cdots, m-1\}, m \in \mathbb{N}, m \geq 2$, the following operators:

$$R_k : C\left(I_{k+1}, \mathbb{B}\right) \to X_1 \times \cdots \times X_{k+1},\ u \longmapsto \left(u_{|_{D_1}}, \cdots, u_{|_{D_{k+1}}}\right).$$

We remark that $R_k : C\left(I_{k+1}, \mathbb{B}\right) \to U_k$ is an isometry, for each $k \in \{1, \cdots, m-1\}$ and $m \in \mathbb{N}, m \geq 2$.

Now we consider the following operators defined in terms of operators V_k, induced by the operator V:

$$T_1 : X_1 \to X_1,\ T_1(u_1)(x) := V_1(u_1)(x), x \in D_1,$$

$$T_2 : U_1 \to X_2,\ T_2(u_1, u_2)(x) := V_2(R_1^{-1}(u_1, u_2))(x), x \in D_2,$$

$$\vdots$$

$$T_m : U_{m-1} \to X_2,\ T_m(u_1, \cdots, u_m)(x) := V_m\left(R_{m-1}^{-1}(u_1, \cdots, u_m)\right)(x), x \in D_m$$

and the triangular operator induced by $T_1, \cdots, T_m$

$$T : U_{m-1} \to U_{m-1},\ T(u_1, \cdots, u_m) := \left(T_1(u_1), T_2(u_1, u_2), \cdots, T_m(u_1, \cdots, u_m)\right).$$

We notice that T satisfies the conditions in Theorem 2. Thus, by Theorem 2, T is a Picard operator. On the other hand, we observe that

$$V = R_{m-1}^{-1} T R_{m-1} \text{ and } V^n = R_{m-1}^{-1} T^n R_{m-1}.$$

Hence, V is a Picard operator.

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