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ASYMPTOTIC AND NUMERICAL ASPECTS OF THE GENERALIZED MARCUM FUNCTION OF THE SECOND KIND

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The asymptotic and numerical behavior of the so-called generalized Marcum function of the second kind is considered. By using some known expansions on the modified Bessel function of the second kind, we deduce the asymptotic expansions for the generalized Marcum function of the second kind for large parameters, and all the expansions are obtained when exactly one parameter is large and others are fixed. We also show numerically that the asymptotic formulas obtained in this paper are good approximations.

1. INTRODUCTION

Recently the probability density function

$$f_m(t) = C|t|^m e^{-pt^2} K_m\left(\frac{|t|}{b}\right),$$

based on the modified Bessel function of the second kind appeared during the study of Bayesian inference of an inverse Gaussian sample [**4**]. Here,

$$\frac{1}{C} = \frac{\Gamma(m+1)}{b^m 2^{m+2} p^{m+1}} e^{\frac{1}{4pb^2}} \Gamma\left(-m, \frac{1}{4pb^2}\right)$$

with $b, p > 0$, $m > 1$, $t \in \mathbb{R}$, and $\Gamma(\nu, z)$ denotes the upper incomplete gamma function, defined by [**5**, p. 174, 8.2.2]

$$\Gamma(\nu, z) = \int_z^\infty e^{-t} t^{\nu-1} dt. \tag{1}$$

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As mentioned in [**1**], for the choice $b = \frac{1}{a}$, $p = \frac{1}{2}$ and replacing m by $m-1$, the above probability density function can be written as

$$f_{m-1}(t) = \frac{2}{\Gamma(m)\Gamma\left(1-m, \frac{a^2}{2}\right)} \frac{1}{a^{m-1}} |t|^{m-1} e^{-\frac{t^2+a^2}{2}} K_{m-1}(a|t|).$$

The so-called generalized Marcum function of the second kind for $x > 0$, $\nu > 0$ and $y \geq 0$ is defined by

$$R_\nu(x,y) = c_{x,\nu} \cdot x^{1-\nu} \int_y^\infty t^\nu e^{-\frac{t^2+x^2}{2}} K_{\nu-1}(xt)dt, \tag{2}$$

where

$$c_{x,\nu} = \frac{2}{\Gamma(\nu)\Gamma\left(1-\nu, \frac{x^2}{2}\right)}$$

and this is the survival function of the truncated version of the above distribution for $p = \frac{1}{2}$.

In [**1**], Baricz et al. obtained various properties of the generalized Marcum function of the second kind like monotonicity properties, convexity properties, log-convexity properties with respect to the parameters x, y and ν. Similarly, they also obtained tight lower and upper bounds for the generalized Marcum function of the second kind in terms of elementary special functions in [**1**] as well as in [**2**]. However, the numerical evaluation of the generalized Marcum function of the second kind is one of the important aspects from the application point of view.

In this paper, asymptotic expansions for the generalized Marcum function of the second kind are obtained for large parameters x, y and ν. Based on the asymptotic expansions, $R_\nu(x,y)$ is calculated numerically. The obtained numerical values of $R_\nu(x,y)$ are compared with values of $R_\nu(x,y)$ calculated using Mathematica 8.0. From the comparison tables, we can conclude that obtained asymptotic expansions are efficient ones.

It is interesting to note the following notation of generalized Marcum function of the second kind

$$\tilde{R}_\nu(x,y) = \tilde{c}_{x,\nu} \cdot x^{\frac{1-\nu}{2}} \int_y^\infty t^{\frac{\nu-1}{2}} e^{-t-x} K_{\nu-1}(2\sqrt{xt})dt, \tag{3}$$

where

$$\tilde{c}_{x,\nu} = \frac{2}{\Gamma(\nu)\Gamma(1-\nu, x)}.$$

This can be obtained by replacing x and y by $\sqrt{2x}$ and $\sqrt{2y}$ in (2), respectively. Also note that the complementary function

$$\tilde{S}_\nu(x,y) = \tilde{c}_{x,\nu} \cdot x^{\frac{1-\nu}{2}} \int_0^y t^{\frac{\nu-1}{2}} e^{-t-x} K_{\nu-1}(2\sqrt{xt})dt$$

satisfies the relation

$$\tilde{R}_\nu(x,y) + \tilde{S}_\nu(x,y) = 1.$$

This paper deals with asymptotic expansions for $R_\nu(x,y)$ when x, y and ν are large. To obtain these expansions, we use particularly the technique called integration by parts and Laplace's method. The following theorem [**6**, Theorem 1, p. 58] in which we deal with Laplace type integrals is an important tool to obtain the asymptotic expansions.

Theorem 1. *For the integral*

$$I(\lambda) = \int_a^b \phi(x) e^{-\lambda h(x)} dx,$$

we assume that the integral $I(\lambda)$ converges absolutely for all sufficiently large λ; $h(x) > h(a)$ for all $x \in (a,b)$, and for every $\delta > 0$ the infimum of $h(x) - h(a)$ in $[a+\delta, b)$ is positive; $h'(x)$ and $\phi(x)$ are continuous in a neighborhood of $x = a$, except possibly at a; the expansions

$$h(x) \sim h(a) + \sum_{s=0}^{\infty} a_s (x-a)^{s+\mu} \tag{4}$$

and

$$\phi(x) \sim \sum_{s=0}^{\infty} b_s (x-a)^{s+\alpha-1} \tag{5}$$

hold when $x \to a^+$, and the expansion (4) *can be term-wise differentiable. Then*

$$I(\lambda) \sim e^{-\lambda h(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\alpha}{\mu}\right) \frac{d_s}{\lambda^{\frac{s+\alpha}{\mu}}} \quad \textit{as } \lambda \to +\infty,$$

where the coefficients d_s are expressible in terms of a_s and b_s. The first three coefficients are explicitly given by

$$d_0 = \frac{b_0}{\mu a_0^{\frac{\alpha}{\mu}}}, \quad d_1 = \left(\frac{b_1}{\mu} - \frac{(\alpha+1)a_1 b_0}{\mu^2 a_0}\right) \frac{1}{a_0^{\frac{\alpha+1}{\mu}}}$$

and

$$d_2 = \left[\frac{b_2}{\mu} - \frac{(\alpha+2)a_1 b_1}{\mu^2 a_0} + ((\alpha+\mu+2)a_1^2 - 2\mu a_0 a_2)\frac{(\alpha+2)b_0}{2\mu^3 a_0^2}\right] \frac{1}{a_0^{\frac{\alpha+2}{\mu}}}.$$

The paper is organized as follows: in Section 2 we provide some recurrence relations as well as asymptotic expansions for the generalized Marcum function of the second kind. Section 3 contains the proofs of the theorems which are given in Section 2, while in Section 4 we discuss how good are these expansions to calculate the value of the function $R_\nu(a,b)$ for large parameters. To support the argument, tables and figures are presented in Section 4.

2. RECURRENCE RELATIONS AND ASYMPTOTIC EXPANSIONS

This section is devoted to the study of some recurrence relations for the generalized Marcum function of the second kind as well as some asymptotic expansions which help to study the behaviour of the generalized Marcum function of the second kind when exactly one parameter is large and others are fixed.

2.1 Recurrence relation

The properties of the generalized Marcum function of the second kind are discussed in details in [**1**]. In this subsection we derive some interesting recurrence relations. Using integration by parts and the asymptotic formula [**5**, p. 249, 10.25.3]

$$K_\nu(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}} \text{ as } x \to \infty \tag{6}$$

in equation (3), we arrive at

$$\frac{\tilde{R}_\nu(x,y)}{\tilde{c}_{x,\nu}} + \frac{\tilde{R}_{\nu-1}(x,y)}{\tilde{c}_{x,\nu-1}} = \left(\frac{y}{x}\right)^{\frac{\nu-1}{2}} K_{\nu-1}(2\sqrt{xy})e^{-x-y}. \tag{7}$$

The above recurrence relation and the complementary relation implies that

$$\frac{\tilde{S}_\nu(x,y)}{\tilde{c}_{x,\nu}} + \frac{\tilde{S}_{\nu-1}(x,y)}{\tilde{c}_{x,\nu-1}} = \frac{\Gamma(\nu-1)}{2}e^{-x}x^{1-\nu} - \left(\frac{y}{x}\right)^{\frac{\nu-1}{2}} K_{\nu-1}(2\sqrt{xy})e^{-x-y}.$$

Replacing ν by $\nu+1$ in (7), we see that

$$\frac{\tilde{R}_{\nu+1}(x,y)}{\tilde{c}_{x,\nu+1}} + \frac{\tilde{R}_\nu(x,y)}{\tilde{c}_{x,\nu}} = \left(\frac{y}{x}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{xy})e^{-x-y}. \tag{8}$$

From equations (7) and (8), we find that

$$\frac{\tilde{R}_{\nu+1}(x,y)}{\tilde{c}_{x,\nu+1}} + (1-a_\nu(x,y))\frac{\tilde{R}_\nu(x,y)}{\tilde{c}_{x,\nu}} - a_\nu(x,y)\frac{\tilde{R}_{\nu-1}(x,y)}{\tilde{c}_{x,\nu-1}} = 0,$$

where

$$a_\nu(x,y) = \left(\frac{y}{x}\right)^{\frac{1}{2}} \frac{K_\nu(2\sqrt{xy})}{K_{\nu-1}(2\sqrt{xy})}.$$

Similarly, we arrive at

$$\begin{aligned}\frac{\tilde{S}_{\nu+1}(x,y)}{\tilde{c}_{x,\nu+1}} &+ (1-a_\nu(x,y))\frac{\tilde{S}_\nu(x,y)}{\tilde{c}_{x,\nu}} - a_\nu(x,y)\frac{\tilde{S}_{\nu-1}(x,y)}{\tilde{c}_{x,\nu-1}} \\ &= \frac{\Gamma(\nu)}{2}e^{-x}x^{-\nu} - \frac{\Gamma(\nu-1)}{2}e^{-x}x^{1-\nu}a_\nu(x,y).\end{aligned}$$

From [**3**], we can see that the generalized Marcum Q-function $Q_\nu(x,y)$ and the corresponding cumulative distribution function $P_\nu(x,y)$ satisfy the same third order recurrence relation. Whereas the generalized Marcum function of the second kind $\tilde{R}_\nu(x,y)$ and the corresponding cumulative distribution function $\tilde{S}_\nu(x,y)$ satisfy different third order recurrence relations. Consequently, the evaluation of $\tilde{R}_\nu(x,y)$ and $\tilde{S}_\nu(x,y)$ as the maximal and the minimal solution of a third order recurrence relation may not work here.

Remark 1. Using

$$K_{\nu+1}(x) = K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x),$$

one can eliminate the terms involving the modified Bessel function of the second kind and obtain the following four term recurrence relation

$$x\frac{R_{\nu+2}(x,y)}{c_{x,\nu+2}} = (\nu - x)\frac{R_{\nu+1}(x,y)}{c_{x,\nu+1}} + (\nu + y)\frac{R_\nu(x,y)}{c_{x,\nu}} + y\frac{R_{\nu-1}(x,y)}{c_{x,\nu-1}}.$$

The above equation can be written as

$$\frac{x\frac{R_{\nu+2}(x,y)}{c_{x,\nu+2}} - (\nu - x)\frac{R_{\nu+1}(x,y)}{c_{x,\nu+1}}}{(\nu + y)\frac{R_\nu(x,y)}{c_{x,\nu}} + y\frac{R_{\nu-1}(x,y)}{c_{x,\nu-1}}} = 1 \text{ when } x \geq \nu, \tag{9}$$

and

$$\frac{x\frac{R_{\nu+2}(x,y)}{c_{x,\nu+2}}}{(\nu - x)\frac{R_{\nu+1}(x,y)}{c_{x,\nu+1}} + (\nu + y)\frac{R_\nu(x,y)}{c_{x,\nu}} + y\frac{R_{\nu-1}(x,y)}{c_{x,\nu-1}}} = 1 \text{ when } x \leq \nu. \tag{10}$$

Equations (9) and (10) are used for testing the accuracy of the numerical results obtained by the proposed asymptotic methods to evaluate the generalized Marcum function of the second kind.

2.2 Asymptotic expansions

In this subsection, we present the asymptotic expansions for the generalized Marcum function of the second kind when exactly one parameter is large and others are fixed. The following theorem will provide an asymptotic expansion when y is large and others are fixed.

Theorem 2. *Let y be large and other parameters be fixed. Then the asymptotic expansion for the generalized Marcum function of the second kind is given by*

$$R_\nu(x,y) \sim c_{x,\nu} \cdot e^{-\frac{x^2+y^2}{2}} \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{x}{y}\right)^{n-\nu} K_{\nu-n}(xy). \tag{11}$$

Remark 2. **(i)** The "o-notation" has its usual meaning i.e., $f(x) = o(g(x))$ near $x = x_0$ means $\lim_{x\to x_0}(f(x)/g(x)) = 0$.

(ii) In particular, when $\nu = 1$, the asymptotic expansion (11) leads to the next asymptotic expansion for the Marcum function of the second kind $R_1(x,y)$ for large y

$$R_1(x,y) \sim c_{x,1} \cdot e^{-\frac{x^2+y^2}{2}} \sum_{i=0}^{\infty} \left(-\frac{x}{y}\right)^i K_i(xy).$$

(iii) The next asymptotic formula

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2xy}}\, e^{-\frac{(x+y)^2}{2}} \left(\frac{y}{x}\right)^{\nu-1} = {}_1A_\nu(x,y) \tag{12}$$

is also valid for large y.

(iv) It is worth mentioning that in [**1**] Baricz et al. deduced the following asymptotic expansion

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} \int_{x+y}^{\infty} e^{-\frac{t^2}{2}}\, dt \tag{13}$$

for large y.

(v) In general, the estimates obtained by using the asymptotic formulas (12) and (13) are not comparable but under certain conditions these formulas coincide. To see this, observe that in view of the complementary error function [**5**, p. 160, 7.2.2]

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\, dt,$$

equation (13) can be rewritten as

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} \sqrt{\frac{\pi}{2}}\, \operatorname{erfc}\left(\frac{x+y}{\sqrt{2}}\right),$$

and by using the asymptotic formula [**5**, 7.12.1]

$$\operatorname{erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi} z} \text{ as } z \to \infty, \tag{14}$$

we readily see that

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} \frac{e^{-\frac{(x+y)^2}{2}}}{(x+y)}.$$

If $x + y \sim y$, then this asymptotic result reduces to

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} \frac{e^{-\frac{(x+y)^2}{2}}}{y} = c_{x,\nu} \cdot \sqrt{\frac{\pi}{2xy}} \left(\frac{y}{x}\right)^{\nu-1} e^{-\frac{(x+y)^2}{2}},$$

which is same as the asymptotic expansion (12).

The next two theorems discuss the asymptotic expansions when the parameters x and ν are large.

Theorem 3. *If $\nu \neq n - \frac{1}{2}, n \in \mathbb{N}$, then the following asymptotic expansion is valid when x is large and others parameters are fixed*

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{a_k(\nu-1)}{x^k} f_k(x,y;\nu), \tag{15}$$

where

$$f_k(x,y;\nu) = \int_y^\infty t^{\nu-\frac{1}{2}-k} e^{-\frac{t^2}{2}} e^{-xt} dt, \tag{16}$$

$a_0(\nu) = 1$ *and for $k \in \mathbb{N}$ the coefficients $a_k(\nu)$ are defined by*

$$a_k(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\dots(4\nu^2-(2k-1)^2)}{k!8^k}. \tag{17}$$

Remark 3. **(i)** By expanding $e^{-\frac{t^2}{2}}$ in powers of t in equation (16), we find that

$$\begin{aligned} f_k(x,y;\nu) &= \int_y^\infty t^{\nu-k-\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{t^2}{2}\right)^n \right) e^{-xt} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \int_y^\infty t^{2n+\nu-k-\frac{1}{2}} e^{-xt} dt. \end{aligned}$$

In view of (1), the above equation reduces to

$$f_k(x,y;\nu) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{\Gamma\left(2n+\nu-k+\frac{1}{2}, xy\right)}{x^{2n+\nu-k+\frac{1}{2}}}, \tag{18}$$

which is useful to calculate the first few values of f_k. Using the integration by parts in (16), we can show that the term $f_k(x,y;\nu)$ satisfies the following recurrence relation

$$\left(\nu - k + \frac{1}{2}\right) f_k(x,y;\nu) = -\frac{y^{\nu-k+\frac{1}{2}}}{e^{\frac{y^2}{2}+xy}} + f_{k-2}(x,y;\nu) + x f_{k-1}(x,y;\nu), \tag{19}$$

which can be used to calculate the other values of f_k.

(ii) Consider the first term of the asymptotic expansion (15), that is,

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} f_0(x,y;\nu). \tag{20}$$

Clearly, from equation (18), we have

$$f_0(x,y;\nu) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{\Gamma\left(2n+\nu+\frac{1}{2}, xy\right)}{x^{2n+\nu+\frac{1}{2}}}. \tag{21}$$

Now, in view of [**5**, 8.11.2] $\Gamma(a,z) \sim z^{a-1}e^{-z}$ as $z \to \infty$, equation (21) becomes

$$\begin{aligned} f_0(x,y;\nu) &\sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \frac{(xy)^{2n+\nu-\frac{1}{2}}e^{-xy}}{x^{2n+\nu+\frac{1}{2}}} \\ &= \frac{e^{-xy}}{x}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} y^{2n}\right) y^{\nu-\frac{1}{2}} \\ &= \frac{y^{\nu-\frac{1}{2}}}{x} e^{-\frac{y^2}{2}-xy}. \end{aligned}$$

Using this in (20), we obtain the following asymptotic formula

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{(x+y)^2}{2}}}{x} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} = {}_2A_\nu(x,y). \tag{22}$$

(iii) In view of equation (16), the asymptotic expansion (20) becomes

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} \int_y^\infty t^{\nu-\frac{1}{2}} e^{-\frac{t^2}{2}} e^{-xt} dt. \tag{23}$$

If we compare the integral that appears in (23) with the integral given in Theorem 1, we obtain

$$h(t) = t \text{ and } \phi(t) = t^{\nu-\frac{1}{2}} e^{-\frac{t^2}{2}}.$$

Clearly, these functions satisfy all the conditions of Theorem 1 and have the following expansion around $t = y$

$$h(t) = y + (t-y)$$

and

$$\phi(t) = \phi(y) + \phi'(y)(t-y) + \phi''(y)(t-y)^2 + \cdots .$$

Again, comparing the above expansions with the asymptotic expansions given in (4) and (5), we arrive at $h(y) = y$, $a_0 = 1$, $a_s = 0$ for all $s \in \mathbb{N}$; $\mu = 1$, $b_0 = \phi(y)$, $b_s = \phi^{(s)}(y)$ for all $s \in \mathbb{N}$, and $\alpha = 1$. Thus, by Theorem 1, the integral given in (23) has the following asymptotic expansion

$$\int_y^\infty t^{\nu-\frac{1}{2}} e^{-\frac{t^2}{2}} e^{-xt} dt \sim e^{-xy} \sum_{s=0}^{\infty} \Gamma(s+1) \frac{b_s}{x^{s+1}} \text{ as } x \to \infty. \tag{24}$$

In view of (23) and (24), we have the following asymptotic expansion

$$R_\nu(x,y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} e^{-xy} \sum_{s=0}^{\infty} \Gamma(s+1) \frac{b_s}{x^{s+1}} \text{ as } x \to \infty, \tag{25}$$

where $b_0 = \phi(y), b_s = \phi^{(s)}(y)$ for all $s \in \mathbb{N}$ and $\phi(t) = t^{\nu-\frac{1}{2}} e^{-\frac{t^2}{2}}$.

(iv) The asymptotic expansion (25) gives the same asymptotic formula as given in equation (22).

(v) Consider the case when $\nu = k - \frac{1}{2}$ for $k = n > 2$ and $n \in \mathbb{N}$. In this case $a_{n-1}(\nu - 1) = 0$, and thus equation (15) reduces to

$$R_\nu(x, y) \sim c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} \sum_{i=0}^{n-2} \frac{a_i(\nu - 1)}{x^i} f_i(x, y; \nu),$$

where f_i is given by (16) and satisfies the recurrence relation (19).

(vi) If $\nu = k - \frac{1}{2}$ and $k \in \{1, 2\}$, then $\nu \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$. Then we find that the functions $R_{\frac{1}{2}}(x, y)$ and $R_{\frac{3}{2}}(x, y)$ involve $K_{-\frac{1}{2}}(xt)$ and $K_{\frac{1}{2}}(xt)$ respectively, which are equal in view of [**5**, 10.27.3] $K_{-\nu}(z) = K_\nu(z)$. The asymptotic expansion (31) for $\nu = \frac{1}{2}$ reduces to equation (6). Therefore, we conclude that the asymptotic expansion of $R_\nu(x, y)$ for $\nu \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$ when x is large is given by the equation (25).

(vii) It is easy to verify that if x is large and $x+y \sim x$, then in view of the equation (14), the asymptotic formula (13) is same as the asymptotic formula (22). More precisely, the asymptotic formula (13) is also valid for large x.

Theorem 4. *The asymptotic expansion for the generalized Marcum function of the second kind when ν is large and x, y are fixed, is*

$$\tilde{R}_\nu(\nu x, \nu y) \sim \tilde{c}_{\nu x,\nu} \left(\frac{K_\nu(\nu\xi)}{e^{-\nu(\beta+\eta(\xi))}} - \sqrt{\frac{\nu\pi}{2}} \frac{e^{-\nu(x+h(\xi))}}{(2x)^{\nu+1}} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} (-1)^k \frac{\Gamma(s+1) d_{k,s}}{\nu^{s+k+1}} \right),$$

where $h(\xi) = -\log\xi + \frac{\xi^2}{4x} + \eta(\xi), \beta = \frac{1}{2}\log\left(\frac{y}{x}\right) - x - y - \eta(\xi)$, $\eta(\xi) = \sqrt{1+\xi^2} + \log\left(\frac{\xi}{1+\sqrt{1+\xi^2}}\right)$ *and* $\xi = 2\sqrt{xy}$. *The coeffients $d_{k,s}$ are given in Theorem 1.*

Remark 4. The first three coefficients $d_{k,s}$ can be evaluated as follows

$$d_{k,0} = \frac{\phi_k(\xi)}{h(\xi)}, \quad d_{k,1} = \left(\phi_k'(\xi) - \frac{2h'(\xi)\phi_1(\xi)}{h(\xi)}\right) \frac{1}{h^2(\xi)} \tag{26}$$

and

$$d_{k,2} = \left(\phi_k''(\xi) - \frac{3h'(\xi)\phi_k'(\xi)}{h(\xi)} + 3(2(h'(\xi))^2 - h(\xi)h''(\xi))\frac{\phi_k(\xi)}{h^2(\xi)}\right) \frac{1}{h^3(\xi)}, \tag{27}$$

where

$$\phi_k(z) = \frac{z}{(1+z^2)^{\frac{1}{4}}} u_k(t),$$

and the first coefficients $u_k(t)$ are

$$u_0(t) = 1, \quad u_1(t) = \frac{3t - 5t^3}{24}, \quad u_2(t) = \frac{81t^2 - 462t^4 + 385t^6}{1152}, \tag{28}$$

and the other coefficients can be obtained by applying the formula

$$u_{k+1}(t) = \frac{1}{2}t^2(1-t^2)u_k'(t) + \frac{1}{8}\int_0^t (1-5s^2)u_k(s)ds, \quad k \in \{0,1,2,\ldots\}. \tag{29}$$

Remark 5. The asymptotic formula for $\tilde{R}_\nu(\nu x, \nu y)$ when ν is large, is given by

$$\tilde{R}_\nu(\nu x, \nu y) \sim \tilde{c}_{\nu x,\nu} \cdot \sqrt{\frac{\pi}{2\nu}} \left(\frac{e^{\nu\beta}}{(1+\xi^2)^{\frac{1}{4}}} - \frac{e^{-\nu(x+h(\xi))}\phi_0(\xi)}{(2x)^{\nu+1}h(\xi)} \right) = {}_3A_\nu(x,y), \tag{30}$$

where $h(\xi) = -\log\xi + \frac{\xi^2}{4x} + \eta(\xi)$, $\phi_0(\xi) = \frac{\xi}{(1+\xi^2)^{\frac{1}{4}}}$ and $\beta = \frac{1}{2}\log\left(\frac{y}{x}\right) - x - y - \eta(\xi)$.

3. PROOF OF THE MAIN RESULTS

This section deals with the proofs of Theorems stated in Section 2. To prove Theorem 2 we use the integration by parts technique.

Proof of Theorem 2. Using the formula [**5**, p. 252, 10.29.4]

$$(x^{-\nu}K_\nu(x))' = -x^{-\nu}K_{\nu+1}(x) \text{ and } (x^\nu K_\nu(x))' = -x^\nu K_{\nu-1}(x),$$

and integrating by parts repeatedly, equation (2) leads to the following equation

$$R_\nu(x,y) = c_{x,\nu}\frac{e^{-\frac{x^2}{2}}}{x^{\nu-1}}\left[e^{-\frac{y^2}{2}}y^{\nu-1}\sum_{n=1}^{N}\left(-\frac{x}{y}\right)^{n-1}K_{\nu-n}(xy) + r_N(x,y)\right],$$

where

$$r_N(x,y) = (-x)^N\int_y^\infty t^{\nu-N}e^{-\frac{t^2}{2}}K_{\nu-(N+1)}(xt)dt,$$

is a remainder term. It can be proved that

$$r_N(x,y) = o(y^{-N}) \text{ for all } N \in \mathbb{N}.$$

Hence

$$\begin{aligned} R_\nu(x,y) &\sim c_{x,\nu}\cdot e^{-\frac{x^2+y^2}{2}}\left(\frac{y}{x}\right)^{\nu-1}\sum_{n=1}^{\infty}\left(-\frac{x}{y}\right)^{n-1}K_{\nu-n}(xy) \\ &= c_{x,\nu}\cdot e^{-\frac{x^2+y^2}{2}}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{x}{y}\right)^{n-\nu}K_{\nu-n}(xy), \end{aligned}$$

where y is large. □

Proof of Theorem 3. By using the asymptotic expansion [**5**, p. 255, 10.40.2]

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k} \text{ as } z \text{ is large,} \tag{31}$$

where $a_k(\nu)$ is given by the equation (17), we arrive at

$$\begin{aligned} R_\nu(x,y) &\sim c_{x,\nu} \cdot \frac{e^{-\frac{x^2}{2}}}{x^{\nu-1}} \int_y^\infty t^\nu e^{-\frac{t^2}{2}} \left(\frac{\pi}{2xt}\right)^{\frac{1}{2}} e^{-xt} \left(\sum_{k=0}^{\infty} \frac{a_k(\nu-1)}{(xt)^k}\right) dt \\ &= c_{x,\nu} \cdot \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{a_k(\nu-1)}{x^k} \int_y^\infty t^{\nu-\frac{1}{2}-k} e^{-\frac{t^2}{2}} e^{-xt} dt \\ &= c_{x,\nu} \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x^2}{2}}}{x^{\nu-\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{a_k(\nu-1)}{x^k} f_k(x,y;\nu), \end{aligned}$$

as $x \to \infty$, where $f_k(x,y;\nu)$ is given by (16). □

Proof of Theorem 4. To obtain the expansion for the generalized Marcum function of the second kind we replace x, y and ν by $\nu x, \nu y$ and $\nu+1$ in (3), and we find that

$$\tilde{R}_{\nu+1}(\nu x, \nu y) = \tilde{c}_{\nu x,\nu+1} \cdot \int_{\nu y}^{\infty} \left(\frac{t}{\nu x}\right)^{\frac{\nu}{2}} e^{-t-\nu x} K_\nu(2\sqrt{\nu x t}) dt, \tag{32}$$

where

$$\tilde{c}_{\nu x,\nu+1} = \frac{2}{\Gamma(\nu+1)\Gamma(-\nu,\nu x)}.$$

By using the substitution $2\sqrt{\nu x t} \mapsto \nu z$, equation (32) becomes

$$\tilde{R}_{\nu+1}(\nu x, \nu y) = \tilde{c}_{\nu x,\nu+1} \cdot \frac{\nu e^{-\nu x}}{(2x)^{\nu+1}} \int_\xi^\infty z^{\nu+1} e^{-\frac{\nu}{4x} z^2} K_\nu(\nu z) dz, \tag{33}$$

where $\xi = 2\sqrt{xy}$. Now, we use the following asymptotic expansion for the modified Bessel function of the second kind [**5**, p. 256, 10.41.4]

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta(z)}}{(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} (-1)^k \frac{u_k(t)}{\nu^k}, \tag{34}$$

as $\nu \to \infty$ and $z \geq 0$, where

$$t = \frac{1}{\sqrt{1+z^2}}, \quad \eta(z) = \sqrt{1+z^2} + \log \frac{z}{1+\sqrt{1+z^2}}. \tag{35}$$

and $u_k(t)$ is given by the equations (28) and (29). Now, we write (33) as

$$\tilde{R}_{\nu+1}(\nu x, \nu y) = \tilde{c}_{\nu x,\nu+1} \cdot \frac{\nu e^{-\nu x}}{(2x)^{\nu+1}} \int_\xi^\infty z e^{-\nu h(z)} e^{\nu\eta(z)} K_\nu(\nu z) dz,$$

where

$$h(z) = -\log z + \frac{z^2}{4x} + \eta(z).$$

By using (34) in the above equation, we have

$$\tilde{R}_{\nu+1}(\nu x, \nu y) \sim \tilde{c}_{\nu x,\nu+1} \cdot \frac{e^{-\nu x}}{(2x)^{\nu+1}} \sqrt{\frac{\nu\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \frac{f_k(\nu)}{\nu^k}, \tag{36}$$

where

$$f_k(\nu) = \int_{\xi}^{\infty} \frac{z}{(1+z^2)^{\frac{1}{4}}} u_k(t) e^{-\nu h(z)} dz.$$

Using Theorem 1 for large ν, we calculate the asymptotic expansion for $f_k(\nu)$. Clearly, the functions $h(z) = -\log z + \frac{z^2}{4x} + \eta(z)$ and $\phi_k(z) = \frac{z}{(1+z^2)^{\frac{1}{4}}} u_k(t)$ satisfy all the conditions of Theorem 1 and have the following expansions around ξ

$$h(z) = h(\xi) + (z-\xi)h'(\xi) + (z-\xi)^2 h''(\xi) + \cdots$$

and

$$\phi_k(z) = \phi_k(\xi) + (z-\xi)\phi_k'(\xi) + (z-\xi)^2 \phi_k''(\xi) + \cdots.$$

Comparing these expansions with (4) and (5), respectively, we have

$$\mu = 1, \alpha = 1, a_0 = h(\xi), a_i = h^{(i)}(\xi), b_{k,0} = \phi_k(\xi), b_{k,i} = \phi_k^{(i)}(\xi) \text{ for all } i \in \{1, 2, \ldots\}.$$

Hence,

$$f_k(\nu) \sim e^{-\nu h(\xi)} \sum_{s=0}^{\infty} \Gamma(s+1) \frac{d_{k,s}}{\nu^{s+1}}, \ \nu \to \infty, \tag{37}$$

where the first few coefficients $d_{k,s}$ are given by (26) and (27). In view of (36) and (37), we obtain that

$$\begin{aligned} \tilde{R}_{\nu+1}(\nu x, \nu y) &\sim \tilde{c}_{\nu x,\nu+1} \cdot \sqrt{\frac{\nu\pi}{2}} \frac{e^{-\nu(x+h(\xi))}}{(2x)^{\nu+1}} \left(\sum_{k=0}^{\infty} (-1)^k \frac{1}{\nu^k} \sum_{s=0}^{\infty} \frac{\Gamma(s+1) d_{k,s}}{\nu^{s+1}} \right) \\ &= \tilde{c}_{\nu x,\nu+1} \cdot \sqrt{\frac{\nu\pi}{2}} \frac{e^{-\nu(x+h(\xi))}}{(2x)^{\nu+1}} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} (-1)^k \frac{\Gamma(s+1) d_{k,s}}{\nu^{s+k+1}}, \end{aligned} \tag{38}$$

and to obtain the asymptotic expansion for $\tilde{R}_\nu(\nu x, \nu y)$, we use the recurrence relation (7) in the following form

$$\frac{\tilde{R}_{\nu+1}(\nu x, \nu y)}{\tilde{c}_{\nu x,\nu+1}} + \frac{\tilde{R}_{\nu}(\nu x, \nu y)}{\tilde{c}_{\nu x,\nu}} = e^{\nu\eta(\xi)} K_\nu(\nu\xi) e^{\nu\beta},$$

where

$$\beta = \frac{1}{2} \log\left(\frac{y}{x}\right) - x - y - \eta(\xi)$$

and $\eta(z)$ is defined by equation (35). Hence, as $\nu \to \infty$ we arrive at

$$\tilde{R}_\nu(\nu x, \nu y) \sim \tilde{c}_{\nu x,\nu} \left(\frac{K_\nu(\nu\xi)}{e^{-\nu(\beta+\eta(\xi))}} - \sqrt{\frac{\nu\pi}{2}} \frac{e^{-\nu(x+h(\xi))}}{(2x)^{\nu+1}} \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} (-1)^k \frac{\Gamma(s+1)d_{k,s}}{\nu^{s+k+1}} \right).$$

□

Table 1: Values of $R_\nu(x,y)$, of the asymptotic expansion (11) and the corresponding errors

y	$R_\nu(x,y)$	Asymptotic expansion (11)	Absolute error	Relative error
1.0	0.598181	0.597695	4.86234×10^{-4}	8.12854×10^{-4}
1.1	0.536993	0.536556	4.36339×10^{-4}	8.12561×10^{-4}
1.2	0.477135	0.476747	3.87547×10^{-4}	8.12239×10^{-4}
1.3	0.419614	0.419274	3.40681×10^{-4}	8.11890×10^{-4}
1.4	0.365256	0.364959	2.96410×10^{-4}	8.11514×10^{-4}

Numerical values calculated by Wolfram Mathematica 8.0 for the values $\nu = 20$ and $x = 1$.

4. NUMERICAL RESULTS

Now we show that the asymptotic expansions obtained in this paper are good approximations to the function $R_\nu(x,y)$. To support the argument we provide some tables and figures.

Table 1 deals with the absolute as well as relative error of the asymptotic expansion (11) in comparison to the generalized Marcum function of the second kind when $\nu = 20$ and $x = 1$. It clearly shows that the relative error goes down when y is increasing. Similarly, the absolute and relative errors corresponding to the asymptotic expansion (25) for the generalized Marcum function of the second kind when $\nu = 2$ and $y = 0.5$ are provided in Table 2. In both tables the asymptotic values are calculated by taking two terms of the respective asymptotic expansions.

In Table 3 we discuss the numerical results for the asymptotic formula of $\tilde{R}_{\nu+1}(x,y)$ when $\nu = 2000$ and $y = 0.3$, and which is obtained from (38).

Figure 1 shows the comparison between $R_2(1,y)$ and ${}_1A_2(1,y)$, which denotes the asymptotic expansion given in (12) for $\nu = 2$, $x = 1$ and y large.

Figure 2 shows the comparison between $R_1\left(x, \frac{1}{2}\right)$ and ${}_2A_2\left(x, \frac{1}{2}\right)$, which denotes the asymptotic expansion given in (22) for $\nu = 2$, $y = \frac{1}{2}$ and x large.

Finally, Figure 3 shows the comparison between $\tilde{R}_\nu(2\nu, 2\nu)$ and ${}_3A_\nu(2,2)$, which denotes the asymptotic expansion given in (30) for $x = 2$, $y = 2$ and ν large. Figure 3 and Table 3 discuss the different form of the same asymptotic

Table 2: Values of $R_\nu(x,y)$, of the asymptotic expansion (25) and the corresponding errors

x	$R_\nu(x,y)$	Asymptotic expansion (25)	Absolute error	Relative error
3.1	0.535169	0.558691	2.35220×10^{-2}	4.39524×10^{-2}
3.3	0.509526	0.524576	1.50497×10^{-2}	2.95368×10^{-2}
3.5	0.484397	0.492936	8.53870×10^{-3}	1.76275×10^{-2}
3.7	0.459861	0.463416	3.55449×10^{-3}	7.72949×10^{-3}
3.9	0.435984	0.435749	2.34856×10^{-4}	5.38680×10^{-4}

Numerical values calculated by Wolfram Mathematica 8.0 for the values $\nu = 2$ and $y = 0.5$.

Table 3: Values of $\tilde{R}_{\nu+1}(x,y)$, of the asymptotic expansion (38) and the corresponding errors

x	$\tilde{R}_{\nu+1}(x,y)$	Expansion (38)	Absolute error	Relative error
2.8	0.7405069870669	0.790602	5.0095×10^{-2}	6.76496×10^{-2}
2.9	0.7404958740074	0.776878	3.63826×10^{-2}	4.91328×10^{-2}
3.0	0.7404847611150	0.763847	2.33627×10^{-2}	3.15505×10^{-2}
3.1	0.7404736483900	0.751453	1.09789×10^{-2}	1.48269×10^{-2}
3.2	0.7404625358327	0.739644	8.18787×10^{-4}	1.10578×10^{-3}

Numerical values calculated by Wolfram Mathematica 8.0 for the values $\nu = 2000$ and $y = 0.3$.

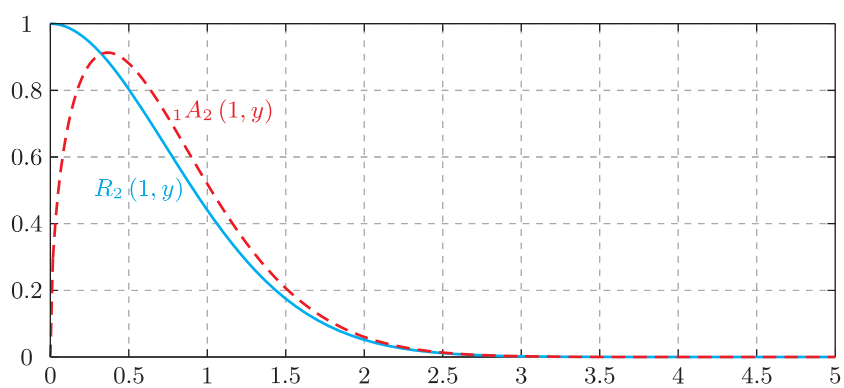

Figure 1: Comparison between $R_\nu(x,y)$ and its asymptotic formula (12) for $\nu = 2$, $x = 1$ and $y \in (0,5)$.

expansion which is valid for large ν. All three figures illustrate the situation when the generalized Marcum function of the second kind attains value near zero, while tables demonstrate the situation when the same function attains value near one.

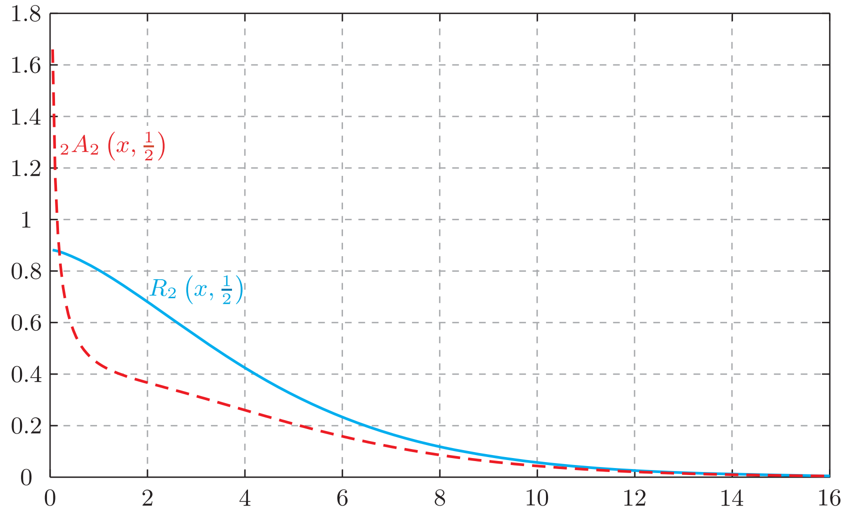

Figure 2: Comparison between $R_\nu(x,y)$ and its asymptotic formula (22) when $\nu=2$, $y=\frac{1}{2}$ and $x\in(0,16)$.

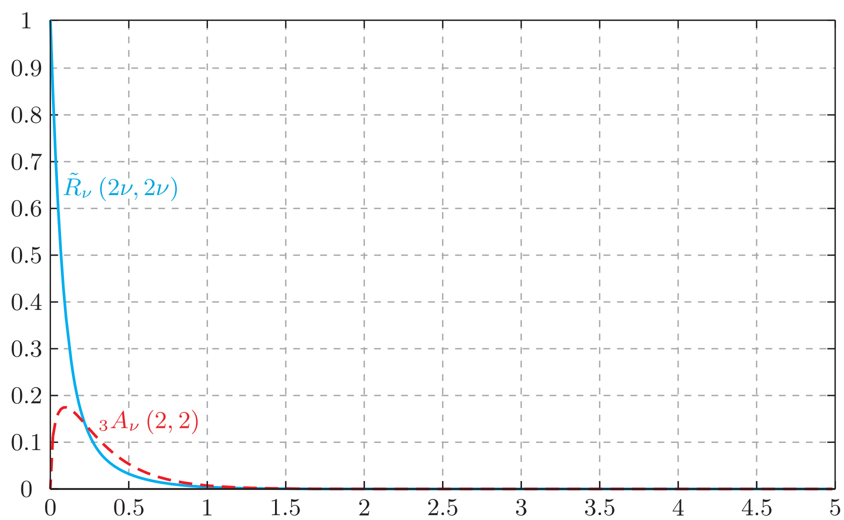

Figure 3: Comparison between $\tilde{R}_\nu(\nu x,\nu y)$ and its asymptotic formula (30) when $x=2$, $y=2$ and $\nu\in(0,5)$.

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