

GRANULAR COMPUTING ON BASIC DIGRAPHS

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In the present paper we investigate (p, q) -directed complete bipartite graphs $\vec{K}_{p,q}$, n -directed paths \vec{P}_n and n -directed cycles \vec{C}_n from the perspective of Granular Computing. For each model, we establish the general form of all possible indiscernibility relations, analyze the classical rough approximation functions of rough set theory and provide a close formula for the global accuracy average. Finally, we completely determine the attribute dependency function and the global dependency average for both \vec{C}_n and $\vec{K}_{p,q}$.

1. INTRODUCTION

1.1 General Premise

An *information system* [55] is a structure \mathcal{I} given by a non-empty finite object set U , a non-empty finite attribute (or condition) set Att and a map F assigning to any pair $(u, a) \in U \times Att$ a quantitative or qualitative value chosen in a fixed value set Val . In particular, if $Val = \{0, 1\}$ we say that \mathcal{I} is a *Boolean information system*. Moreover, it is usual to identify the information system \mathcal{I} with a matrix whose rows and columns are respectively indexed by U and Att , and containing the values $F(u, a)$ in the corresponding entries.

Information systems are also known as *relational tables* in database theory [64] and they are usually studied when a huge amount of data needs to be classified in a table according to some criterion of subdivision.

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In his classical monograph [55], Pawlak introduced the so called *rough set theory* (abbreviated RST) in order to better analyze and reduce the complexity of a generic information table. The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data like probability distributions in statistics, basic probability assignments in Dempster-Shafer theory [63], a degree of membership or the value of possibility in fuzzy set theory [86]. Actually, RST is considered a consistent part of a new emerging research field named *granular computing* (abbreviated GrC) [56, 57].

GrC deals with representing and processing information in the form of some type of aggregates. These aggregates are generally called *information granules* or simply granules and they arise in the process of data abstraction and knowledge derivation from data. In GrC, several theoretical tools have been recently developed in order to make less complex the data interpretation for the values entered in a very large information system. More in detail, sophisticated GrC techniques have been used in order to:

- provide new concrete and theoretical tools for classification problems [44, 45, 46, 84, 85];
- determine total or partial dependencies between attribute subsets in relational databases [39, 40, 41, 42, 65];
- establish new links between mereology and RST [58, 59, 60, 61];
- determine characteristic and significant attribute subsets for the data tabular interpretation [47, 48, 49];
- generate appropriate decision rules [66, 67, 68, 69];
- study discrete dynamical systems and interactive information systems from a granular standpoint [1, 2, 5, 6, 12, 31, 71, 72];
- analyze various models taken from bioinformatics and biomedical imaging [43, 76, 77].

In the present paper we are concerned with the study of basic digraph families by means of GrC techniques. To this regard, it is convenient to highlight that currently the studies regarding applications and links between GrC and graph and digraph theory are various and intersect several areas of research. More specifically, some of them deal with the investigation of structural properties of graphs [4, 14, 15, 17, 62] and digraphs [20, 21, 81], others are inherent to applicative contexts [35, 51, 78, 82] and others provide characterizations of graphs and hypergraphs in a morphological perspective [73, 74, 75].

The notion of information system has a relevant role in GrC by virtue of a very natural equivalence relation which any attribute subset induces on the universe set.

To be more specific, if \mathcal{I} is an information system and $A \subseteq Att$, it is usual to consider the binary relation \equiv_A on the universe set U defined as follows (see

[55, 64]): if $u, u' \in U$, then $u \equiv_A u'$ if $F(u, a) = F(u', a)$ for any $a \in A$. The binary relation \equiv_A is an equivalence relation on U and it is called *A-indiscernibility relation*. For any $u \in U$, let $[u]_A$ *A-indiscernibility class* of u , i.e. the equivalence class of u with respect to \equiv_A . In addition, we denote by $\pi_A(\mathcal{I})$ the set partition on U induced by the equivalence relation \equiv_A and we call it the *A-indiscernibility partition* of the information system \mathcal{I} .

Through the indiscernibility notion, the information system theory (that is RST) can be considered a sub-field of study in the scope of GrC. To this regard, in [7, 16, 23] to any information system has been associated a micro-macro granular structure whose set systems of its maximal and minimal members are respectively a closure system and an abstract simplicial complex on the attribute set. Then, in a granular perspective, the basic aim of such a structure is that to simplify the visualization and the interpretation of the data appearing in the given information system (for details see [19, 24]). On the other hand, in a purely mathematical perspective, the aforementioned structure and others connected to it have several properties of algebraic and geometrical nature (for details see [8, 27, 28, 29, 34]).

Another basic notion in RST is the *exactness* of an object subset relatively to a given attribute subset. This notion has been introduced by Pawlak in [55] and, actually, it is a capital notion in RST and GrC [52, 53, 54]. It depends on the choice of a fixed attribute subset A through which it is possible to define two set operators lw_A and up_A on U , respectively called *lower approximation operator* and *upper approximation operator* and defined by $lw_A(Y) := \{u \in U \mid [u]_A \subseteq Y\}$ and $up_A(Y) := \{u \in U \mid [u]_A \cap Y \neq \emptyset\}$. The subset Y is called *A-exact* if and only if $lw_A(Y) = up_A(Y)$ and *A-rough* otherwise.

The lower approximation represents the elements that *certainly*, with respect to our knowledge expressed by A , belongs to Y . On the other hand, the upper approximation is the set of objects *possibly* belonging to A .

In [15, 18, 20, 62] the adjacency matrix of simple undirected graphs has been interpreted and studied by means of GrC methodologies. In this perspective of study, in [16, 18] several types of hypergraph families and order structures that are related between them by means of matroidal properties have been introduced (to this regard see also a series of very recent studies concerning the link between matroids and data tables [38, 70, 79, 80, 87]). One among the most interesting of such structures is the *granular partition lattice* of an information system [83]. GrC on graphs and digraphs can be addressed from two complementary perspectives: the determination of the granular partition lattice and its related structures (reducts, essentials, dissymmetry hypergraph, maximum partitioner hypergraph, minimal partitioner hypergraph [15, 16, 18, 25, 26]), and, on the other hand, the analysis of functional dependency and the determination of the rough approximation functions [17, 20, 22, 62].

In the first perspective, a relevant scope is the determination of the *reducts* of graphs and digraphs and, more in general, of any information table. The difficulty arising during the computation of the reducts consists in the fact that they form the minimal transversals family of the discernibility hypergraph of the given

information system (for details see [15, 18]).

In general, we recall that the problem to determine the transversal hypergraph of given hypergraphs is an important mathematical problem in computer science [32, 33, 36, 37] and in graph theory [9, 10, 11]. Moreover, it is easy to see that the minimum cardinality of the reducts of an information table induced by the vertex shortest distance in a connected graph agrees with the metric dimension of such a graph (see [13, 50]). Therefore, we can consider the determination of specific types of reducts as a generalization concerning the study of the metric dimension of connected graphs.

In this paper we continue to work on the interpretation of the digraph adjacency matrix in terms of information systems. We develop some aspects of GrC on digraphs [20, 21] by investigating the notion of indiscernibility for Boolean information tables that are the exactly adjacency matrices of some basic digraph families: the complete bipartite digraph $\vec{K}_{p,q}$, the n -directed cycle \vec{C}_n and the n -directed path \vec{P}_n . To be more specific, in the first part of this paper, we give a complete description of the general form of the indiscernibility partitions for the information systems induced by the previous digraph families. Furthermore, we also determine the lower and upper approximation functions associated with each of these digraphs. In the next subsection we describe in more detail the results of the second part of our paper.

1.2 Attribute Dependency and Related Notions

A well studied notion in database theory is the *attribute dependency*. If \mathcal{I} is an information system and A and B are two subsets of Att , two important notions (see [55, 64]) in RST are the *A -positive region* of B , defined by $\Gamma_A(B) = \{u \in U \mid [u]_A \subseteq [u]_B\}$ and the *A -attribute dependency function* γ_A which is the mapping assinging the number $\frac{|\Gamma_A(X)|}{|U|}$ to each attribute subset X , usually named the number $\gamma_A(B)$ the *quality of approximation* of B by A .

The A -positive region of B can be considered as the set of all objects in the universe set U which can be properly classified by means of blocks of $\pi_B(\mathcal{I})$ using only the knowledge expressed by classification $\pi_A(\mathcal{I})$.

The quality of approximation of B by A is a measure of the representability degree of the knowledge provided from the attribute subset B in terms of the knowledge provided from the other attribute subset A . If we want to establish an analogy with the probabilistic language, we can interpret the number $\gamma_A(B)$ as the "probability" which whenever two objects are A -indiscernible then they also are B -indiscernible.

Now, when we study the general form of the dependency function γ_A in an information system \mathcal{I} , it becomes necessary to visualize all values $\gamma_A(B)$. Therefore we will call *attribute dependency table* of \mathcal{I} , denoted by $T_{dep}(\mathcal{I})$, the $2^n \times 2^n$ table having as rows and columns all subsets of Att and in the place corresponding to row A and column B the value $\gamma_A(B)$.

Let us note that the attribute dependency table is a source of very useful information when one studies the properties of \mathcal{I} . In fact, it may be easily observed that all the order relations between indiscernibility relations can be deduced from the table $T_{dep}(\mathcal{I})$. In particular, a complete knowledge of the attribute dependency table allows us build the granular partition lattice [16] of the information system \mathcal{I} . We can also think the table $T_{dep}(\mathcal{I})$ as a type of numerical completion of the granular partition lattice of \mathcal{I} .

Now, in order to have a global measure associated with the attribute dependency table, it is convenient to introduce the following numerical averages for any information system (in fact, the study of average numbers is very frequent both in digraph context [3] and in GrC [30]). To this regard, we call:

- *average attribute dependency from A in \mathcal{I}* the number $\lambda_r(A) := \frac{1}{2^n} \sum_{B \subseteq Att} \gamma_A(B)$;
- *average attribute dependency to A in \mathcal{I}* the number $\lambda_c(A) := \frac{1}{2^n} \sum_{B \subseteq Att} \gamma_B(A)$;
- *average attribute dependency of \mathcal{I}* the number $\lambda(\mathcal{I}) := \frac{1}{2^n} \sum_{A \subseteq Att} \lambda_r(A) = \frac{1}{2^n} \sum_{A \subseteq Att} \lambda_c(A)$.

Let us note that the number $\lambda_r(A)$ [$\lambda_c(A)$] can be considered, by using a statistical terminology, a *marginal average* on the row [column] corresponding to the attribute subset A in the dependency table of \mathcal{I} . Therefore we can interpret the number $\lambda(\mathcal{I})$ as the global average capacity of transmit (or, equivalently, receive) dependency in the information system \mathcal{I} . Nevertheless, it should be noted that the exact computation of $\lambda(\mathcal{I})$ can be very hard in general.

In the second part of this paper we firstly determine the general form of the attribute dependency function for the digraphs $\vec{K}_{p,q}$, \vec{C}_n and \vec{P}_n . Next we investigate the value of $\lambda(\mathcal{I})$ when the information system \mathcal{I} is induced respectively by the out-adjacency matrices of the above digraph families. To be more specific, we are able to provide an exact closed formula of $\lambda(\mathcal{I})$ for the information system associated with the digraphs $\vec{K}_{p,q}$ and \vec{C}_n and, with regard to these two digraphs, we also provide an asymptotic estimate when the number of their vertices grows indefinitely.

1.3 Content of the paper

Now we briefly describe the contents of all sections in this paper.

In Section 2 we recall the basic notions concerning digraphs and information systems which we will use in this paper.

In Section 3 we present the first basic results concerning such a new granular geometry on digraphs. Specifically, we interpret a generic indiscernibility relation in the digraph context and next we show as this interpretation is naturally connected to the action of the digraph automorphism group. Moreover, we determine

the complete form of the indiscernibility partition for four fundamental digraph families which will be widely studied in this paper: the complete bipartite digraph $\vec{K}_{p,q}$, the n -directed path \vec{P}_n and the n -directed cycle \vec{C}_n .

In Section 4 we study the Pawlak lower and upper approximation functions in the specific digraph context and we completely determine roughness and exactness for all the digraphs $\vec{K}_{p,q}$, \vec{P}_n and \vec{C}_n . Furthermore, we introduce a quantity taking account of the global accuracy of the digraph and that we call *out-accuracy average* and compute it for the above digraph families.

In Section 5 we study the attribute dependency for digraphs. More in detail, we completely determine the attribute dependency function in the cases $\vec{K}_{p,q}$ and \vec{C}_n .

In Section 6 we firstly adapt to the digraph case the general averages defined through the attribute dependency and that we introduced in the previous subsection for any information system. Next, we completely determine such average dependencies when $D = \vec{K}_{p,q}$ and $D = \vec{C}_n$. In particular, we provide a precise asymptotic estimate for the global average dependency number.

2. NOTATIONS AND PRELIMINARIES

Let X be a given finite set, $\mathcal{P}(X)$ its powerset and $\mathcal{P}(X)^* := \mathcal{P}(X) \setminus \{\emptyset\}$. We call any subset $\mathcal{F} \subseteq \mathcal{P}(X)$ a *set system* on X . If $A \in \mathcal{P}(X)$ and X is clear from the context, we denote by A^c the complementary subset of A in X , that is $X \setminus A$. We denote by $|X|$ the number of elements of X . A non-empty set system π on X for which $B \neq \emptyset$ for any $B \in \pi$, is said a *set partition* of X if $\bigcup\{B \mid B \in \pi\} = X$, $B \cap B' = \emptyset$ for any $B, B' \in \pi$ such that $B \neq B'$. If π is a set partition of X , its elements $B \in \pi$ are usually called *blocks* of π and, for each $x \in X$, we will denote by $\pi(x)$ the unique block of π containing x .

We will often use the classical combinatorial notation in order to denote a given set partition of X . More in detail, we write $\pi = B_1 | \dots | B_m$ in order to denote that π is a set partition of X whose distinct blocks are B_1, \dots, B_m .

2.1 Digraphs

We call an ordered pair $D = (V(D), Arc(D))$ a *digraph*, where:

- $V(D)$ is a finite set $\{w_1, \dots, w_n\}$, whose elements are said *vertices* of D ;
- $Arc(D)$ is a finite set $\{e_1, \dots, e_m\} \subseteq V(D) \times V(D)$, whose elements are said *arcs* of D ;
- for any arc $e_k \in Arc(D)$ there exist two distinct vertices w_i, w_j in $V(D)$ for which $e_k = (w_i, w_j)$.

When the digraph D is clear from the context, we will write simply V instead of $V(D)$.

If $(v, w) \in \text{Arc}(D)$, we also use the notation $v \rightarrow w$. It is clear that any digraph D is uniquely determined by its *adjacency matrix*, here denoted by $\text{Adj}(D)$, which is the $n \times n$ matrix (a_{ij}) defined by $a_{ij} := 1$ if $(w_i, w_j) \in \text{Arc}(D)$ and $a_{ij} := 0$ otherwise. If $v \in V(D)$, we respectively set $N_D^+(v) := \{z \in V(D) \mid (v, z) \in \text{Arc}(D)\}$ and $N_D^-(v) := \{u \in V(D) \mid (u, v) \in \text{Arc}(D)\}$. Usually $N_D^+(v)$ is called the *out-neighborhood* of v in D and $N_D^-(v)$ is called the *in-neighborhood* of v in D . If $X \in \mathcal{P}(V(D))$, the *generated subdigraph* by X in D , denoted by $D[X]$, is the digraph whose vertices are exactly those of X and such that for any $x, y \in X$, with $x \neq y$, it results that $(x, y) \in \text{Arc}(D[X])$ if and only if $(x, y) \in \text{Arc}(D)$.

The *converse digraph* D^* of the digraph D is defined as the digraph whose vertex set agrees with $V(D)$ and such that $(w_i, w_j) \in \text{Arc}(D^*)$ if and only if $(w_j, w_i) \in \text{Arc}(D)$.

In what follows, we will study the following basic digraph families:

- the (p, q) -complete bipartite digraph $\vec{K}_{p,q}$. It is the digraph with $V(\vec{K}_{p,q}) = \{x_1, \dots, x_p, y_1, \dots, y_q\}$ and $\text{Arc}(\vec{K}_{p,q}) = \{(x_i, y_j) \mid i = 1, \dots, p, j = 1, \dots, q\}$;
- the n -directed path \vec{P}_n . It is the digraph with $V(\vec{P}_n) = \{w_1, \dots, w_n\}$ and $\text{Arc}(\vec{P}_n) = \{(w_i, w_{i+1}) \mid i = 1, \dots, n-1\}$;
- the n -directed cycle \vec{C}_n . It is the digraph with $V(\vec{C}_n) = \{w_1, \dots, w_n\}$ and $\text{Arc}(\vec{C}_n) = \{(w_i, w_{i+1}) \mid i = 1, \dots, n\}$, where the index sums are taken modulo n .

2.2 Basics on Information Systems and Rough Set Theory

In the present subsection we provide the basics of information system theory.

Definition 2.1. [55] An information system is a structure $\mathcal{I} = \langle U, \text{Att}, \text{Val}, F \rangle$, where $U = \{u_1, u_2, \dots, u_m\}$ is a non-empty finite set called universe set, $\text{Att} = \{a_1, a_2, \dots, a_n\}$ is a non-empty finite set called attribute set, Val is the value set and $F : U \times \text{Att} \rightarrow \text{Val}$ is called information map. The elements of U are called objects and the elements of Att are called attributes. In particular, if $\text{Val} = \{0, 1\}$ we say that \mathcal{I} is a Boolean information system. The information table $T(\mathcal{I})$ of \mathcal{I} is the $m \times n$ matrix whose rows are indexed with all objects in U , whose columns are indexed with all attributes in Att and which contains the value $F(u_i, a_j)$ in the place (i, j) .

For each $u, u' \in U$, we set

$$u \equiv_A u' \iff \forall a \in \text{Att} [F(u, a) = F(u', a)]$$

The relation \equiv_A turns out to be an equivalence relation on U , which we call *A-indiscernibility relation* on U . If $u \in U$, we denote by $[u]_A$ the equivalence class of u with respect to \equiv_A (i.e. the *A-indiscernibility class* of u). When u and u' are not A -indiscernible, we write $u \not\equiv_A u'$. We also set

$$\pi_A(\mathcal{I}) := \{[u]_A \mid u \in U\}$$

to denote the set partition on U induced by \equiv_A . We call it the the A -indiscernibility partition of \mathcal{I} .

We recall the two basic *rough approximation functions* leading to the notion of exactness.

Definition 2.2. [55] Let $A \in \mathcal{P}(\text{Att})$ and $Y \subseteq U$. We call A -lower approximation of Y the subset of U defined by:

$$lw_A(Y) := \{u \in U \mid [u]_A \subseteq Y\} = \bigcup\{C \in \pi_A(\mathcal{I}) \mid C \subseteq Y\}$$

Moreover, we call A -upper approximation of Y the subset of U defined by:

$$up_A(Y) := \{u \in U \mid [u]_A \cap Y \neq \emptyset\} = \bigcup\{C \in \pi_A(\mathcal{I}) \mid C \cap Y \neq \emptyset\}$$

The subset Y is called A -exact if and only if $lw_A(Y) = up_A(Y)$ and A -rough otherwise.

We recall now the basic notion of *attribute dependency*.

Definition 2.3. [55] Let $A, B \in \mathcal{P}(\text{Att})$. We call A -positive region of B the subset of U given by:

$$\Gamma_A(B) = \{u \in U \mid [u]_A \subseteq [u]_B\}$$

Moreover, we call A -attribute dependency function the mapping γ_A defined as follows:

$$\gamma_A : X \in \mathcal{P}(\text{Att}) \mapsto \gamma_A(X) := \frac{|\Gamma_A(X)|}{|U|}$$

Let us note that if A and B are two any attribute subsets of \mathcal{I} and \preceq is the usual partial order between set partitions, we have

$$(1) \quad B \subseteq A \implies \pi_A(\mathcal{I}) \preceq \pi_B(\mathcal{I})$$

and

$$\pi_A(\mathcal{I}) \preceq \pi_B(\mathcal{I}) \iff \Gamma_A(B) = U \iff \gamma_A(B) = 1$$

If $k = \gamma_A(B)$, then it results that $0 \leq k \leq 1$ and it is usual to write $A \Rightarrow_k B$. In particular, if $k = 1$, it is said that B totally depends from A ; if $0 < k < 1$, it is said that B partially depends from A , finally, if $k = 0$ it is said that B is totally independent from A .

3. DIGRAPHS AS BOOLEAN INFORMATION TABLES

In this section we interpret the adjacency matrix of a digraph D as a Boolean information system and, next, provide a complete description for the indiscernibility partitions induced by any vertex subsets of $\vec{K}_{p,q}$, \vec{P}_n and \vec{C}_n .

Definition 3.1. Let $D = (V(D), Arc(D))$ be a digraph. We will consider the Boolean information system $[D]^+ = (V(D), V(D), F_D^+, \{0, 1\})$, where

$$F_D^+(u, v) := \begin{cases} 1 & \text{if } u \rightarrow v \\ 0 & \text{otherwise} \end{cases}$$

We call the above information system the out-information system of D .

If $A \in \mathcal{P}(V(D))$, we will use the notation $\pi_A^+(D) := \pi_A([D]^+)$ to denote the A^+ -indiscernibility partition of D and call any block of $\pi_A^+(D)$ an A^+ -granule of D . Furthermore, we will use the symbol \equiv_A^+ to denote the A^+ -indiscernibility relation on D ; for each $v \in V(D)$, we denote by $[v]_A^+$ the A^+ -indiscernibility class of v and call it the A^+ -indiscernibility class of v in D .

Similarly, we may consider the information system $[D]^- = (V(D), V(D), F_D^-, \{0, 1\})$, where

$$F_D^-(u, v) := \begin{cases} 1 & \text{if } v \rightarrow u \\ 0 & \text{otherwise} \end{cases}$$

For the previous information system, we use the terminology in-information system of D . We will use the above notations in order to denote the indiscernibility relation and partitions, only changing the sign + with the sign -.

Relatively to a digraph D , it results that

$$(2) \quad [D]^- = [D^*]^+ \text{ and } T([D]^-) = T([D]^+)^t = Adj(D)^t,$$

where t denotes the transposition of a matrix. Consequently, by taking into account the identities provided in (2), in what follows we will describe most of the results relative to a digraph D only for the corresponding out-information system $[D]^+$, which we will denote simply with the letter D , leaving for granted that the given results may be translated for the information system $[D]^-$.

Now we describe the first elementary GrC characterization on a generic digraph D . The next result allows us to establish a basic link between the indiscernibility relation and its equivalent geometric characterization.

Theorem 3.2. Let $A \in \mathcal{P}(V(D))$ and $v, v' \in V(D)$. The following conditions are equivalent:

- (i) $v \equiv_A^+ v'$;
- (ii) for all $z \in A$ it results that $v \rightarrow z$ if and only if $v' \rightarrow z$;
- (iii) $N_D^+(v) \cap A = N_D^+(v') \cap A$.

Consequently

$$[v]_A^+ = \{w \in V(D) \mid N_D^+(v) \cap A = N_D^+(w) \cap A\},$$

and if $v \equiv_A^+ v'$ and $v' \in A$, then $v \not\rightarrow v'$.

Proof. See [20]. □

Remark 3.3. In view of Theorem 3.2, for any vertex v of D we have that $v' \in [v]_A^+$ if and only if v and v' “out-see” all vertices of the subset A in a same way. Formally: $v' \in [v]_A^+$ if, for any $z \in A$, it results that $v \rightarrow z$ if and only if $v' \rightarrow z$. Therefore, any A^+ -granule of $[D]^+$ is exactly a vertex subset of D whose elements have the same out-adjacency relation with respect the vertices of A . In such a way, if a vertex subset $A \in \mathcal{P}(V(D))$ is given, it is appropriate to interpret A as a type of symmetry block against which to examine the out/in behavior of all vertices in D . Geometrically, this means that the choice of a given vertex subset $A \in \mathcal{P}(V(D))$ can be thought of as the choice of an “out-symmetry axis” relatively to the digraph D . Analogously, the choice of the blocks of $\pi_A^+(D)$ as the choice of “out-symmetry point subsets” with respect to A . Thus it is natural to call all the A^+ -granules $B \in \pi_A^+(D)$ the A^+ -symmetry blocks of D .

Let us now see what happens to the A^+ -indiscernibility when we consider it in terms of induced subdigraph automorphisms. To this regard, let $A \in \mathcal{P}(V(D))$ and $v, v' \in V(D)$ be distinct vertices. Set $X := A \cup \{v, v'\}$ and we define the map $\psi_{v,v'}^A : X \rightarrow X$ setting

$$\psi_{v,v'}^A(x) = \begin{cases} x & \text{if } x \in A \setminus \{v, v'\} \\ v' & \text{if } x = v \\ v & \text{if } x = v', \end{cases}$$

for any $x \in X$.

The next result shows that if $v \not\rightarrow v'$ and $v' \not\rightarrow v$ or if $A \cap \{v, v'\} = \emptyset$, then v and v' are A^+ -indiscernible if and only if A is a type of “symmetry axis” with respect to v and v' .

Theorem 3.4. Let $A \in \mathcal{P}(V(D))$, $v, v' \in V(D)$ be distinct vertices and $X := A \cup \{v, v'\}$. Then, if $v \not\rightarrow v'$ and $v' \not\rightarrow v$ or if $A \cap \{v, v'\} = \emptyset$, it results that

$$\psi_{v,v'}^A \in \text{Aut}(D[X]) \iff v \equiv_A^+ v'$$

Proof. See [20]. □

At this point, we close the present section determining the general form of the indiscernibility partitions for the Boolean information tables induced by some basic digraph families. In the next results we establish the form of any A^+ -indiscernibility partition for the three digraph families $\vec{K}_{p,q}$, \vec{P}_n and \vec{C}_n .

We begin to describe the A^+ -indiscernibility partitions of the digraph $\vec{K}_{p,q}$.

Proposition 3.5. Let $D := \vec{K}_{p,q} = (B_1 | B_2)$, where $B_1 = \{x_1, \dots, x_p\}$ and $B_2 = \{y_1, \dots, y_q\}$. Let moreover $A \in \mathcal{P}(V(D))$ and $w_i, w_j \in V(D)$ with $w_i \neq w_j$. Then

$$w_i \equiv_A^+ w_j \iff A \subseteq B_1 \vee \{w_i, w_j\} \subseteq B_1 \vee \{w_i, w_j\} \subseteq B_2.$$

Proof. See Proposition 3.12 of [20]. □

In the next results we will exhibit the A^+ -indiscernibility partitions for n -cycles and n -paths.

Proposition 3.6. *Let $D := \vec{C}_n$, $A \in \mathcal{P}(V(D))$ and $w_i, w_j \in V(D)$ be distinct. Then*

$$w_i \equiv_A^+ w_j \iff \{w_{i+1}, w_{j+1}\} \cap A = \emptyset,$$

where the sum is taken modulo n .

Proof. See [20]. □

Proposition 3.7. *Let $D := \vec{P}_n$, $A \in \mathcal{P}(V(D))$ and $w_i, w_j \in V(D)$ be distinct $w_i \neq w_j$. Then*

$$w_i \equiv_A^+ w_j \iff \{w_{i+1}, w_{j+1}\} \cap A = \emptyset$$

Proof. See [20]. □

Corollary 3.8. *Let $D = \vec{P}_n$ and $A \in \mathcal{P}(V(D))$ be of the form $A = \{w_{i_1}, \dots, w_{i_k}\}$, where $1 < i_1 < \dots < i_k \leq n$. Then $\pi_A^+(D) = \pi_{A \cup \{w_1\}}^+(D)$.*

Proof. The case of \vec{P}_n follows immediately from Proposition 3.7. □

4. ROUGH APPROXIMATIONS IN SOME BASIC DIGRAPHS

In this section we determine the general form of the rough approximation functions for the same digraphs of the previous section. Moreover, we compute the classical A -accuracy for these digraphs and then introduce a quantity evaluating the *accuracy average* of a digraph.

To this purpose, we need to express the general definitions of lower and upper approximation within our digraph context.

Definition 4.1. *If $D = (V(D), \text{Arc}(D))$ is a digraph and $A, X \in \mathcal{P}(V(D))$, we say that:*

- $lw_A^+(X) := \{v \in V(D) \mid [v]_A^+ \subseteq X\}$ is the A^+ -lower approximation of X ;
- $up_A^+(X) := \{v \in V(D) \mid [v]_A^+ \cap X \neq \emptyset\}$ is the A^+ -upper approximation of X ;
- X is A^+ -symmetric if $lw_A^+(X) = up_A^+(X)$ and A^+ -rough otherwise;
- the quantity

$$\alpha_A^+(X) := \frac{|lw_A^+(X)|}{|up_A^+(X)|}$$

is the A^+ -accuracy of X , when $X \neq \emptyset$.

The A^+ -accuracy evaluates the degree of out-accuracy of X with respect to A . It may be easily verified that $\alpha_A^+(X) \in [0, 1]$ and, in particular, that a $X \in \mathcal{P}(V(D))^*$ is A^+ -symmetric if and only if $\alpha_A^+(X) = 1$. To provide a global out-accuracy measure depending only by the structure of D we will consider the following average

$$\rho^+(D) := \frac{1}{(2^n - 1)^2} \sum_{A, X \in \mathcal{P}(V(D))^*} \alpha_A^+(X)$$

Then $\rho^+(D)$ is a number between 0 and 1 and we call it *out-accuracy average* of D .

We also set

$$\rho_r^+(X) := \frac{1}{2^n - 1} \sum_{A \in \mathcal{P}(V(D))^*} \alpha_A^+(X)$$

and

$$\rho_c^+(A) := \frac{1}{2^n - 1} \sum_{X \in \mathcal{P}(V(D))^*} \alpha_A^+(X)$$

In what follows, we deal with the case of \vec{P}_n , \vec{C}_n and $\vec{K}_{p,q}$. For these digraph families, it is possible to determine completely the A^+ -accuracy and, furthermore, to exhibit closed formulas for their out-accuracy averages and evaluations for their asymptotic estimates.

4.1 Out-Accuracy for \vec{P}_n and \vec{C}_n

In this subsection we investigate the behaviour of the digraphs \vec{P}_n and \vec{C}_n . Indeed, we will compute the A^+ -accuracy for such digraphs when A varies over $\mathcal{P}(V(D))^*$.

Theorem 4.2. *Let $D = \vec{P}_n$ or $D = \vec{C}_n$ and let $A, X \in \mathcal{P}(V(D))^*$. Then:*

$$(3) \quad \alpha_A^+(X) = \begin{cases} 1 & \text{if } X \supseteq N_D^-(A)^c \vee X \subseteq N_D^-(A), \\ \frac{|N_D^-(A) \cap X|}{|N_D^-(A)^c \cup X|} & \text{otherwise.} \end{cases}$$

Proof. The claim follows directly by Propositions 3.6 and 3.7. \square

In order to evaluate the expression of the out-accuracy average, we need to provide an expression for the quantity $\rho_c^+(A)$ in the two cases.

Proposition 4.3. *Let $D = \vec{P}_n$ or $D = \vec{C}_n$. Let $A \in \mathcal{P}(V(D))^*$ be such that $|N_D^-(A)| = k$. Then we have that:*

$$(4) \quad \rho_c^+(A) = \begin{cases} \frac{1}{2^{n-1}} \left(2^{k+1} - 1 + (2^{n-k} - 2) \sum_{i=0}^k \binom{k}{i} \frac{i}{n+i-k} \right) & \text{if } 1 \leq k \leq n-2, \\ \frac{1}{2^{n-1}} & \text{if } k = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $V := V(D)$. Clearly, by (3), if $k \geq n - 1$, then it follows that $\rho_c^+(A) = 1$. Assume now that $1 \leq k \leq n - 2$. Let \mathcal{F}_A be the set system on V of all the vertex subsets $X \in \mathcal{P}(V)^*$ such that $X \not\subseteq N_D^-(A)$ and $N_D^-(A)^c \not\subseteq X$. It is immediate to verify that

$$\mathcal{F}_A = \{X \in \mathcal{P}(V) \mid \emptyset \subsetneq N_D^-(A)^c \cap X \subsetneq N_D^-(A)^c\}.$$

Now, for each $X \in \mathcal{F}_A$, we set $Y_X := X \cap N_D^-(A)^c$ and $Z_X := X \cap N_D^-(A)$. Thus $\emptyset \subsetneq Y_X \subsetneq N_D^-(A)^c$ and $\emptyset \subseteq Z_X \subseteq N_D^-(A)$.

Hence

$$|\mathcal{F}_A| = 2^k(2^{n-k} - 2).$$

Let us compute now $\rho_c^+(A)$. It holds

$$\rho_c^+(A) = \frac{1}{2^n - 1} \left(2^{k+1} - 1 + \sum_{X \in \mathcal{F}_A} \alpha_A^+(X) \right)$$

If $X \in \mathcal{F}_A$, we have that

$$\alpha_A^+(X) = \frac{|Z_X|}{|Z_X| + n - k}$$

Then

$$\sum_{X \in \mathcal{F}_A} \alpha_A(X) = \sum_{Z \in \mathcal{P}(N_D^-(A))} \left(\sum_{\substack{X \in \mathcal{F}_A \\ Z_X = Z}} \frac{|Z|}{|Z| + n - k} \right).$$

Fix $Z \in \mathcal{P}(N_D^-(A))$ and consider the following subfamily of \mathcal{F}_A :

$$\mathcal{G}_A^Z := \{X \in \mathcal{F}_A \mid Z_X = Z\}$$

Observe that $|\mathcal{G}_A^Z| = 2^{n-k} - 2$ since Y_X and Z_X uniquely determine X . Therefore the quantity

$$\sum_{Z \in \mathcal{P}(N_D^-(A))} \left(\sum_{X \in \mathcal{G}_A^Z} \frac{|Z|}{|Z| + n - k} \right)$$

equals to the quantity

$$\sum_{Z \in \mathcal{P}(N_D^-(A))} (2^{n-k} - 2) \frac{|Z|}{|Z| + n - k}$$

In other terms, we get

$$\begin{aligned}
\sum_{X \in \mathcal{F}_A} \alpha_A(X) &= \sum_{Z \in \mathcal{P}(N_D^-(A))} \left(\sum_{\substack{X \in \mathcal{F}_A \\ Z_X = Z}} \frac{|Z|}{|Z| + n - k} \right) = \\
&= \sum_{Z \in \mathcal{P}(N_D^-(A))} \left(\sum_{X \in \mathcal{G}_A^Z} \frac{|Z|}{|Z| + n - k} \right) = \sum_{Z \in \mathcal{P}(N_D^-(A))} \frac{(2^{n-k} - 2)|Z|}{|Z| + n - k} = \\
&= (2^{n-k} - 2) \sum_{i=0}^k \binom{k}{i} \frac{i}{n + i - k}.
\end{aligned}$$

Thus we get

$$\rho_c^+(A) = \frac{1}{2^n - 1} \left(2^{k+1} - 1 + (2^{n-k} - 2) \sum_{i=0}^k \binom{k}{i} \frac{i}{n + i - k} \right)$$

This concludes the evaluation for $1 \leq k \leq n - 2$. Assume now that $k = 0$. In this case, we clearly have $D = \vec{P}_n$ and $A = \{v_1\}$. In such a case, we get $up_A^+(X) = V$. Moreover, it results that $lw_A^+(X) = \emptyset$ if and only if $X \neq V$; otherwise $lw_A^+(V) = V$. Hence, we easily conclude that

$$\rho_c^+(A) = \frac{1}{2^n - 1}$$

□

Proposition 4.3 enables us to evaluate the quantity $\rho^+(\vec{C}_n)$. In the next result, we also give an asymptotic estimate for such a quantity.

Theorem 4.4. *We have that*

$$(5) \quad \rho^+(\vec{C}_n) = \frac{1}{(2^n - 1)^2} \left[\frac{1}{3} 4^n - 3^n + 2^n - \frac{1}{3} \right] + \frac{n+1}{2^n - 1} + \frac{2 \cdot 3^n - (n+3) \cdot 2^n + n}{(2^n - 1)^2},$$

so that

$$\rho^+(\vec{C}_n) \rightarrow \frac{1}{3}$$

Proof. Let $V := V(\vec{C}_n)$. In order to compute the quantity $\rho^+(\vec{C}_n)$, we will use the evaluation given in (4) for the quantity $\rho_c^+(A)$. In particular, we get

$$\begin{aligned}
\rho^+(\vec{C}_n) &= \frac{1}{2^n - 1} \left[\sum_{k=1}^{n-2} \binom{n}{k} \frac{1}{2^n - 1} \left(2^{k+1} - 1 + (2^{n-k} - 2) \sum_{i=0}^k \binom{k}{i} \frac{i}{n + i - k} \right) \right] + \\
&\quad + \frac{n+1}{2^n - 1}
\end{aligned}$$

The quantity $n + 1$ occurs because of the fact that whenever we choose $A \in \mathcal{P}(V(\vec{C}_n))$ such that $|A| \geq n - 1$, Equation (4) ensures that $\rho_c^+(A) = 1$, since $\alpha_A^+(X) = 1$ for any $X \in \mathcal{P}(V)^*$.

Let us now compute the quantity

$$\sum_{k=1}^{n-2} \binom{n}{k} (2^{k+1} - 1)$$

It may be easily shown by means of simple algebraic manipulations that the previous quantity agrees with the following one:

$$2 \cdot 3^n - (n + 3) \cdot 2^n + n.$$

On the other hand, let us provide an evaluation for the sum

$$\sum_{k=1}^{n-2} \binom{n}{k} (2^{n-k} - 2) \sum_{i=1}^k \binom{k}{i} \frac{i}{n-k+i}$$

Reindex the above sum, setting $s := n - k + i$. In this way, we obtain the equivalent sum

$$\sum_{s=3}^n \binom{n}{s} \sum_{i=1}^{s-2} \binom{s-1}{i-1} (2^{s-i} - 2)$$

It holds

$$\begin{aligned} \sum_{i=1}^{s-2} \binom{s-1}{i-1} 2^{s-i} &= \sum_{i=0}^{s-3} \binom{s-1}{i} 2^{s-1-i} = \sum_{i=0}^{s-1} \binom{s-1}{i} 2^{s-1-i} - 2(s-1) - 1 = \\ &= 3^{s-1} - 2(s-1) - 1 \end{aligned}$$

Similarly, it results

$$\sum_{i=1}^{s-2} \binom{s-1}{i-1} = \sum_{i=0}^{s-3} \binom{s-1}{i} = \sum_{i=0}^{s-1} \binom{s-1}{i} - (s-1) - 1 = 2^{s-1} - (s-1) - 1$$

Thus

$$\sum_{s=3}^n \binom{n}{s} \sum_{i=1}^{s-2} \binom{s-1}{i-1} (2^{s-i} - 2) = \sum_{s=3}^n \binom{n}{s} (3^{s-1} - 2^s + 1) = \frac{1}{3} 4^n + 3^n - 2^n - \frac{1}{3}$$

Hence, we easily get (5) and, furthermore, the evaluation of the limit of $\rho^+(\vec{C}_n)$ is now immediate. \square

Though Proposition 4.3 holds also for the digraph \vec{P}_n , the results we obtain in this case is completely different from the corresponding computations for the digraph \vec{C}_n .

Theorem 4.5. *We have that*

$$(6) \quad \rho^+(\vec{P}_n) = \frac{1}{(2^n - 1)^2} \left[\left(\frac{1}{6} - \frac{2}{9n} \right) 4^n - \left(-1 - \frac{1}{2n} \right) 3^n - 2n - \frac{5}{18n} - 2^n \right],$$

and, in particular, $\lim_{n \rightarrow \infty} \rho^+(\vec{P}_n) = \frac{1}{6}$.

Proof. Let $V := V(\vec{P}_n)$. In order to compute the quantity $\rho^+(\vec{P}_n)$, we will use the evaluation given in (4) for the quantity $\rho_c^+(A)$. In particular, we get

$$\begin{aligned} \rho^+(\vec{P}_n) &= \frac{1}{2^n - 1} \left[\sum_{k=1}^{n-2} m_k \frac{1}{2^n - 1} \left(2^{k+1} - 1 + (2^{n-k} - 2) \sum_{i=0}^k \binom{k}{i} \frac{i}{n+i-k} \right) \right] + \\ &\quad + \frac{1}{2^n - 1} \left(\frac{1}{2^n - 1} + 2 \right), \end{aligned}$$

where m_k denotes the number of vertex subsets $A \in \mathcal{P}(V)^*$ such that $|N_{\vec{P}_n}^-(A)| = k$, with $1 \leq k \leq n-2$.

The summand 2 in the above sum occurs because of the fact that there are only two vertex subsets A for which $\alpha_A^+(X) = 1$ for any $X \in \mathcal{P}(V)^*$, namely $A = V$ and $A = V \setminus \{v_1\}$. Similarly, the summand $\frac{1}{2^n - 1}$ comes from the case $A = \{v_1\}$, already discussed in Proposition 4.3.

Let us firstly note that

$$m_k = 2 \binom{n-1}{k}$$

since, in view of the definition of $N_{\vec{P}_n}^-(A)$, it results that $|N_{\vec{P}_n}^-(A)| = |N_{\vec{P}_n}^-(A \cup \{v_1\})|$. Thus, we must evaluate the quantity

$$\begin{aligned} \rho^+(\vec{P}_n) &= \frac{1}{(2^n - 1)^2} \left[2 \sum_{k=1}^{n-2} \binom{n-1}{k} \left(2^{k+1} - 1 + (2^{n-k} - 2) \sum_{i=0}^k \binom{k}{i} \frac{i}{n+i-k} \right) \right] + \\ &\quad + \frac{2^{n+1} - 1}{(2^n - 1)^2} \end{aligned}$$

Through simple algebraic manipulations, we may easily show the relation

$$2 \sum_{k=1}^{n-2} \binom{n-1}{k} (2^{k+1} - 1) = \frac{4}{3} \cdot 3^n - 3 \cdot 2^n.$$

At this point, let us compute the sum

$$\sum_{k=1}^{n-2} 2 \binom{n-1}{k} (2^{n-k} - 2) \sum_{i=1}^k \binom{k}{i} \frac{i}{n-k+i}.$$

Reindex the above sum setting $s := n - k + i$. In this way, we obtain the equivalent sum

$$\sum_{s=3}^n \sum_{i=1}^{s-2} \frac{s-i}{n} \binom{n}{s} \binom{s-1}{i-1} (2^{s-i} - 2).$$

Using some simple algebraic manipulations, it is straightforward to prove that

$$\sum_{i=1}^{s-2} \frac{s-i}{n} \binom{s-1}{i-1} (2^{s-i} - 2) = \frac{s-1}{n} (2 \cdot 3^{s-2} - 2^{s-1}).$$

Therefore, we get

$$\begin{aligned} \sum_{s=3}^n \sum_{i=1}^{s-2} \frac{s-i}{n} \binom{n}{s} \binom{s-1}{i-1} (2^{s-i} - 2) &= \sum_{s=3}^n \binom{n}{s} \frac{s-1}{n} (2 \cdot 3^{s-2} - 2^{s-1}) = \\ &= \left(\frac{3n-4}{18n} \right) 4^n - \left(\frac{2n-3}{2n} \right) 3^n - 2n + 2 - \frac{5}{18n} \end{aligned}$$

Thus (6) may be easily obtained holds and the evaluation of the limit of $\rho^+(\vec{P}_n)$ is now trivial. \square

4.2 Out-accuracy and Averages for $\vec{K}_{p,q}$

We investigate in this subsection the case of the complete bipartite digraph $\vec{K}_{p,q}$. In the next result, we give an expression for the quantity $\alpha_A^+(X)$ when $A, X \in \mathcal{P}(V(\vec{K}_{p,q}))^*$.

Theorem 4.6. *Let $A, X \in \mathcal{P}(V(\vec{K}_{p,q}))^*$. Then:*

$$(7) \quad \alpha_A^+(X) = \begin{cases} 1 & \text{if } X = V \vee [(X = B_1 \vee X = B_2) \wedge A \not\subseteq B_1], \\ \frac{p}{n} & \text{if } B_1 \subsetneq X \subsetneq V \wedge A \not\subseteq B_1, \\ \frac{q}{n} & \text{if } B_2 \subsetneq X \subsetneq V \wedge A \not\subseteq B_1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is an immediate consequence of Proposition 4.4 of [20]. \square

We are now able to compute $\rho^+(\vec{K}_{p,q})$ and to provide for it an asymptotic estimate.

Theorem 4.7. *We have that*

$$(8) \quad \rho^+(\vec{K}_{p,q}) = \frac{1}{(2^n - 1)^2} \left\{ 2^p - 1 + 2^p (2^q - 1) \left[1 + (2^q - 2) \frac{p}{n} + (2^p - 2) \frac{q}{n} \right] \right\}.$$

Moreover, if p is a fixed integer, one has $\lim_{q \rightarrow \infty} \rho^+(\vec{K}_{p,q}) = 0$. Analogously, if q is a fixed integer, one has that $\lim_{p \rightarrow \infty} \rho^+(\vec{K}_{p,q}) = 0$.

Proof. Let us fix $A \in \mathcal{P}(V) \setminus \{\emptyset\}$. Assume that $A \subseteq B_1$. We clearly have $lw_A^+(X) = \emptyset$ if and only if $X \neq V$. Hence, in this case

$$\rho_c^+(A) = \frac{1}{2^n - 1}$$

Assume now that $A \not\subseteq B_1$. By (7), we have that $\alpha_A^+(X) = 1$ if $X = V$. We have to study the other possibilities. Again by (7), it readily follows that there are $2^q - 2$ subsets X such that $B_1 \subsetneq X \wedge X \neq V$ and, on the other hand, there are $2^p - 2$ subsets X such that $B_2 \subsetneq X \wedge X \neq V$. Hence, one has

$$\rho_c^+(A) = \frac{1}{2^n - 1} \left[1 + (2^q - 2) \frac{p}{n} + (2^p - 2) \frac{q}{n} \right]$$

Therefore, we have shown that

$$\rho_c^+(A) = \begin{cases} \frac{1}{2^n - 1} & \text{if } A \subseteq B_1, \\ \frac{1}{2^n - 1} \left[1 + (2^q - 2) \frac{p}{n} + (2^p - 2) \frac{q}{n} \right] & \text{otherwise.} \end{cases}$$

Let us note that there are $2^p - 1$ non-empty subsets A of B_1 . Therefore, it follows that

$$\rho^+(\vec{K}_{p,q}) = \frac{2^p - 1}{(2^n - 1)^2} + \frac{2^p(2^q - 1)}{(2^n - 1)^2} \left[1 + (2^q - 2) \frac{p}{n} + (2^p - 2) \frac{q}{n} \right]$$

and this proves (8).

Finally, the evaluations of the limits when p (or q) is a fixed integer are immediate. \square

5. ATTRIBUTE DEPENDENCY IN OUR DIGRAPH FAMILIES

We interpret in the present section the classical notion of attribute dependency on digraphs. Let D be a digraph and let A, B be two subsets of the vertex subset $V(D)$. By virtue of our interpretation of the indiscernibility class in terms of symmetry blocks, for any vertex $v \in V(D)$ we can see $[v]_A^+$ as the A^+ -symmetry block of v in $V(D)$. Hence the inclusion $[v]_A^+ \subseteq [v]_B^+$ is equivalent to say that the A^+ -symmetry block of v becomes a part of the B^+ -symmetry block of v . For example, let us consider the digraph in Figure 1. Let $A = \{w_1, w_2, w_4\}$. Then $[w_0]_A^+ = \{w_0, w_7\}$. Moreover, if we take $B = \{w_4, w_6\}$ we have $[w_0]_B^+ = \{w_0, w_5, w_7\}$. This means that v delivers its A^+ -symmetric point w also in a B^+ -symmetric point.

By virtue of the above interpretation, it is natural to introduce the following new notions in digraph theory.

Table 1: The Out Symmetry Transmission Quality Table of $[\vec{C}_3]^+$.

	\emptyset	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	V
\emptyset	1	0	0	0	0	0	0	0
$\{w_1\}$	1	1	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$
$\{w_2\}$	1	$1/3$	1	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$
$\{w_3\}$	1	$1/3$	$1/3$	1	$1/3$	$1/3$	$1/3$	$1/3$
$\{w_1, w_2\}$	1	1	1	1	1	1	1	1
$\{w_1, w_3\}$	1	1	1	1	1	1	1	1
$\{w_2, w_3\}$	1	1	1	1	1	1	1	1
V	1	1	1	1	1	1	1	1

Definition 5.1. We say that a vertex $v \in V(D)$ is an $(A, B)^+$ -symmetry transmitter if $[v]_A^+ \subseteq [v]_B^+$. Moreover we set

$$\Gamma_A^+(B) := \{v \in V(D) \mid [v]_A^+ \subseteq [v]_B^+\} \text{ and } \gamma_A^+(B) := \frac{|\Gamma_A^+(B)|}{|V(D)|}$$

We can interpret $\Gamma_A^+(B)$ as the subset of all the vertices in D which transmit out-symmetry from the vertex subset A towards the vertex subset B . Analogously, we can interpret $\gamma_A^+(B)$ as the $(A, B)^+$ -symmetry transmission quality in D .

For example, when \mathcal{I} is the information system $[\vec{C}_3]^+$ the attribute dependency table is given in Table 1.

Let us examine now, for example, the values on the row corresponding to the vertex subset $\{w_1\}$. By means of our previous interpretation, we can say that $\{w_1\}$ transmits out-symmetry towards all other vertex subsets with sequence $1, 1, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3$. Hence it is also natural to say that $\{w_1\}$ has a transmission out-symmetry average capacity in \vec{C}_3 equal to the average of the above sequence, that is $1/2$.

Based on the above discussion we introduce the following terminology, that is specific for Boolean information tables induced from out adjacency matrices of digraphs.

Definition 5.2. Let D be a digraph such that $V(D) = V = \{w_1, \dots, w_n\}$ and $A \in \mathcal{P}(V(D))$. We call out symmetry transmission quality table of D , the table $T_{sym}^+(D) := T_{dep}([D]^+)$.

In this section we study the dependency function relatively to the information systems $[\vec{K}_{p,q}]^+$, $[\vec{C}_n]^+$ and $[\vec{P}_n]^+$.

Let us firstly recall the case of the directed n -path \vec{P}_n .

Proposition 5.3. Let $V := V(\vec{P}_n)$ and $A, B \in \mathcal{P}(V)$ be two distinct vertex subsets.

Then

$$\Gamma_A^+(B) = \begin{cases} V & \text{if } B = \emptyset \vee B = \{w_1\} \vee A = V \vee A = \{w_2, \dots, w_n\} \vee B \subseteq A \\ \emptyset & \text{if } (A = \emptyset \vee A = \{w_1\}) \wedge (B \neq \emptyset \wedge B \neq \{w_1\}) \\ N_D^-(A) & \text{otherwise} \end{cases}$$

Moreover

$$\gamma_A^+(B) = \begin{cases} 0 & \text{if } (A = \emptyset \vee A = \{w_1\}) \wedge (B \neq \emptyset \wedge B \neq \{w_1\}) \\ \frac{k-1}{n} & \text{if } 1 \leq |A| = k \leq n-2 \wedge w_1 \in A \\ \frac{k}{n} & \text{if } 1 \leq |A| = k \leq n-2 \wedge w_1 \notin A \\ 1 & \text{if } B = \emptyset \vee B = \{w_1\} \vee A = V \vee A = \{w_2, \dots, w_n\} \vee B \subseteq A, \end{cases}$$

Proof. See [21]. \square

At this point, we will devote our attention to the case of the complete bipartite digraph $\vec{K}_{p,q}$.

Proposition 5.4. *Let $V := V(\vec{K}_{p,q})$ and $A, B \in \mathcal{P}(V)$ be two distinct vertex subsets. Then*

$$\Gamma_A^+(B) = \begin{cases} \emptyset & \text{if } ((A = \emptyset \vee A \subseteq B_1) \wedge B \not\subseteq B_1) \vee (A \not\subseteq B_1 \wedge (B = \emptyset \vee B \subseteq B_1)) \\ V & \text{otherwise} \end{cases}$$

Moreover

$$\gamma_A^+(B) = \begin{cases} 0 & \text{if } ((A = \emptyset \vee A \subseteq B_1) \wedge B \not\subseteq B_1) \vee (A \not\subseteq B_1 \wedge (B = \emptyset \vee B \subseteq B_1)) \\ 1 & \text{otherwise} \end{cases}$$

Proof. We recall that $\Gamma_A^+(B) = \{v \in V \mid [v]_A^+ \subseteq [v]_B^+\}$. In view of Proposition 3.5, it is clear that if $(A = \emptyset \vee A \subseteq B_1) \wedge (B \neq \emptyset \wedge B \not\subseteq B_1)$, then $\Gamma_A^+(B) = \emptyset$ since the $\pi_A^+ = V$ while $\pi_B^+ = B_1|B_2$. The case $(A \neq \emptyset \wedge A \not\subseteq B_1) \wedge (B = \emptyset \vee B \subseteq B_1)$ is similar. In the other cases, the A^+ -indiscernibility partition and the B^+ -indiscernibility partition coincide, so $\Gamma_A^+(B) = V$. \square

Now we focus our attention to the case of the directed n -cycle \vec{C}_n .

Proposition 5.5. *Let $V := V(\vec{C}_n)$ and $A, B \in \mathcal{P}(V)$ be two distinct vertex subsets. Then:*

$$(9) \quad \Gamma_A^+(B) = \begin{cases} V & \text{if } B = \emptyset \vee A = V \vee |A| = n-1 \vee B \subseteq A \\ \emptyset & \text{if } A = \emptyset \wedge B \neq \emptyset \\ N_D^-(A) & \text{otherwise} \end{cases}$$

Moreover

$$(10) \quad \gamma_A^+(B) = \begin{cases} 1 & \text{if } B = \emptyset \vee A = V \vee |A| = n-1 \vee B \subseteq A \\ 0 & \text{if } A = \emptyset \wedge B \neq \emptyset \\ \frac{|A|}{n} & \text{otherwise} \end{cases}$$

Proof. In view of Proposition 3.6, we have that if $B = \emptyset$, then $\pi_B^+(\vec{C}_n) = V$. Therefore if A is any subset of V and $v \in V$ is any vertex, it follows that $[v]_A \subset [v]_B = V$. Similarly, if $A = V$ or if $|A| = n - 1$, we have $\pi_A^+(\vec{C}_n) = w_1 | \dots | w_n$. In other words, in these situations $\Gamma_A^+(B) = V$. Furthermore, by (1), if $B \subseteq A$, we deduce that $\Gamma_A^+(B) = V$. Again by Proposition 3.6, it is clear that if $A = \emptyset$ and $B \neq \emptyset$, we have $[v]_A = V$ for every $v \in V$, but $[v]_B \subsetneq V$. So, $\Gamma_A^+(B) = \emptyset$. Let $|A| = k$ and $|B| = l$. In view of Proposition 3.7, we have $\pi_A^+(\vec{C}_n) = w_{i_1-1} | \dots | w_{i_k-1} | N_D^-(A)^c$ and $\pi_B^+(\vec{C}_n) = w_{j_1-1} | \dots | w_{j_l-1} | N_D^-(B)^c$. This means that $[v]_A \subseteq [v]_B$ only for $x = w_{i_1-1}, \dots, w_{i_k-1}$, therefore $\Gamma_A^+(B) \supseteq \{w_{i_1-1}, \dots, w_{i_k-1}\}$. Let $v \in \Gamma_A^+(B) \setminus N_D^-(A)$. Then $[v]_A^+ = N_D^-(A)^c$. We have two cases: $v \in B$ or $v \notin B$. In the first case, $[v]_B^+$ has a single element, so $[v]_A^+$ can't be contained in $[v]_B^+$ unless $N_D^-(A)^c$ consists of a single element, but this happens if $|A| = n - 1$ and we have excluded this situation. In the second case, we have $[v]_A^+ \subseteq [v]_B^+$ if and only if $N_D^-(A)^c \subseteq N_D^-(B)^c$, i.e. if and only if $N_D^-(B) \subseteq N_D^-(A)$. This means that $B \subseteq A$, but this situation has been excluded. Hence the proof of (9) is complete. Finally, by definition we have $\gamma_A^+(B) = \frac{|\Gamma_A^+(B)|}{|V|}$, therefore (10) follows directly from (9). \square

6. AVERAGE ATTRIBUTE DEPENDENCY

In this section we interpret in digraph context the averages introduced in the conclusive part of the introductory section. The two marginal average attribute dependency numbers $\lambda_r(A)$, $\lambda_c(A)$ and the global average attribute dependency number $\lambda(\mathcal{I})$ defined in the introduction for any information system, can be considered a specific case of their corresponding statistical concepts.

Firstly we formalize the previous average numbers in the particular digraph case. Next, we completely determine such numbers when $D = \vec{C}_n$ and we also provide an asymptotic estimate.

Definition 6.1. Let D be a digraph and $A \in \mathcal{P}(V(D))$. We call the quantity:

- $\lambda_r^+(A) := \frac{1}{2^n} \sum_{B \in \mathcal{P}(V(D))} \gamma_A^+(B)$ the out-average symmetry transmission from A in D ;
- $\lambda_c^+(A) := \frac{1}{2^n} \sum_{B \in \mathcal{P}(V(D))} \gamma_B^+(A)$ the out-average symmetry transmission to A in D ;
- $\lambda^+(D) := \frac{1}{2^n} \sum_{A \in \mathcal{P}(V(D))} \lambda_r^+(A) = \frac{1}{2^n} \sum_{A \in \mathcal{P}(V(D))} \lambda_c^+(A)$ the out-average transmission symmetry of D .

In [21], the computation of the out-average transmission symmetry has been undertaken for the digraph \vec{P}_n . To be more detailed, for the digraph \vec{P}_n it has been provided not only a complete formula for the aforementioned quantity but also an

asymptotic estimate when the number of vertices grows indefinitely. We recall here the expression of the out-average symmetry transmission from any vertex subset A and the out-average transmission symmetry for the digraph \vec{P}_n .

Theorem 6.2. *Let $A \in \mathcal{P}(V(\vec{P}_n))$. Then we have*

$$\lambda_r^+(A) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } A = \emptyset \vee A = \{w_1\} \\ 1 & \text{if } \{w_2, \dots, w_n\} \subseteq A \\ \frac{1}{n} + \frac{3}{2^n} \left(1 - \frac{1}{n}\right) & \text{if } |A| = 1 \wedge A \neq \{w_1\} \\ \frac{1}{2^{n-k}} \left(1 + \frac{k-1}{n} (2^{n-k} - 1)\right) & \text{if } A = \{w_1, w_{i_2}, \dots, w_{i_k}\} \\ \frac{1+2^k+\frac{k}{n}(2^n-2^k-1)}{2^n} & \text{if } 2 \leq |A| = k \leq n-2 \wedge w_1 \notin A \end{cases}$$

Moreover, the quantity $\lambda^+(\vec{P}_n)$ is equal to:

$$\frac{1}{4^n} \left\{ 4 + 2^{n+1} + (n-2) \left(3 + \frac{2^n-3}{n} \right) + \sum_{k=2}^{n-1} \binom{n-1}{k-1} 2^k \left[1 + \frac{k-1}{n} (2^{n-k} - 1) \right] \right\} + \frac{1}{4^n} \left\{ \sum_{k=2}^{n-2} \binom{n-1}{k} \left[1 + 2^k + \frac{k}{n} (2^n - 2^k - 1) \right] \right\}$$

and $\lim_{n \rightarrow \infty} \lambda^+(\vec{P}_n) = \frac{1}{2}$.

Proof. See [21]. □

In this paper, we focus our attention to the digraphs $\vec{K}_{p,q}$ and \vec{C}_n . In particular, for such digraphs, we are able to express both a closed formula for the out-average transmission symmetry and an asymptotic estimate for it.

6.1 The $\vec{K}_{p,q}$ case

In this paragraph we will focus our attention on the case $D = \vec{K}_{p,q}$. Preliminarily, we will compute the out-average symmetry transmission from a vertex subset A in $\vec{K}_{p,q}$.

Lemma 6.3. *Let A be a vertex subset of $V(\vec{K}_{p,q})$ and $n = p + q$. Then:*

$$(11) \quad \lambda_r^+(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \frac{2^q-1}{2^q} & \text{if } A \subseteq B_1 \\ 1 & \text{otherwise} \end{cases}$$

Proof. In view of Proposition 5.4, it is easy to see that in the row corresponding to $A = \emptyset$, there are only 0 entries, while in the rows corresponding to any vertex subset A of B_1 we have exactly 2^p entries equal to 0, namely those corresponding to the columns $B \subseteq B_1$ and the remaining are all 1. Hence we have exactly $\frac{2^n - 2^p}{2^n}$, which after some simplification, becomes $\frac{2^q - 1}{2^q}$. Finally, in the other rows, we have all entries 1. Hence, we conclude that (11) holds. \square

In the next result we determine the out-average transmission symmetry of $\vec{K}_{p,q}$.

Theorem 6.4. *Let $n = p + q$. Then*

$$(12) \quad \lambda^+(\vec{K}_{p,q}) = \frac{4^n + 2^p - 2^n - 2^{2p}}{4^n},$$

therefore $\lim_{n \rightarrow \infty} \lambda^+(\vec{K}_{p,q}) = 1$.

Proof. Just add the quantities $(2^n - 2^p)$ exactly $2^p - 1$ times, corresponding to all non-empty subsets of B_1 , and the quantities 2^n exactly $(2^n - 2^p)$ times. In this way, dividing by 4^n , we obtain (12). \square

6.2 The \vec{C}_n case

We study now the case $D = \vec{C}_n$. Also in this case we establish firstly the following preliminary result.

Lemma 6.5. *Let $A \in \mathcal{P}(V(\vec{C}_n))$. Then*

$$\lambda_r^+(A) = \begin{cases} \frac{1}{2^n} & \text{if } A = \emptyset \\ \frac{1}{2^{n-k}} + \frac{k}{n2^{n-k}}(2^{n-k} - 1) & \text{if } k = |A| < n - 1 \\ 1 & \text{if } |A| \geq n - 1 \end{cases}$$

Proof. If $A = \emptyset$, by (10), in the first row of the table of dependency of \vec{C}_n we have all the entries 0 except the first, which is 1, so $\lambda_r^+(\emptyset) = \frac{1}{2^n}$. Otherwise, if $|A| = r$, where $r = 1, \dots, n - 2$, we have exactly 2^r entries equal to 1, corresponding to all its possible subsets, and the other are equal to $\frac{r}{n}$. finally, when $|A| = n - 1$ or $A = V$, the entries of the corresponding rows in the dependency table are all 1. \square

For the sake of completeness, we also compute the value $\lambda_c^+(B)$ when B is a column label running over $\mathcal{P}(V(\vec{C}_n))$. We make such a computation in order to underline how it can be radically different to express the quantities $\lambda_r^+(\mathcal{I})$ and $\lambda_c^+(\mathcal{I})$. This allows us to prefer one among the averages $\lambda_r^+(\mathcal{I})$ or $\lambda_c^+(\mathcal{I})$ to evaluate the out-average transmission symmetry.

Proposition 6.6. *Let $V := V(\vec{C}_n)$ and $B \in \mathcal{P}(V)$. Then we have:*

$$\lambda_c^+(B) = \begin{cases} 1 & \text{if } B = \emptyset \\ \frac{3n-1}{4n} + \frac{1}{n2^n} & \text{if } |B| = 1 \\ \frac{1}{2^k} + \frac{k}{2^n} + \frac{1}{2^n} \left[\sum_{l=1}^{k-1} \binom{n}{l} \frac{l}{n} + \sum_{l=k}^{n-2} \left(\binom{n}{l} - \binom{n-k}{l-k} \right) \frac{l}{n} \right] & \text{if } 1 < k = |B| < n \\ \frac{1}{2^n} + \frac{1}{2} & \text{if } |B| = n \end{cases}$$

Proof. Assume firstly that $B = \emptyset$. In view of (10), $\gamma_A^+(B) = 1$ for any $A \in \mathcal{P}(V)$. Let now B be a singleton, say $B := \{w_i\}$ for some $i = 1, \dots, n$. Then, if $B \subseteq A$, the corresponding entries of the dependency table are 1. This occurs 2^{n-1} times. Moreover, it is true also if $|A| = n - 1$. Now, we compute the other values. There are exactly $\binom{n-1}{k}$ k -subsets of V not containing B and the corresponding value of the dependency table is $\frac{k}{n}$. Therefore we have

$$\lambda_c^+(B) = 2^{n-1} + 1 + \sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k}{n}$$

As $\sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k}{n} = \frac{n-1}{n} (2^{n-2} - 1)$, through algebraic manipulation we get

$$\lambda_c^+(B) = \frac{3n-1}{4n} + \frac{1}{n2^n}.$$

Let us assume now that B has k elements, where $1 < k := |B| < n$ elements. Notice that there are exactly 2^{n-k} subsets of V containing B and $\binom{n}{n-1} - \binom{n-k}{n-k-1} = k$ subsets of cardinality $n - 1$ not containing B . To them, the function $\gamma_A^+(B)$ equals 1. On the other hand, there are no l -subsets containing B for $l = 1, \dots, k - 1$ and, for each of them, the corresponding value in the dependency table is $\frac{l}{n}$, i.e.

we can sum the quantity $\sum_{l=1}^{k-1} \binom{n}{l} \frac{l}{n}$. Finally, assume that $k \leq l \leq n - 2$: we will find exactly $\binom{n}{l} - \binom{n-k}{l-k}$ subsets of cardinality l not containing B and in correspondence of them, the value taken in the dependency table is $\frac{l}{n}$. So, we get:

$$\lambda_c^+(B) = \frac{1}{2^k} + \frac{k}{2^n} + \frac{1}{2^n} \left[\sum_{l=1}^{k-1} \binom{n}{l} \frac{l}{n} + \sum_{l=k}^{n-2} \left(\binom{n}{l} - \binom{n-k}{l-k} \right) \frac{l}{n} \right]$$

The remaining case is when $|B| = n$. Clearly, we have $n + 1$ entries equal to 1. Furthermore, there are no l -subsets containing B for $l = 1, \dots, n - 2$ and, for each of them, the corresponding value in the dependency table is $\frac{l}{n}$. So $\lambda_c^+(V) = \frac{n+1}{2^n} + \frac{1}{2^n} \sum_{l=1}^{n-2} \binom{n}{l} \frac{l}{n}$. After adding and subtracting the same quantity, namely $n(n-1) + n$,

and as $\sum_{k=0}^n \binom{n}{k} k = n2^{n-1}$, by means of algebraic manipulations we get $\lambda_c^+(V) = \frac{1}{2^n} + \frac{1}{2}$. \square

We are also able to provide a complete determination of the out-average transmission symmetry and a corresponding asymptotic estimate.

Theorem 6.7. *We have that*

$$\lambda^+(\vec{C}_n) = \frac{1}{4^n} \left\{ \sum_{k=0}^{n-2} \binom{n}{k} \left[2^k + k \frac{2^n - 2^k}{n} \right] + 2^n(n+1) \right\}$$

and $\lim_{n \rightarrow \infty} \lambda^+(\vec{C}_n) = \frac{1}{2}$.

Proof. Set $k := |A|$. Then Lemma 6.5 asserts that the sum of elements in the same row is $2^k + \frac{k2^k}{n}(2^{n-k} - 1)$ for $k = 1, \dots, n-2$. We observe that if $k = 0$, then the previous formula gives the result 1, again in view of Lemma 6.5. So, we get

$$(13) \quad \lambda^+(\vec{C}_n) = \frac{1}{4^n} \left\{ \sum_{k=0}^{n-2} \binom{n}{k} \left[2^k + k \frac{2^n - 2^k}{n} \right] + 2^n(n+1) \right\}.$$

Observe now that $\lim_{n \rightarrow \infty} \frac{2^n(n+1)}{4^n} = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{4^n} \sum_{k=0}^{n-2} \binom{n}{k} 2^k = 0$, as $\sum_{k=0}^{n-2} \binom{n}{k} 2^k = \sum_{k=0}^n \binom{n}{k} 2^k - n2^{n-1} - 2^n = 3^n - n2^{n-1} - 2^n$. Moreover

$$\sum_{k=0}^{n-2} \binom{n}{k} \frac{k}{n} (2^n - 2^k) \leq \sum_{k=0}^{n-2} \binom{n}{k} \frac{k2^n}{n} = \frac{2^n}{n} \left[\sum_{k=0}^n k \binom{n}{k} - n(n-1) - n \right] = 2^{2n-1} - n2^n$$

Thus

$$\lim_{n \rightarrow \infty} \lambda^+(\vec{C}_n) \leq \lim_{n \rightarrow \infty} \frac{2^{2n-1} - n2^n}{4^n} = \frac{1}{2}.$$

Similarly, as $\sum_{k=0}^n \binom{n}{k} k = n2^{n-1}$, in view of (13) we get

$$\lambda^+(\vec{C}_n) = \frac{1}{4^n} \left\{ \sum_{k=0}^{n-2} \binom{n}{k} \left[k \frac{2^n}{n} + 2^k \left(1 - \frac{k}{n} \right) \right] + 2^n(n+1) \right\} \geq \frac{2^{2n-1} + 2^n(1+n)}{4^n}$$

Therefore

$$\lim_{n \rightarrow \infty} \lambda^+(\vec{C}_n) \geq \lim_{n \rightarrow \infty} \frac{1}{4^n} [2^{2n-1} + 2^n(1+n)] = \frac{1}{2}$$

and our conclusion follows. \square

7. CONCLUSIONS

In the present work we studied of some classical digraph families [21, 62, 73] by means of GrC techniques. More in detail, we investigated the out-adjacency matrix of a digraph as a specific case of Boolean information table. To this regard, we determined the out-dependency average and the out-accuracy average relatively to the directed n -cycle and the directed complete bipartite digraph. Moreover, we also computed the out-accuracy average for the n -directed path, observing some similarities with the case of the n -directed cycle.

The above analysis has implications related to both information sciences and RST on graphs, as highlighted in [20]. Therefore, the present paper represents a further contribution to the study of families of graphs and digraphs which are relevant in both discrete mathematics and computer science through the GrC and RST methods.

Finally, with regard to the future potential applications of the results obtained in the present work, in the following we provide a possible interpretation of some numerical parameters treated in this paper. Our simple example could be a starting point for ideas addressed to the development of the connections between GrC on digraphs and virological models.

In order to illustrate to the reader a possible real case where the GrC methods in digraph theory could be applied, we briefly discuss a digraph model derived from a possible viral transmission situation.

Let us consider as an example the analysis of the transmission of a virus when a certain number of potentially infected people are in contact in a restricted environment. In the model we give in Figure 1, we suppose that there are exactly eight people $w_0, w_1, \dots, w_6, w_7$ and that the existence of a direct arc from a vertex w_i into a vertex w_j means that individual w_i transmits a specific viral load to individual w_j . Let \vec{D} be the corresponding digraph drawing in Figure 1.

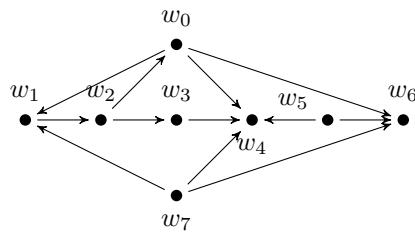


Figure 1: Viral Load Digraph

Take now the vertex set $A := \{w_0, w_4, w_6\}$ as a sample of people against which to evaluate the viral load transmission (in bound or outbound) of all individuals (including the same w_0, w_4 and w_6). To this end, we may easily obtain the induced indiscernibility partition given by $\pi_A(\vec{D}) = w_0w_5w_7|w_1w_4w_6|w_2|w_3$. Indeed, take

any vertex in the first block: notice that it transmits its viral load to both w_4 and w_6 . On the other hand, any vertex of the second block does not transmit its viral load to anyone between w_0 , w_4 and w_6 , and, finally, w_2 infects only w_0 , while w_3 infects only w_4 .

Let us modify the sample, passing for instance from the vertex set A to $B := \{w_2, w_3, w_4\}$. In this case we obtain a new indiscernibility partition, given by $\pi_B(\vec{D}) = w_0w_3w_5w_7|w_1|w_2|w_4w_6$. we consider now the probability that any two people having the same transmissibility behavior with respect to A , also have the same transmissibility behavior with respect to B . In our previous terminology, this is equivalent to determine the number $\gamma_A(B)$ (which is clearly $\gamma_A(B) = 5/8$ in the example just discussed).

Now, let us note that the previous measure is substantially of *local* type, as it is defined relatively to two fixed individual subsets A and B . Therefore, a natural question arises: what is the most natural number that provides a type of *global* dependency measure between two any arbitrary and not fixed vertex subsets of the digraph \vec{D} ? With regard to our previous theoretical premises, such a number agrees with $\lambda(\vec{D})$, and clearly it depends only on the digraph structure.

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