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COMPARISON INEQUALITIES OF BERNSTEIN-TYPE BETWEEN POLYNOMIALS WITH RESTRICTED ZEROS

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In this paper, we establish certain comparison inequalities of Bernstein-type for a linear operator between complex polynomials under certain constraints on their zeros. A variety of interesting results follow as special cases from our results.

1. INTRODUCTION

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and $P'(z)$ is the derivative of $P(z)$. The study of comparison inequalities that relate the norm between polynomials on the disk $|z| = R,\ R > 0$, and their various versions are a classical topic in analysis. Various results of majoriazation can be found in the comprehensive books of Milovanović et al. [**9**], Marden [**8**] and Rahman and Schmeisser [**11**], where some approaches to obtaining polynomial inequalities are developed on applying the methods and results of the geometric function theory. A classical majorization result due to Bernstein [**4**] is that, for two polynomials $f, F \in \mathcal{P}_n$ with $\deg f \leq \deg F$ and $F(z) \neq 0$ for $|z| > 1$, the majorization $|f(z)| \leq |F(z)|$ on the unit circle $|z| = 1$ implies the majorization of their derivatives $|f'(z)| \leq |F'(z)|$ on $|z| = 1$. In particular, this majorization result allows one to establish the famous Bernstein inequality [**3**] for the sup-norm on the unit circle: namely, if $P \in \mathcal{P}_n$, then it is true that

$$\max_{|z|=1}\big|P'(z)\big| \leq n \max_{|z|=1}\big|P(z)\big|. \tag{1}$$

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On the other hand, concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geq 1$, we have another classical result known as Bernstein-Walsh lemma ([**11**], Corollary 12.1.3), which states that, if $f, F \in \mathcal{P}_n$ with $\deg f \leq \deg F$ and $F(z) \neq 0$ for $|z| > 1$, the majorization $|f(z)| \leq |F(z)|$ on the unit circle $|z| = 1$ implies that $|f(z)| < |F(z)|$ for $|z| > 1$, unless $f(z) = e^{i\theta} F(z)$, $\theta \in \mathbb{R}$. From this, one can deduce that if $P \in \mathcal{P}_n$, then for $R \geq 1$,

$$\max_{|z|=R} \big|P(z)\big| \leq R^n \max_{|z|=1} \big|P(z)\big|. \tag{2}$$

The inequalities (1) and (2) are related with each other and it was observed by Bernstein [**4**] that (1) can be deduced from (2) by making use of Gauss-Lucas theorem and the proof of this fact was given by Govil et al. [**6**]. If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ with $P(z) \neq 0$ in $|z| < 1$, then (1) and (2) can be respectively replaced by

$$\max_{|z|=1} \big|P'(z)\big| \leq \frac{n}{2} \max_{|z|=1} \big|P(z)\big|, \tag{3}$$

and

$$\max_{|z|=R} \big|P(z)\big| \leq \frac{R^n + 1}{2} \max_{|z|=1} \big|P(z)\big|, \ R \geq 1. \tag{4}$$

Inequality (3) was conjectured by Erdös and later proved by Lax [**7**], whereas Ankeny and Rivlin [**1**] used (3) to prove (4). Over the last four decades many different authors produced a large number of different versions and generalizations of the above mentioned inequalities by introducing various operators which preserve such type of inequalities between polynomials. It is an interesting problem, as pointed out by Professor Q. I. Rahman to characterize all such operators and as part of this characterization, Rahman [**10**] (see also Rahman and Schmeisser [**11**], page 538-551) introduced a class B_n of operators B that maps $P \in \mathcal{P}_n$ into $B[P] \in \mathcal{P}_n$. Very recently, Rather et al. [**12**] considered a generalized B_n-operator $\mathcal{N}$ which carries $P \in \mathcal{P}_n$ into $\mathcal{N}[P] \in \mathcal{P}_n$, defined by

$$\mathcal{N}[P](z) := \sum_{j=0}^{m} \lambda_j \left(\frac{nz}{2}\right)^j \frac{P^{(j)}(z)}{j!}, \tag{5}$$

where $\lambda_j;\ j = 0, 1, 2, ..., m$, are such that all the zeros of

$$\phi(z) = \sum_{j=0}^{m} \binom{n}{j} \lambda_j z^j, \ m \leq n, \tag{6}$$

lie in the half plane

$$\Re z \leq n/4. \tag{7}$$

It is important to mention that by taking $\lambda_j = 0$ in (5) and (6) for $3 \leq j \leq m$, the operator $\mathcal{N}$ reduces to the B-operator. They established the following results concerning the upper bound of the modulus of $\mathcal{N}[P]$ for $|z| \geq 1$, thereby giving generalizations of (1) - (4).

Theorem 1. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in \mathcal{P}_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then*

$$|\mathcal{N}[P](z)| \leq |\mathcal{N}[f](z)| \text{ for } |z| \geq 1. \tag{8}$$

Equality in (8) holds for $P(z) = e^{i\alpha} f(z), \alpha \in \mathbb{R}$.

Theorem 2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then*

$$|\mathcal{N}[P](z)| \leq \frac{1}{2}\Big(|\mathcal{N}[\psi_n](z) + |\lambda_0|\Big) \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1, \tag{9}$$

where $\psi_n(z) = z^n$.

Equality in (9) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

For suitable choices of $\lambda_j;\ 0 \leq j \leq m$, Theorem 1 yields inequalities (1) and (2), whereas Theorem 2 yields inequalities (3) and (4).

While making an attempt towards the generalizations of the above inequalities, the author considered a more general problem of investigating the dependence of $|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]|$ on the maximum of $|P(z)|$ on $|z| = 1$ for every $|\beta| \leq 1$, $R \geq r \geq 1$, and develop a unified method for arriving at these results.

2. MAIN RESULTS

Here, we shall first establish the following comparison inequality between complex polynomials involving the operator $\mathcal{N}$, when the zeros of one of the polynomials are restricted. The obtained inequality gives compact generalizations of (1) and (2) and includes Theorem 1 as a special case.

Theorem 3. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in \mathcal{P}_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then for any complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \leq \Big|\mathcal{N}[f(Rz)] - \beta\mathcal{N}[f(rz)]\Big| \text{ for } |z| \geq 1. \tag{10}$$

Equality in (10) holds for $P(z) = e^{i\alpha} f(z), \alpha \in \mathbb{R}$. The following result immediately follows from Theorem 3, if we take $r = 1$.

Corollary 1. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in \mathcal{P}_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then for any complex number β with $|\beta| \leq 1$ and $R \geq 1$, we have*

$$\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(z)]\Big| \leq \Big|\mathcal{N}[f(Rz)] - \beta\mathcal{N}[f(z)]\Big| \text{ for } |z| \geq 1. \tag{11}$$

Equality in (11) holds for $P(z) = e^{i\alpha} f(z), \alpha \in \mathbb{R}$.

Remark 1. *Taking $\beta = 0$ in Corollary 1, we get Theorem 1. If in Theorem 3, we take $f(z) = Mz^n$, where $M = \max_{|z|=1}|P(z)|$, then we get the following result.*

Corollary 2. *If $P \in \mathcal{P}_n$, then for every $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \leq |R^n - \beta r^n||\mathcal{N}[\psi_n](z)| \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1, \tag{12}$$

where $\psi_n(z) = z^n$.

Equality in (12) holds for $P(z) = \gamma z^n, \gamma \neq 0$.

If in (12), after substituting the value of $\mathcal{N}[\psi_n](z)$, we get for every $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \\ &\leq |R^n - \beta r^n||z|^n \left|\sum_{k=0}^{m} \lambda_k \left(\frac{n}{2}\right)^k \binom{n}{k}\right| \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1, \end{aligned} \tag{13}$$

where λ_k ; $0 \leq k \leq m$, are such that all the zeros of $\phi(z)$ defined by (6) lie in the half plane (7). Taking $\lambda_k = 0,\ k = 1, 2, ..., m$, in (13) and noting that $\mathcal{N}[P](z) = \lambda_0 P(z)$, we get the following result.

Corollary 3. *If $P \in \mathcal{P}_n$, then for every $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$|P(Rz) - \beta P(rz)| \leq |R^n - \beta r^n||z|^n \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1. \tag{14}$$

Equality in (14) holds for $P(z) = \gamma z^n, \gamma \neq 0$.

If in (14), we take $\beta = r = 1$ and divide both sides of it by $R - 1$ and make $R \to 1$, we get

$$|P'(z)| \leq |z|^{n-1} \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1,$$

which in particular yields (1), whereas (2) is a special case of (14), if we take $\beta = 0$. Next, we prove the following extension of Theorem 2, which in turn provides generalizations of (3) and (4) as well.

Theorem 4. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \\ &\leq \frac{1}{2}\Big(|R^n - \beta r^n||\mathcal{N}[\psi_n](z)| - |1 - \beta||\lambda_0|\Big) \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1, \end{aligned} \tag{15}$$

where $\psi_n(z) = z^n$.

Equality in (15) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta| \neq 0$.

Remark 2. *For $\beta = 0$, Theorem 4 in particular gives Theorem 2 and for suitable choices of $\lambda_j;\ 0 \le j \le m$, it yields inequalities* (3) *and* (4) *as well.*

We now prove the following more refined result which besides strengthens Theorem 4, also provides extensions of Theorem 2 and some results of Aziz and Dawood [**2**].

Theorem 5. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \le 1$ and $R \ge r \ge 1$, we have*

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \\ &\quad \le \frac{1}{2}\Bigg\{\Big(|R^n - \beta r^n||\mathcal{N}[\psi_n](z)| + |1-\beta||\lambda_0|\Big)\max_{|z|=1}|P(z)| \\ &\quad - \Big(|R^n - \beta r^n||\mathcal{N}[\psi_n](z)| - |1-\beta||\lambda_0|\Big)\min_{|z|=1}|P(z)|\Bigg\} \text{ for } |z| \ge 1, \end{aligned} \tag{16}$$

where $\psi_n(z) = z^n$.

Equality in (16) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta| \neq 0$.

Again, as before, if in (16) after substituting the value of $\mathcal{N}[\psi_n](z)$, we get for every $|\beta| \le 1$, $R \ge r \ge 1$,

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \\ &\le \frac{1}{2}\Bigg\{\left[|R^n - \beta r^n||z|^n\left|\sum_{k=0}^{m}\lambda_k\left(\frac{n}{2}\right)^k\binom{n}{k}\right| + |1-\beta||\lambda_0|\right]\max_{|z|=1}|P(z)| \\ &- \left[|R^n - \beta r^n||z|^n\left|\sum_{k=0}^{m}\lambda_k\left(\frac{n}{2}\right)^k\binom{n}{k}\right| - |1-\beta||\lambda_0|\right]\min_{|z|=1}|P(z)|\Bigg\} \text{ for } |z| \ge 1, \end{aligned} \tag{17}$$

where $\lambda_k;\ 0 \le k \le m$, are such that all the zeros of $\phi(z)$ defined in (6) lie in the half plane (7). Taking $\lambda_k = 0;\ k = 1, 2, ..., m$, in (17) and noting that $\mathcal{N}[P](z) = \lambda_0 P(z)$, we get the following result which is of independent interest, because besides giving generalizations and refinements of (3) and (4), it also provides generalizations of some results of Aziz and Dawood [**2**].

Corollary 4. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \le 1$ and $R \ge r \ge 1$, we have*

$$\begin{aligned} &\big|P(Rz) - \beta P(rz)\big| \\ &\quad \le \frac{1}{2}\Bigg\{\Big(|R^n - \beta r^n||z|^n + |1-\beta|\Big)\max_{|z|=1}|P(z)| \\ &\quad - \Big(|R^n - \beta r^n||z|^n - |1-\beta|\Big)\min_{|z|=1}|P(z)|\Bigg\} \text{ for } |z| \ge 1. \end{aligned} \tag{18}$$

Equality in (18) holds for $P(z)=\alpha z^n+\beta, |\alpha|=|\beta|\neq 0$.

Remark 3. *Taking $\beta=r=1$ in (18) and divide both sides of it by $R-1$ and let $R\to 1$, we get in particular a result of Aziz and Dawood ([**2**], Theorem 4), whereas by taking $\beta=0$ and $r=1$ in (18), it yields a result ([**2**], Theorem 3) of Aziz and Dawood.*

A polynomial $P\in\mathcal{P}_n$ is said to be self-inversive if $P(z)=\zeta Q(z)$, $|\zeta|=1$, where as usual $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$. Finally, we prove the following result for self-inversive polynomial.

Theorem 6. *If $P\in\mathcal{P}_n$ is self-inversive, then for every $|\beta|\leq 1$ and $R\geq r\geq 1$, we have*

$$\begin{aligned}&\big|\mathcal{N}[P(Rz)]-\beta\mathcal{N}[P(rz)]\big|\\ &\leq\frac{1}{2}\Big(\big|R^n-\beta r^n\big|\big|\mathcal{N}[\psi_n](z)\big|+|1-\beta||\lambda_0|\Big)\max_{|z|=1}|P(z)| \text{ for } |z|\geq 1,\end{aligned}\tag{19}$$

where $\psi_n(z)=z^n$.

Equality in (19) holds for $P(z)=z^n+1$.

Remark 4. *For $\beta=0$, the above result in particular reduces to a result of Rather et al. ([**12**], Theorem 1.4). By taking $\lambda_k=0$; $k=1,2,..,m$, in (19) and noting that $\mathcal{N}[P](z)=\lambda_0 P(z)$, we easily get the following result, which as special cases, include some known polynomial inequalities for self-inversive polynomials.*

Corollary 5. *If $P\in\mathcal{P}_n$ is self-inversive, then for $R\geq r\geq 1$, we have*

$$\big|P(Rz)-\beta P(rz)\big|\leq\frac{1}{2}\Big(\big|R^n-\beta r^n\big||z|^n+|1-\beta|\Big)\max_{|z|=1}|P(z)| \text{ for } |z|\geq 1.\tag{20}$$

Equality in (20) holds for $P(z)=z^n+1$.

Remark 5. *Taking $\beta=r=1$ in (20) and dividing both sides of it by $R-1$ and letting $R\to 1$, we get that the inequality (3) is true for self-inversive polynomials as well, where as, for $\beta=0$, the inequality (20) shows that the inequality (4) also holds for self-inversive polynomials.*

3. AUXILIARY RESULTS

For the proofs of these theorems, we require the following lemmas.

Lemma 1. *If $P\in\mathcal{P}_n$ and $P(z)$ has all its zeros in $|z|\leq 1$, then for every $R\geq r\geq 1$, and $|z|=1$,*

$$\big|P(Rz)\big|\geq\left(\frac{R+1}{r+1}\right)^n\big|P(rz)\big|.$$

The proof of this lemma is similar to the proof of Lemma 2.1 of Govil et al. [**5**], and hence we omit the details.

If we take $r = s = 1$ and $\sigma = \frac{n}{2}$ in Theorem 1.1 of Rather et al. [**12**], we get the following:

Lemma 2. *If all the zeros of polynomial $f \in \mathcal{P}_n$ lie in $|z| \leq 1$, then all the zeros of $\mathcal{N}[f](z)$ defined by* (5) *also lie in $|z| \leq 1$.*

Lemma 3. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\left|\mathcal{N}[P(Rz)] - \beta \mathcal{N}[P(rz)]\right| \leq \left|\mathcal{N}[Q(Rz)] - \beta \mathcal{N}[Q(rz)]\right| \text{ for } |z| \geq 1, \tag{21}$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof. Since $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, therefore, $|P(z)| = |Q(z)|$ for $|z| = 1$. Also, since $P(z) \neq 0$ in $|z| < 1$ and hence $Q(z) \neq 0$ in $|z| > 1$, it follows by the Maximum Modulus Principle that $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. Hence, the inequality (21) holds trivially for $R = r$, therefore, we now assume that $R > r$. By Rouché's theorem, the polynomial $g(z) = P(z) - \delta Q(z)$, with $|\delta| > 1$, has all its zeros in $|z| \leq 1$. If we apply Lemma 1 to the polynomial $g(z)$, we get for $R > r \geq 1$ and for each $0 \leq \theta < 2\pi$,

$$\left|g(Re^{i\theta})\right| \geq \left(\frac{R+1}{r+1}\right)^n \left|g(re^{i\theta})\right|. \tag{22}$$

Since $g(Re^{i\theta}) \neq 0$ and $\frac{R+1}{r+1} > 1$, for every $R > r \geq 1$, it follows from (22) that

$$\begin{aligned} \left|g(Re^{i\theta})\right| &> \left(\frac{r+1}{R+1}\right)^n \left|g(Re^{i\theta})\right| \\ &\geq \left|g(re^{i\theta})\right|, \end{aligned}$$

which is equivalent to

$$|g(rz)| < |g(Rz)| \text{ for } |z| = 1 \text{ and } R > r \geq 1. \tag{23}$$

If β is any complex number with $|\beta| \leq 1$, then from (23), it follows that $|\beta g(rz)| < |g(Rz)|$ for $|z| = 1$ and $R > r \geq 1$. Since $g(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$, by Rouché's theorem again, the polynomial

$$\begin{aligned} T(z) &= g(Rz) - \beta g(rz) \\ &= P(Rz) - \beta P(rz) - \delta\Big(Q(Rz) - \beta Q(rz)\Big), \end{aligned}$$

has all its zeros in $|z| < 1$, for every $|\delta| > 1$ and $R > r \geq 1$. Applying Lemma 2 to $T(z)$ and noting that $\mathcal{N}$ is a linear operator, we conclude that the polynomial

$$\mathcal{N}[T](z) = \mathcal{N}[P(Rz)] - \beta \mathcal{N}[P(rz)] - \delta\Big(\mathcal{N}[Q(Rz)] - \beta \mathcal{N}[Q(rz)]\Big),$$

has all its zeros in $|z|<1$, for every $|\delta|>1$ and $R>r\geq 1$. This implies that

$$\big|\mathcal{N}[P(Rz)]-\beta\mathcal{N}[P(rz)]\big|\leq\big|\mathcal{N}[Q(Rz)]-\beta\mathcal{N}[Q(rz)]\big| \text{ for } |z|\geq 1.$$

This completes the proof of Lemma 3. □

Lemma 4. *If $P\in\mathcal{P}_n$, then for every $|\beta|\leq 1$ and $R\geq r\geq 1$, we have*

$$\begin{aligned}\big|\mathcal{N}[P(Rz)]-\beta\mathcal{N}[P(rz)]\big|&+\big|\mathcal{N}[Q(Rz)]-\beta\mathcal{N}[Q(rz)]\big|\\ &\leq\big(|R^n-\beta r^n||\mathcal{N}[\psi_n](z)|+|1-\beta||\lambda_0|\big)\max_{|z|=1}|P(z)| \textit{ for } |z|\geq 1,\end{aligned}$$

where $\psi_n(z)=z^n$ and $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof. Let $M=\max\limits_{|z|=1}\big|P(z)\big|$, then $\big|P(z)\big|\leq M$ for $|z|=1$. If α is any complex number with $|\alpha|>1$, the polynomial $P(z)+\alpha M$ does not vanish in $|z|<1$. Applying Lemma 3 to the polynomial $P(z)+\alpha M$ and noting that $\mathcal{N}$ is a linear operator with $\mathcal{N}[1]=\lambda_0$, we obtain for $|\beta|\leq 1,\ R\geq r\geq 1$,

$$\begin{aligned}&\Big|\mathcal{N}[P(Rz)]-\beta\mathcal{N}[P(rz)]+\alpha(1-\beta)\lambda_0 M\Big|\\ &\qquad\leq\Big|\mathcal{N}[Q(Rz)]-\beta\mathcal{N}[Q(rz)]+\overline{\alpha}(R^n-\beta r^n)M\mathcal{N}[\psi_n](z)\Big| \text{ for } |z|\geq 1,\end{aligned}\tag{24}$$

where $\psi_n(z)=z^n$.
Choosing the argument of α on the right hand side of (24) suitably, we get

$$\begin{aligned}&\Big|\mathcal{N}[P(Rz)]-\beta\mathcal{N}[P(rz)]\Big|-|\alpha||1-\beta||\lambda_0|M\\ &\qquad\leq|\alpha||R^n-\beta r^n||\mathcal{N}[\psi_n](z)|M-\Big|\mathcal{N}[Q(Rz)]-\beta\mathcal{N}[Q(rz)]\Big| \text{ for } |z|\geq 1.\end{aligned}\tag{25}$$

The fact that the right hand side of (25) is non-negative follows from Corollary 2, if we apply it to the polynomial $Q\in\mathcal{P}_n$. Now, if in (25), we make $|\alpha|\to 1$, we get for $|\beta|\leq 1$ and $R\geq r\geq 1$,

$$\begin{aligned}\Big|\mathcal{N}[P(Rz)]-\beta\mathcal{N}[P(rz)]\Big|&+\Big|\mathcal{N}[Q(Rz)]-\beta\mathcal{N}[Q(rz)]\Big|\\ &\leq(|R^n-\beta r^n||\mathcal{N}[\psi_n](z)|+|1-\beta||\lambda_0|)\max_{|z|=1}|P(z)| \text{ for } |z|\geq 1.\end{aligned}$$

This proves Lemma 4 completely. □

4. PROOFS OF THEOREMS

Proof of Theorem 3. As the result holds trivially for $R = r$, by virtue of Theorem 1, therefore, we now assume that $R > r$. By Rouché's theorem, the polynomial $g(z) = P(z) - \delta f(z)$, with $|\delta| > 1$, has all its zeros in $|z| \leq 1$. On applying Lemmas 1 and 2 and proceeding similarly as in the proof of Lemma 3, the result follows. Hence, we omit the details. □

Proof of Theorem 4. Since $P(z) \neq 0$ in $|z| < 1$. If $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then by Lemma 3, we have

$$\big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\big| \leq \big|\mathcal{N}[Q(Rz)] - \beta\mathcal{N}[Q(rz)]\big| \text{ for } |z| \geq 1.$$

The above inequality in conjunction with Lemma 4 gives (15). This completes the proof of Theorem 4. □

Proof of Theorem 5. Recall that $P(z) \neq 0$ in $|z| < 1$. If $m = \min_{|z|=1} |P(z)|$, them $m \leq |P(z)|$ for $|z| = 1$. If $P(z)$ has some zeros on $|z| = 1$, then clearly $m = 0$ and in this case the result follows from Theorem 4. Henceforth, we suppose that $P(z)$ has no zeros in $|z| \leq 1$, so that $m > 0$. Then clearly, for any α with $|\alpha| < 1$, the polynomial $h(z) = P(z) + \alpha m z^n$ has no zeros $|z| \leq 1$, and that the polynomial $t(z) = z^n \overline{h\left(\frac{1}{\bar{z}}\right)} = Q(z) + \overline{\alpha} m$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, has all its zeros in $|z| \leq 1$. Therefore, on applying Lemma 3 and noting that $\mathcal{N}$ is a linear operator with $\mathcal{N}[1] = \lambda_0$, we get for every $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\Big|\mathcal{N}[h(Rz)] - \beta\mathcal{N}[h(rz)]\Big| \leq \mathcal{N}[t(Rz)] - \beta\mathcal{N}[t(rz)]| \text{ for } |z| \geq 1.$$

Equivalently,

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)] + \alpha m(R^n - \beta r^n)\mathcal{N}[\psi_n](z)\Big| \\ &\qquad \leq \Big|\mathcal{N}[Q(Rz)] - \beta\mathcal{N}[Q(rz)] + \overline{\alpha} m(1-\beta)\lambda_0\Big| \text{ for } |z| \geq 1, \end{aligned} \tag{26}$$

where $\psi_n(z) = z^n$.
If we now choose the argument of α in the left hand side of (26) such that

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)] + \alpha m(R^n - \beta r^n)\mathcal{N}[\psi_n](z)\Big| \\ &\qquad = \Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| + m|\alpha||R^n - \beta r^n||\mathcal{N}[\psi_n](z)|, \end{aligned}$$

we get for every $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\begin{aligned} &\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| + m|\alpha||R^n - \beta r^n||\mathcal{N}[\psi_n](z)| \\ &\qquad \leq \Big|\mathcal{N}[Q(Rz)] - \beta\mathcal{N}[Q(rz)]\Big| + m|\alpha||1-\beta||\lambda_0| \text{ for } |z| \geq 1. \end{aligned} \tag{27}$$

If in (27), we let $|\alpha| \to 1$, we obtain for every $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| \leq \Big|\mathcal{N}[Q(Rz)] - \beta\mathcal{N}[Q(rz)]\Big| - \Big(|R^n - \beta r^n||\mathcal{N}[\psi_n](z)| - |1-\beta||\lambda_0|\Big)m \text{ for } |z| \geq 1.$$

The above inequality in conjunction with Lemma 4 gives (16) and this completes the proof of Theorem 5. □

Proof of Theorem 6. If $P \in \mathcal{P}_n$ is a self-inversive polynomial, therefore $P(z) = \zeta Q(z)$, where $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ and $|\zeta| = 1$. It gives for every $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\Big|\mathcal{N}[P(Rz)] - \beta\mathcal{N}[P(rz)]\Big| = \Big|\mathcal{N}[Q(Rz)] - \beta\mathcal{N}[Q(rz)]\Big| \text{ for all } z.$$

This when combined with Lemma 4 yields (19). This completes the proof of Theorem 6. □

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