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CONNECTED DOMINATION GAME: PREDOMINATION, STALLER-START GAME, AND LEXICOGRAPHIC PRODUCTS

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The connected domination game was introduced in 2019 by Borowiecki, Fiedorowicz and Sidorowicz as another variation of the domination game. We answer a problem from their paper regarding the relation between the number of moves in a game where Dominator/Staller starts the game. Additionally, we study the relation to the diameter and present graphs with small game connected domination number. We also determine the value on the lexicographic products, and consider the effect of predomination of a vertex.

1. INTRODUCTION

The *connected domination game* was introduced in [**1**] as a game played on a connected graph G by two players, Dominator and Staller, with the following rules. Dominator's goal is to finish the game with the smallest possible number of moves, while Staller aims to prolong the game. The players alternately select vertices such that each move dominates at least one vertex that was not dominated by the previous moves, and the selected vertex is connected to at least one already played vertex. The game ends when no more moves are possible, i.e. the selected vertices form a connected dominating set. Note that we will only consider the game on connected graphs.

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If both players play optimally, the number of moves on a given connected graph is the *game connected domination number*[1] $\gamma_{\rm cg}(G)$ if Dominator starts the game on G. In this case we say for short that a *D-game* is played on G. If Staller starts the game (shortly, *S-game*), the number of moves is the *Staller-start game connected domination number* $\gamma'_{\rm cg}(G)$.

The connected domination game is a variation of the classical domination game, which was introduced in 2010 by Brešar, Klavžar and Rall [**3**] and has been widely studied ever since (see for example [**11, 16, 20, 21, 24, 25, 26, 27**]), along with the total domination game which was introduced in [**12**] (see also [**7, 10, 14, 17, 18**]). Recently, three other variations of the game have been introduced [**2**].

In the first paper on the connected domination game [**1**], the game connected domination number of trees and 2-paths is determined, along with an upper bound on 2-trees. The game is studied on the Cartesian product graphs. Some results about the Staller-start and Staller-pass game are also presented. In the Staller-pass game Staller is allowed to pass some moves instead of playing a vertex. The moves when she passes do not count into the number of moves in the game. The number of moves in a game on G when Staller can pass at most k times is $\hat{\gamma}_{cg}^k(G)$. It was proved in [**1**] that $\gamma_{\rm cg}(G) \le \hat{\gamma}_{cg}^k(G) \le \gamma_{\rm cg}(G) + k$. Several additional results about the connected domination game played on Cartesian product graphs are established in [**5**]. The total variation of the game is defined and studied in [**6**].

Another important variation of the connected domination game presented in [**1**] is the *connected domination game with Chooser* which turns out to be an extremely useful tool for proving some bounds for the game connected domination number. The rules of the game are essentially the same as in the usual game, except that there is another player, Chooser, who can make zero, one or more moves after any move of Dominator or Staller. The rules for his move to be legal are the same as for Dominator and Staller. Note that Chooser has no specific goal, he can help either Dominator or Staller. For later use we also state the Chooser Lemma from [**1**].

Lemma 1 (Chooser Lemma)**.** *Consider the connected domination game with Chooser on a connected graph G. Suppose that in the game Chooser picks k vertices, and that both Dominator and Staller play optimally. Then at the end of the game the number of played vertices is at most $\gamma_{\rm cg}(G) + k$ and at least $\gamma_{\rm cg}(G) - k$.*

We now recall some basic definitions. The *(closed) neighborhood* of a vertex v in a graph G is denoted by $N[v]$. The (closed) neighborhood of a set of vertices $S \subseteq V(G)$ is $N[S] = \bigcup_{v\in S} N[v]$. Similarly, the *open neighborhood* of v is $N(v) = N[v] \setminus \{v\}$. A set $S \subseteq V(G)$ is a *dominating set* of a graph G if $N[S] = V(G)$. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$. The *connected dominating set* is a dominating set S which induces a connected subgraph. The smallest cardinality of such set is the *connected domination number*

[1] Note that the terminology "connected game domination number" is also used in the literature, but we use "game connected domination number" to be consistent with the terminology of other domination games (see [**4**]).

$\gamma_c(G)$. Recall also that a *join* of graphs G and H is a graph with vertex set $V(G) \cup V(H)$ and edges $E(G) \cup E(H) \cup \{gh\ ;\ g \in V(G), h \in V(H)\}$. Additionally, we denote $[n] = \{1, \ldots, n\}$, and $\Delta(G)$ as the maximum degree of vertices of G. Sometimes we will use the notation $u \sim_G v$ or $u \sim v$ to denote that vertices u and v are adjacent in graph G.

Let G and H be graphs. The *lexicographic product* $G[H]$ of graphs G and H has the vertex set $V(G) \times V(H)$ and the edges $(g_1, g_2) \sim (h_1, h_2)$ if $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Note that $G[H]$ can be obtained by substituting each vertex g of G with a copy of H (denoted by H_g) and connecting all vertices in H_{g_1} with all vertices in H_{g_2} if and only if $g_1g_2 \in E(G)$. The *Cartesian product* $G \,\square\, H$ of G and H has the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$.

In this paper we first characterize graphs with small game connected domination number and observe a relation between the diameter of a graph and values of $\gamma_{cg}(G)$, $\gamma'_{cg}(G)$. Next we solve Problem 1 from [**1**] regarding the difference between $\gamma_{cg}(G)$ and $\gamma'_{cg}(G)$. In the next section, we discuss the connected domination game on the lexicographic product of graphs. The last topic we study is the predomination of vertices and its effect on the game connected domination number. We conclude the paper with a discussion about possibilities how to define connected game domination critical graphs.

2. GRAPHS WITH SMALL GAME CONNECTED DOMINATION NUMBER AND THE RELATION TO THE DIAMETER OF A GRAPH

Motivated by [**23**] where graphs with small game domination numbers were studied, we consider graphs with the game connected domination number 1 and 2. (The characterization of graphs with game connected domination number 3 seems rather technical, thus we omit it.)

Let G be a graph and $v \in V(G)$. Recall that v is *universal* if $N[v] = V(G)$. The vertex v is 2-*universal* if $\mathrm{ecc}(v) = 2$ and there is a join between $S_2(v)$ and the neighbors of $S_2(v)$ in $S_1(v)$ (see Figure 1). Let $U(v)$ denote vertices in $S_1(v)$ which have no neighbors in $S_2(v)$. Note that a 2-universal vertex is defined similarly as a 2-dense vertex in [**23**], but without an additional condition that $S_2(v)$ induces a clique. Observe that if Dominator's first move on G is on v, then $U(v)$ are all unplayable, since they have no undominated neighbors.

Proposition 2. *Let G be a graph.*

1. $\gamma_{cg}(G) = 1 \iff$ *G has a universal vertex;*
2. $\gamma'_{cg}(G) = 1 \iff$ *G is complete;*
3. $\gamma_{cg}(G) = 2 \iff$ *G has a 2-universal vertex, but not a universal vertex;*

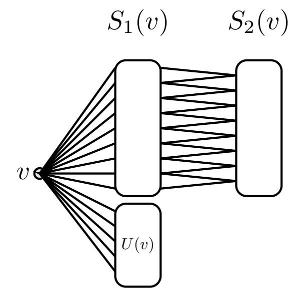

Figure 1: A schematic presentation of a 2-universal vertex v.

4. $\gamma'_{\rm cg}(G) = 2 \iff$ *G is not complete and every vertex lies in a connected dominating set of order* 2.

Proof. Parts (1) and (2) are obvious.

To prove part (3), first assume that $\gamma_{\rm cg}(G) = 2$ and let v be the optimal first move of Dominator. Clearly, ${\rm ecc}(v) \geq 2$, otherwise the connected domination game would finish in one move. Similarly, we conclude that there is no universal vertex. On the other hand, since Staller's first and final move can only be on $S_1(v)$, we have ${\rm ecc}(v) \leq 2$. The only playable vertices for Staller lie in $S_1(v) \setminus U(v)$. Since she ends the game no matter which vertex she plays, there must be a join between $S_1(v) \setminus U(v)$ and $S_2(v)$. Note that $S_1(v) \setminus U(v)$ are exactly neighbors of $S_2(v)$ in $S_1(v)$.

To conclude the proof of part (3), suppose that v is a 2-universal vertex in G and that G does not contain a universal vertex. The latter implies that $\gamma_{\rm cg}(G) \geq 2$. Moreover, Dominator can start the game on v, thus forcing Staller to finish the game in her first move, hence we have $\gamma_{\rm cg}(G) \leq 2$.

The proof of part (4) follows from the observation that, $\gamma'_{\rm cg}(G) = 2$ if and only if G is not complete and for every $u \in V(G)$ there exists a playable neighbor $v \in N(u)$ such that $\{u, v\}$ is a connected dominating set of G. $\square$

Similarly as in [**23**] for the domination game, we can find the relation between diameter of the graph and its game connected domination number.

Proposition 3. *For a connected graph G it holds that*

$$\mathrm{diam}(G) \leq \gamma_{\rm cg}(G) + 1 \qquad \text{and} \qquad \mathrm{diam}(G) \leq \gamma'_{\rm cg}(G).$$

Proof. Let $\mathrm{diam}(G) = k$ and let P be the longest induced path in G, $P = v_0 v_1 \dots v_k$. Let C be the set of vertices played in the course of a connected domination game on G, and let C' be the subgraph of G induced on C.

As C is a dominating set of G, there exists a vertex $x \in C$ such that $x = v_0$ or $x \sim_G v_0$. Similarly, there exists a vertex $y \in C$ such that $y = v_k$ or $y \sim_G v_k$. As no path between v_0 and v_k is shorter than P, it holds that $d_G(x, y) \geq k-2$. But we also have $d_{C'}(x, y) \geq d_G(x, y)$. As C' is connected, it follows that $|V(C')| = |C| \geq k-1$. Hence, $\gamma_{\rm cg}(G) \geq {\rm diam}(G) - 1$.

To prove the result for the S-game consider the following strategy of Staller. She starts the game on v_0 and then plays optimally. As the whole path P must be dominated at the end of the game by a connected set, at least k vertices are played. Hence, $\gamma'_{\rm cg}(G) \geq {\rm diam}(G)$. □

Note that the bound in Proposition 3 is best possible, as $\gamma_{\rm cg}(P_n) = n - 2 = {\rm diam}(P_n) - 1$ and $\gamma'_{\rm cg}(P_n) = n - 1 = {\rm diam}(P_n)$. On the other hand, the game connected domination number cannot be bounded from above by the diameter of a graph. The following proposition shows that given a diameter r there exists a graph with arbitrary large game connected domination number. Note that if $n_1, \ldots, n_r \geq 2$, then ${\rm diam}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) = r$.

Proposition 4. *If $r \geq 2$, $n_r \geq \cdots \geq n_1 \geq 2$ and $n_r \geq 2n_1 \cdots n_{r-1}$, then*

$$\gamma_{\rm cg}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) = 2n_1 \cdots n_{r-1} - 1 \quad \textit{and}$$
$$\gamma'_{\rm cg}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) = 2n_1 \cdots n_{r-1} - 2.$$

Proof. Iteratively using [**1**, Theorem 5] we have $\gamma_{\rm cg}((K_{n_1} \,\square\, \cdots \,\square\, K_{n_{r-1}}) \,\square\, K_{n_r}) \leq 2n_1 \cdots n_{r-1} - 1$. To prove the equality, consider the following strategy of Staller. She always plays in a copy of K_{n_r} where Dominator made his first move. This is legal until the game is finished and ensures that at least $\min\{n_r, 2n_1 \cdots n_{r-1} - 1\} = 2n_1 \cdots n_{r-1} - 1$ moves are played.

Similarly, we prove the result for the S-game. If Dominator plays all his moves in the copy of $K_{n_1} \,\square\, \cdots \,\square\, K_{n_{r-1}}$ where Staller made her first move, he ensures that $\gamma'_{\rm cg}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) \leq 2n_1 \cdots n_{r-1} - 2$. If Staller plays her first move on an arbitrary vertex and all her other moves in the copy of K_{n_r} where Dominator made his first move, she ensures that $\gamma'_{\rm cg}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) \geq 2n_1 \cdots n_{r-1} - 2$. □

3. THE UPPER BOUND FOR STALLER-START GAME CONNECTED DOMINATION NUMBER

For the classical domination game, we have $|\gamma_{\rm g}(G) - \gamma'_{\rm g}(G)| \leq 1$ [**3, 22**]. An analogous result also holds for the total domination game [**12**]. For the connected domination game it holds that

$$\gamma'_{\rm cg}(G) \geq \gamma_{\rm cg}(G) - 1$$

for all connected graphs G [**1**, Theorem 8]. Moreover, the bound is tight. No upper bound for $\gamma'_{\rm cg}(G)$ was previously known. Below we present the solution to the following problem.

Problem 5 ([**1**], Problem 1)**.** Find the maximum k for which there exists a graph G satisfying $\gamma'_{\rm cg}(G) = \gamma_{\rm cg}(G) + k$.

In the following we show that such k does not exist. However, for a fixed graph it can be bounded by $\gamma_{\rm cg}(G)$.

Theorem 6. *For every connected graph G it holds that*

$$\gamma'_{\rm cg}(G) \leq 2\gamma_{\rm cg}(G).$$

Proof. Let $\gamma_{\rm cg}(G) = k$ and let C be the set of played vertices in a D-game on G. Thus C is connected, and for every vertex $v \in V(G)$ it holds that $v \in C$ or v has a neighbor in C. Consider the following strategy for Dominator in the S-game on G. He plays vertices from C until the game is over. Since C is a connected dominating set, he can always chose a legal move from C which is adjacent to an already played vertex, and he also ensures that the game finishes after at most $2|C| = 2k$ moves. □

We now construct a graph that attains the equality in Theorem 6. Let G_n, $n \geq 2$, be a graph with vertices

$$V(G_n) = \{u_0, \ldots, u_n, x_1, y_1, z_1, \ldots, x_{n-1}, y_{n-1}, z_{n-1}\}$$

and edges $u_i \sim u_{i+1}$ for $i \in \{0, \ldots, n-1\}$, $u_i \sim x_i \sim y_i \sim z_i$ and $u_{i+1} \sim x_i, y_i, z_i$ for $i \in \{1, \ldots, n-1\}$. See Figure 2 for the graph G_6.

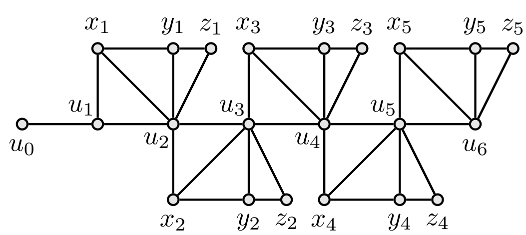

Figure 2: The graph G_6.

Proposition 7. *If $n \geq 2$, then $\gamma_{\rm cg}(G_n) = n$ and $\gamma'_{\rm cg}(G_n) = 2n$.*

Proof. We first prove that $\gamma_{\rm cg}(G_n) = n$. Notice that $\gamma_{\rm c}(G_n) = n$, as $\{u_1, \ldots, u_n\}$ is the smallest connected dominating set. Hence $\gamma_{\rm cg}(G_n) \geq \gamma_{\rm c}(G_n) = n$ by [**1**, Theorem 1].

Observe the following strategy of Dominator. Dominator starts on u_n. The only playable vertex is u_{n-1}, hence Staller plays it. Now Dominator plays u_{n-2} and Staller is forced to reply on u_{n-3}. This continues until the vertex u_1 is played and the game is finished after the nth move. Hence $\gamma_{\rm cg}(G_n) \leq n$.

Next, we prove that $\gamma'_{\rm cg}(G_n) = 2n$. It follows from Theorem 6 and the equality $\gamma_{\rm cg}(G_n) = n$ that $\gamma'_{\rm cg}(G_n) \leq 2n$. Consider the following strategy of Staller to show that the equality is attained. But first set $I_1 = \{u_1, x_1\}$ and $I_j = \{y_{j-1}, u_j, x_j\}$ for $j \in \{2, \dots, n-1\}$.

Staller starts the game on u_0 and thus Dominator can only reply on u_1. From now on, Staller's strategy is to play on $V(G) \setminus \{u_2, \dots, u_n\}$ whenever possible. By this she assures that at least two vertices are played on each I_j, $j \in \{1, \dots, n-1\}$.

Staller's second move is x_1, thus two moves are made on I_1. We now prove that at least two moves are made on I_k, $k \in \{2, \dots, n-2\}$. If Dominator plays u_k in the course of the game, then the vertices x_k, y_k, z_k, u_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}, $u_{k+2}, \dots$ have not been played yet (as Staller started on u_0 and the set of all played vertices must be connected). Hence Staller can reply on x_k, making a second move on I_k.

If Staller plays u_k in the course of the game, then the vertices x_k, y_k, z_k, u_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}, $u_{k+2}, \dots$ have not been played yet. But as Staller's strategy is to avoid playing on $\{u_2, \dots, u_n\}$, this means that she was forced to play on u_k. Thus she was not able to play on $\{x_{k-1}, y_{k-1}, z_{k-1}\}$, which could only happen if Dominator played y_{k-1} just before her move. Hence Staller's move on u_k was already the second move on I_k.

Additionally, to dominate the vertex z_{n-1}, at least one vertex from $\{u_n, y_{n-1}, z_{n-1}\}$ has to be played before the game ends. Hence, considering the first move of Staller on u_0, at least two moves on each I_k and the last move to dominate z_{n-1}, this strategy assures that at least $1 + 2(n-1) + 1 = 2n$ moves are played. □

Note that the family G_n can be simplified to a "smaller" family, which still has the desired property (see [**8**]).

4. LEXICOGRAPHIC PRODUCTS

In this section we present the game connected domination numbers in D- and S-games on the lexicographic product of graphs (for definition see the Introduction). Note that no such result is known for the (total) domination game. We first prove a lemma that will be useful in both games. We say that a first move played on some copy of H is a *new move* and all next moves on the same copy of H are called *duplicate moves*. Note that the first move on the graph $G[H]$ is always a new move, and if $\gamma_{\rm cg}(H) \geq 2$, then the second move in the game can be a duplicate move.

Lemma 8. *In a connected domination game on $G[H]$ it holds that after two new moves are played, no duplicate moves are possible.*

Proof. Assume that at least two new moves were already played in a connected domination game on $G[H]$. Let $S = \{v \in V(G);$ a new move was played on $H_v\}$. Clearly, $|S| \geq 2$ and S induces a connected graph. Hence all vertices in copies H_x

for $x \in N[S]$ are already dominated, thus no other vertex in the copies H_v, $v \in S$, is playable. □

To prove the general result for $G[H]$ we need to introduce another variation of the connected domination game – a game when Staller skips her first move, i.e. Dominator plays two moves, and only then the players start to alternate moves. We denote the number of moves in such a game, when both players play optimally, with $\widetilde{\gamma}_{\rm cg}(G)$, and call the game the *Staller-first-skip connected domination game*. Analogously, we define the *Dominator-first-skip connected domination game* as the Staller-start game in which Dominator skips his first move, i.e., first Staller plays two moves and after that the players alternate taking moves. The number of moves in such a game when both players play optimally is $\widetilde{\gamma}'_{\rm cg}(G)$.

Lemma 9. *For a connected graph G it holds that*

(i) $\gamma_{\rm cg}(G) - 1 \le \widetilde{\gamma}_{\rm cg}(G) \le \gamma_{\rm cg}(G) + 1$,

(ii) $\gamma'_{\rm cg}(G) - 1 \le \widetilde{\gamma}'_{\rm cg}(G) \le \gamma'_{\rm cg}(G) + 1$.

Proof. *(i)* To prove the upper bound, consider the following strategy of Dominator. While the Staller-first-skip game is played on G, he imagines a connected domination game with Chooser on G. Staller is playing optimally in the real game and Dominator is playing optimally in the imagined game. He plays his optimal first move in the imagined game and copies the move to the real game. In the real game he plays an arbitrary legal move and copies it to the imagined game as Chooser's move. From now on he copies each move of Staller from the real to the imagined game, selects his optimal reply in the imagined game and copies it to the real game. Throughout this process the same set of vertices has been played in both games, thus all the copied moves are legal.

The same number of moves was played in the real and in the imagined game, denote it by m. As Staller played optimally in the real game, we have $m \ge \widetilde{\gamma}_{\rm cg}(G)$. Since Dominator player optimally in the imagined game and Chooser made one move, by the Chooser Lemma 1 we have $m \le \gamma_{\rm cg}(G) + 1$. This yields the desired bound $\widetilde{\gamma}_{\rm cg}(G) \le \gamma_{\rm cg}(G) + 1$.

Similarly, we prove the lower bound with the following strategy of Staller. Parallel to the real Staller-first-skip game played on G, she imagines a connected domination game with Chooser on G. Dominator is playing optimally in the real game and Staller is playing optimally in the imagined game. She copies the first move of Dominator from the real game to the imagined game as his first move there. But she copies his second move in the real game as a move of Chooser in the imagined game. From here on, she replies optimally in the imagined game and copies her move to the real game. She also copies each Dominator's move in the real game to the imagined game. Similarly as above, we use the Chooser Lemma and conclude that $\widetilde{\gamma}_{\rm cg}(G) \ge \gamma_{\rm cg}(G) - 1$.

Part *(ii)* is proved analogously. □

The following examples (which have already been studied in [**1**]) demonstrate that the bounds from Lemma 9(i) cannot be improved.

Example 10. 1. For $n \geq 3$ it clearly holds that $\gamma_{\rm cg}(P_n) = \widetilde{\gamma}_{\rm cg}(P_n) = n - 2$.

2. Let F_1 be the fan with $n \geq 7$ vertices and let F_i be obtained from F_{i-1} by identifying one of the vertices of degree 2 and its neighbor of degree 3 in F_1 and F_{i-1}, see Figure 3. Recall from [**1**] that for $i \geq 1$ and all $n \geq 7$, it holds that $\gamma_{\rm cg}(F_i) = 2i - 1$. Staller's strategy in a Staller-pass game in F_i from [**1**] also implies that $\widetilde{\gamma}_{\rm cg}(F_i) = 2i = \gamma_{\rm cg}(F_i) + 1$ for $i \geq 2$.

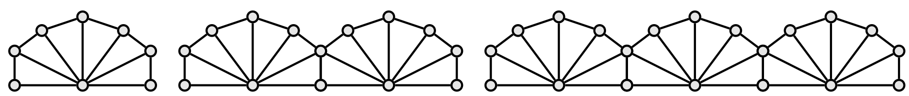

Figure 3: Graphs F_1, F_2 and F_3.

3. Let H_0 be the graph from the left of Figure 4. Let H_i be obtained by identifying two vertices in H_0 and H_{i-1} in the same manner as above with F_i (see Fig. 4). From [**1**] it follows that for all $i \geq 1$, $\gamma_{\rm cg}(H_i) = 4i + 2$ and from Chooser's strategy in [**1**] we can conclude that $\widetilde{\gamma}_{\rm cg}(H_i) = 4i + 1 = \gamma_{\rm cg}(H_i) - 1$.

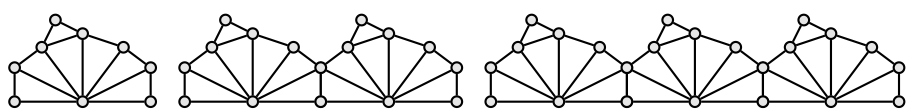

Figure 4: Graphs H_0, H_1 and H_2.

Theorem 11. *If G is a connected graph and H is a graph, then*

$$\gamma_{\rm cg}(G[H]) = \begin{cases} \widetilde{\gamma}_{\rm cg}(G) + 1; & \gamma_{\rm cg}(H) \geq 2 \text{ and } |V(G)| \geq 2, \\ \gamma_{\rm cg}(G); & \gamma_{\rm cg}(H) = 1, \\ \gamma_{\rm cg}(H); & |V(G)| = 1. \end{cases}$$

Proof. First consider the case $\gamma_{\rm cg}(H) \geq 2$ and $|V(G)| \geq 2$. Dominator's strategy is to start on a vertex in H_{d_1}, where d_1 is his optimal first move from the Staller-first-skip connected domination game on G. If Staller replies with a duplicate move on H_{d_1}, then Dominator's strategy is to reply on H_{d_2}, where d_2 is the optimal second move from the Staller-first-skip game on G. After this move, no more duplicate moves are possible. Hence at most $\widetilde{\gamma}_{\rm cg}(G) + 1$ moves are played.

If Staller replies with a new move, say with a vertex in H_{s_1}, then Lemma 8 states that a game on $G[H]$ is essentially just a normal D-game on G. Dominator's strategy from here on is the following. The real game is played on G, where moves

d_1, s_1 are considered to be already played, while Dominator imagines a Staller-first-skip game on G, where d_1 is considered to be already played. Staller plays optimally in the real game and Dominator plays optimally in the imagined game. The set of played vertices in the real (resp. imagined) game is denoted by D_R (resp. D_I), where d_1, s_1 are also considered to be in D_R and d_1 is considered to be in D_I. Dominator plays in such a way (described below), that the following property is preserved until the real game is finished:

$$D_I \subseteq D_R. \tag{1}$$

The property holds at the beginning. Notice that in both games it is Dominator's turn to play next.

At a stage of the game, Dominator selects his optimal reply d_i in the imagined game. If d_i is a legal move in the real game, he plays it and (1) remains valid. Otherwise, since (1) held before he played d_i in the imagined game, the vertex d_i is connected to an already played move also in the real game, but dominates no new vertices. Thus Dominator can play an arbitrary legal move in the real game to preserve (1).

At a stage of the game, Staller selects her optimal move s_i in the real game. If s_i is a legal move in the imagined game, Dominator copies it as the move of Staller to the imagined game and (1) remains valid. Otherwise, since (1) held before Staller played s_i in the real game, the move s_i dominates new vertices in the imagined game, but is not connected to any vertex in D_I. Let P be the shortest path between s_i and D_I in $G\langle D_R \cup \{s_i\}\rangle$ and let $s_i^* \notin D_I$ be the vertex on P which is adjacent to some vertex in D_I. Then Dominator imagines Staller plays s_i^* in the imagined game instead of s_i. Since $s_i^* \in D_R$, the property (1) is preserved.

This strategy of Dominator ensures that the number of moves in the real game is at most the number of moves in the imagined game, which equals $\widetilde{\gamma}_{\rm cg}(G)-1$ (since d_1 is the optimal first move there). Altogether this implies that the number of moves in this case (taking the moves d_1 and s_1 into account) is at most $2+(\widetilde{\gamma}_{\rm cg}(G)-1) = \widetilde{\gamma}_{\rm cg}(G)+1$.

On the other hand, Staller's strategy is to reply to the Dominator's first move with a duplicate move (she can as $\gamma_{\rm cg}(H) \geq 2$). If Dominator's second move is a new move, then they play at least $\widetilde{\gamma}_{\rm cg}(G)+1$ moves. If Dominator's second move is a duplicate move, then Staller's strategy is to reply with a new move (with her optimal strategy from a D-game on G). Thus at least $\gamma_{\rm cg}(G)+2 \geq \widetilde{\gamma}_{\rm cg}(G)+1$ moves are played. This proves the first case.

Next, consider the case $\gamma_{\rm cg}(H) = 1$. As $\gamma_{\rm cg}(H) = 1$, after Dominator's first move on $(d_1, h) \in H_{d_1}$, where d_1 is Dominator's optimal first move on G and h is his optimal first move on H, Staller can only reply with a new move. And from here on, only new moves are possible due to Lemma 8. Hence, just a normal D-game on G is played, so $\gamma_{\rm cg}(G[H]) \leq \gamma_{\rm cg}(G)$. On the other hand, Staller's strategy is to play s_1 as a duplicate move, if this is possible. If it is, then the number of moves is at least $\widetilde{\gamma}_{\rm cg}(G)+1$, which is at least $\gamma_{\rm cg}(G)$ by Lemma 9(i). If it is not possible, then again just a normal D-game is played on G and thus the number of moves is at least $\gamma_{\rm cg}(G)$. This proves the second case.

Lastly, consider the case $|V(G)| = 1$. In this case $G[H] \cong H$, hence

$$\gamma_{\rm cg}(G[H]) = \gamma_{\rm cg}(H). \qquad \square$$

Corollary 12. Let G be a connected graph and let H be a graph. If $\gamma_{\rm cg}(H) \geq 2$ and $|V(G)| \geq 2$, then

$$\gamma_{\rm cg}(G) \leq \gamma_{\rm cg}(G[H]) \leq \gamma_{\rm cg}(G) + 2.$$

Taking a graph H with $\gamma_{\rm cg}(H) \geq 2$ and graphs from Example 10 as G shows that all three possible values in Corollary 12 can be achieved.

Now we focus on the S-game.

Theorem 13. *If G is a connected graph and H is a graph, then*

$$\gamma'_{\rm cg}(G[H]) = \begin{cases} \gamma'_{\rm cg}(G); & \gamma'_{\rm cg}(G) \geq 2, \\ 2; & \gamma'_{\rm cg}(G) = 1, |V(G)| \geq 2 \textit{ and } \gamma'_{\rm cg}(H) \geq 2, \\ \gamma'_{\rm cg}(H); & |V(G)| = 1, \textit{ or } \gamma'_{\rm cg}(G) = 1, |V(G)| \geq 2 \textit{ and } \gamma'_{\rm cg}(H) = 1. \end{cases}$$

Proof. First consider the case $\gamma'_{\rm cg}(G) \geq 2$. Dominator's strategy is to reply to the Staller's first move by playing a new vertex (he can as $\gamma'_{\rm cg}(G) \geq 2$). Lemma 8 claims that in this case, just new moves are possible until the end of the game, thus a normal S-game is played on G. Hence, $\gamma'_{\rm cg}(G[H]) \leq \gamma'_{\rm cg}(G)$. Staller's strategy is to start on H_{s_1}, where s_1 is her optimal first move on G. If Dominator plays a duplicate move, then she replies with a new move. Thus the number of moves is at least $\widetilde{\gamma}'_{\rm cg}(G) + 1 \geq \gamma'_{\rm cg}(G)$ by Lemma 9(ii). If Dominator replies with a new move, the number of moves is at least $\gamma'_{\rm cg}(G)$.

Next, consider the case $\gamma'_{\rm cg}(G) = 1$, $|V(G)| \geq 2$ and $\gamma'_{\rm cg}(H) \geq 2$. Dominator can reply on a new vertex and finish the game in the second move. Staller can start on $(s_1, h) \in H_{s_1}$, where h is her optimal first move on H and s_1 is her optimal first move on G. Thus she forces at least two moves.

Next, in the case $\gamma'_{\rm cg}(G) = 1$, $|V(G)| \geq 2$ and $\gamma'_{\rm cg}(H) = 1$, wherever Staller starts, she dominates the whole graph. But as $1 = \gamma'_{\rm cg}(H)$, this case can be joined with the case $|V(G)| = 1$. In the latter, $G[H] \cong H$ and thus the game is over after $\gamma'_{\rm cg}(H)$ moves. $\square$

5. PREDOMINATION

The behavior of domination games when one or more vertices are considered predominated has been widely studied, see for example [**9, 13, 15, 19**]. Let $S \subseteq V(G)$. When a domination game is played on the graph G with vertices S

predominated, the vertices from S can still be played (if the moves are legal), but they are already considered dominated. We denote such *predominated graph* $G|S$. If $S = \{v\}$, we write $G|v$. A well known fact that holds for the classical domination game is the Continuation Principle [**22**], which states the following. For a graph G and sets $B \subseteq A \subseteq V(G)$ it holds that $\gamma_{\rm g}(G|A) \leq \gamma_{\rm g}(G|B)$ and $\gamma'_{\rm g}(G|A) \leq \gamma'_{\rm g}(G|B)$. Analogous results hold for the total domination game [**12**] as well as for other variations of the game [**2**].

We first observe that a similar statement cannot hold for the connected domination game. For example, consider the path on $n \geq 5$ vertices and let A be the set of all degree 2 vertices on the path. As the played vertices must be connected, the connected domination game on $G|A$ can never be finished.

However, it is still interesting to observe the case when only one vertex is predominated. In this case Dominator can achieve a finite number of moves in a D-game by starting the game on the predominated vertex. But as opposed to the other domination games, it seems that it might not be true that $\gamma_{\rm cg}(G|v) \leq \gamma_{\rm cg}(G)$. We return to this phenomenon in Proposition 14.

Note that in the S-game Staller might be able to prevent the game from ending even if just one vertex is predominated. For example, if a degree 2 vertex on a path is predominated and Staller starts the game on a vertex at distance 2 from it, the game can never end. Hence in the following, we only focus on the D-game on graphs with one vertex predominated.

Proposition 14. *There exists a graph G with a vertex v such that*

$$\gamma_{\rm cg}(G|v) = \gamma_{\rm cg}(G) + 1.$$

Proof. Consider the graph G as presented on Figure 5. Due to the connectedness, the vertices a, b, c, d, e, f, g must be played in any connected domination game on G. Thus $\gamma_{\rm cg}(G) \geq 7$. Consider the following strategy of Dominator to prove the equality. Let $d_1 = g$, then Staller can only reply on $s_1 = f$, Dominator strategy is to play $d_2 = e$, and then the remaining moves of the game are $s_2 = d$, $d_3 = c$, $s_3 = b$ and $d_4 = a$. Hence $\gamma_{\rm cg}(G) = 7$.

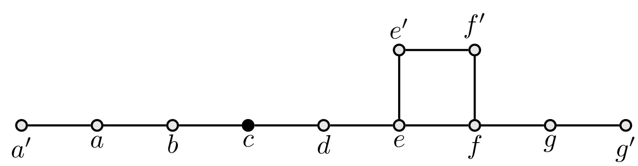

Figure 5: Graph G.

Consider the D-game played on $G|c$. Dominator starts the game on $N[c]$ (otherwise the game does not end). From a simple case analysis it follows that in all these three cases Staller will be able to make a move on e', hence $\gamma_{\rm cg}(G|c) = 8 = \gamma_{\rm cg}(G) + 1$. □

Next, consider some simple examples. Let $n \geq 4$ and P_n be a path with vertices $1, \ldots, n$ and naturally defined edges. We know that $\gamma_{\rm cg}(P_n) = n-2$ [**1**]. If $v \in \{1, n\}$, then it clearly holds $\gamma_{\rm cg}(P_n|v) = n-3$. However, if $v \in \{2, \ldots, n-1\}$, then we have $\gamma_{\rm cg}(P_n|v) = n-2$ (as the set of played vertices must be connected). This reasoning can be extended to trees [**1**]. For a tree T it holds that $\gamma_{\rm cg}(T) = \gamma_{\rm c}(T) = \gamma_{\rm cg}(T|v)$ for every vertex $v \in V(T)$ that is not a leaf.

Let $n \geq 4$ and C_n be a cycle with vertices $1, \ldots, n$ and naturally defined edges. Clearly, it holds that $\gamma_{\rm cg}(C_n) = n-2$. Let $v \in V(C_n)$ and consider the D-game on $C_n|v$. Without loss of generality $v = 1$. Wherever Dominator starts the game, at least the vertices $3, \ldots, n-1$ or any set of $n-2$ vertices must be played during the game. Thus $\gamma_{\rm cg}(C_n|v) \geq n-3$. Dominator's strategy is to start on 3. From this move on, the moves in the game are uniquely determined ($s_1 = 4$, $d_2 = 5$ $\ldots$). Hence, $\gamma_{\rm cg}(C_n|v) = n-3$.

Note that on the contrary to the previous example of cycles where for every vertex $v \in V(G)$ it holds that $\gamma_{\rm cg}(G|v) < \gamma_{\rm cg}(G)$, it cannot happen that $\gamma_{\rm cg}(G|v) > \gamma_{\rm cg}(G)$ for all vertices $v \in V(G)$. Let d_1 be an optimal first move of Dominator in a D-game on G. Then $\gamma_{\rm cg}(G|d_1) \leq \gamma_{\rm cg}(G)$ as Dominator can still start the game on d_1.

But there are also examples where predominating a vertex cannot decrease the connected domination game number.

Proposition 15. *If G is a graph with a cut vertex u (i.e. $G-u$ is not connected), then $\gamma_{\rm cg}(G|u) \geq \gamma_{\rm cg}(G)$.*

Proof. Let $\gamma_{\rm cg}(G) = k$. Suppose $\gamma_{\rm cg}(G|u) \leq k-1$. Let S be the set of played vertices in the game that achieves this bound. If the vertices from S dominate u, then this same strategy can be used on G to show that $\gamma_{\rm cg}(G) \leq k-1$, which is not the case. Thus the vertices from S do not dominate u, meaning that $S \cap N[u] = \emptyset$. But then S cannot be connected, a contradiction. □

Two other families of graphs with the same property as cycles are circular ladders and Möbius ladders. Predomination in the total domination game on those families has already been studied in [**15**]. Recall that the *circular ladder* CL_n, $n \geq 3$, is the Cartesian product $C_n \,\square\, K_2$. Denote the vertices of CL_n by $\{(i,j)\ ;\ i \in [n], j \in [2]\}$ and by C_n^1, C_n^2 the two C_n-layers in CL_n. The *Möbius ladder* ML_n is a graph obtained from C_{2n} by adding edges between opposite vertices.

Theorem 16. *If $n \geq 4$ and $v \in V(\mathrm{CL}_n)$, then*

$$\gamma_{\rm cg}(\mathrm{CL}_n) = 2(n-2) \quad \textit{and} \quad \gamma_{\rm cg}(\mathrm{CL}_n\,|v) = 2(n-2) - 1.$$

Proof. First, we prove that $\gamma_{\rm cg}(\mathrm{CL}_n) = 2(n-2) = 2n-4$. To prove the upper bound, consider the following strategy of Dominator. His first move is $(1,1)$ and as long as possible he plays on $V(C_n^1) \setminus \{(n-1,1), (n,1), (1,1)\}$. If Staller makes no move on C_n^1, her first move can only be on $(1,2)$, so she dominates the vertex $(n,2)$. Thus after Dominator's $(n-2)$th move at most one vertex can be undominated, i.e. $(n-1,2)$. So in this case Staller finishes the game with her next move and at most $2(n-2)$ moves are played. If Staller makes at least one move on C_n^1, then after the players play $n-2$ moves on C_n^1 at most two additional moves might be needed (to dominate possibly undominated vertices $(k,2)$ and $(k+1,2)$, where $(k,1),(k+1,1) \in V(C_n^1)$ have not been played). Thus the number of moves is at most $2(n-3)+2 = 2n-4$.

To prove the lower bound consider the following strategy of Staller. When Dominator plays on (i,j), Staller replies on $(i,3-j)$. Clearly, such a reply is always legal and the game finishes after at least $2(n-2)$ moves are played. Thus $\gamma_{\rm cg}(\mathrm{CL}_n) \geq 2n-4$.

Next, we prove that $\gamma_{\rm cg}(\mathrm{CL}_n\,|v) = 2(n-2)-1 = 2n-5$. Without loss of generality let $v = (1,1)$.

To prove the upper bound consider the following strategy of Dominator. His first move is $(3,2)$ and as long as possible he plays on $V(C_n^2)\setminus\{(1,2),(2,2),(3,2)\}$. If Staller makes no move on C_n^2, her first move can only be on $(3,1)$, so she dominates the vertex $(2,1)$. The game is over after Dominator's $(n-2)$th move (when he plays $(n,2)$). Thus the number of moves is at most $(n-2)+(n-3) = 2n-5$. If Staller makes at least one move on C_n^2, then after the players play $n-2$ moves on C_n^2 one additional move might be needed (to dominate $(2,1)$). Thus the number of moves is at most $2(n-3)+1 = 2n-5$.

To prove the lower bound consider the following strategy of Staller. When Dominator plays on (i,j), Staller replies on $(i,3-j)$. Clearly, such reply is legal unless the game is over, so the game finishes after at least $2(n-2) \geq 2n-5$ moves are played. Thus $\gamma_{\rm cg}(\mathrm{CL}_n) \geq 2n-5$. □

Notice that the strategies described in the proof of Theorem 16 are also valid on Möbius ladders.

Corollary 17. If $n \geq 4$ and $v \in V(\mathrm{ML}_n)$, then

$$\gamma_{\rm cg}(\mathrm{ML}_n) = 2(n-2) \quad \text{and} \quad \gamma_{\rm cg}(\mathrm{ML}_n\,|v) = 2(n-2)-1.$$

5. CONCLUDING REMARKS

Recall that for the domination game (resp. total domination game), critical graphs are defined as graphs G such that for every $v \in V(G)$ it holds that $\gamma_{\rm g}(G|v) < \gamma_{\rm g}(G)$ (resp. $\gamma_{\rm tg}(G|v) < \gamma_{\rm tg}(G)$) [**9, 13**]. On the other hand, it is not clear how to define connected domination game critical graphs. From the previous section, we know that there exist graphs with the property that $\gamma_{\rm cg}(G|v) < \gamma_{\rm cg}(G)$ for all vertices v, and that it can also happen that $\gamma_{\rm cg}(G|v) > \gamma_{\rm cg}(G)$, but this cannot be true on all vertices v. However, the following is not clear.

Problem 18. Does there exist a graph G with the following properties:

- for all $v \in V(G)$: $\gamma_{\rm cg}(G|v) \neq \gamma_{\rm cg}(G)$, and
- there exists a vertex $u \in V(G)$: $\gamma_{\rm cg}(G|u) > \gamma_{\rm cg}(G)$.

If the answer to the Problem 18 is positive, than a sensible definition for critical graphs could be the following. A graph G is connected domination game critical if for every vertex $v \in V(G)$ it holds that $\gamma_{\rm cg}(G|v) \neq \gamma_{\rm cg}(G)$. However, if the answer to the Problem 18 is negative, a definition should go along the same lines as the definition of critical graphs for domination and total domination games. So a graph G would be connected domination game critical if for every vertex $v \in V(G)$ it would hold that $\gamma_{\rm cg}(G|v) < \gamma_{\rm cg}(G)$.

Two other interesting problems arise from the predomination of vertices.

Problem 19.

1. What is the maximal value of k such that there exists a graph G with a vertex $v \in V(G)$ with the property $\gamma_{\rm cg}(G|v) = \gamma_{\rm cg}(G) + k$?
2. What is the maximal value of k such that there exists a graph G with a vertex $v \in V(G)$ with the property $\gamma_{\rm cg}(G|v) = \gamma_{\rm cg}(G) - k$?

Recall that for domination and total domination game the answers to analogous problems is known and equals $k = 0$ for the first part and $k = 2$ for the second part [**9, 19**]. Note that Problem 19 has been recently resolved, see [**8**].

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