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REFINEMENTS OF HUYGENS - WILKER - LAZAROVIĆ INEQUALITIES VIA THE HYPERBOLIC COSINE POLYNOMIALS

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The aim of this paper is to provide new refinements of Huygens - Wilker - Lazarović inequalities using hyperbolic cosine polynomials. We give an unitary approach for both inequalities of trigonometric and hyperbolic functions.

1. INTRODUCTION AND MOTIVATION

The famous Huygens inequality for trigonometric functions states that

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3,\ 0 < x < \frac{\pi}{2},$$

while the Wilker trigonometric inequality asserts

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2,\ 0 < x < \frac{\pi}{2}.$$

The inequality

$$\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\tan x}{x} > 1,\ \ 0 < x < \frac{\pi}{2}$$

is the trigonometric Lazarović inequality.

For more connections between these inequalities see [**15**].

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The hyperbolic counterparts of the Huygens and Wilker trigonometric inequalities are

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3,\ x\neq 0$$

and respectively

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2,\ x\neq 0$$

(see [**15**]).

The hyperbolic Lazarović inequality states that

$$\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}>1,\ \ x\neq 0.$$

(see [**13**]). Recently, some of the above inequalities have been improved using the Taylor's expansions (see [**14**]) or Padé approximant method (see [**3**]).

In [**4**] we improved these inequalities using cosine polynomials for even functions:

For all $x\in\left(0,\frac{\pi}{2}\right)$, one has:

$$2\frac{\sin x}{x}+\frac{\tan x}{x}-3>\frac{3}{5}\left(1-\cos x\right)^2 \tag{1}$$

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2>\frac{32}{45}\left(1-\cos x\right)^2 \tag{2}$$

$$\left(\frac{\sin x}{x}\right)^2\cdot\frac{\tan x}{x}-1>\frac{4}{15}\left(1-\cos x\right)^2. \tag{3}$$

The relevant papers on the topic are also [**3**] - [**20**].

The aim of this paper is to refine the aforesaid inequalities using hyperbolic cosine polynomials method for even functions. The main idea is that the functions $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ are even, so the above inequalities can be expanded as follows

$$2\frac{\sin x}{x}+\frac{\tan x}{x}-3=a+b\cosh x+c\cosh 2x+...$$

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2=a+b\cosh x+c\cosh 2x+...$$

$$\left(\frac{\sin x}{x}\right)^2\cdot\frac{\tan x}{x}-1=a+b\cosh x+c\cosh 2x+...\ .$$

We present the algorithms for the first expansion. We introduce the function

$$F_1(x) = a + b\cosh x + c\cosh 2x.$$

The power series expansion of the function $2\dfrac{\sin x}{x} + \dfrac{\tan x}{x} - 3 - F_1(x)$ near 0 is

$$(-a-b-c) + \left(-\frac{b}{2} - 2c\right)x^2 + \frac{1}{120}(-5b - 80c + 18)x^4 + O\left(x^6\right).$$

In order to increase the speed of the function $F_1(x)$ approximating $2\dfrac{\sin x}{x} + \dfrac{\tan x}{x} - 3$, we vanish the first coefficients as follows:

$$\begin{cases} -a-b-c=0 \\ -\dfrac{b}{2} - 2c = 0 \\ -5b - 80c + 18 = 0 \end{cases}$$

and obtain $a = \dfrac{9}{10}$, $b = -\dfrac{6}{5}$ and $c = \dfrac{3}{10}$.

Therefore we find

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 - \frac{9}{10} + \frac{6}{5}\cosh x - \frac{3}{10}\cosh 2x = \frac{1}{35}x^6 + \frac{1}{50}x^8 + \frac{4867}{554400}x^{10}.$$

In a similar way we deduce

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 - \frac{16}{15} + \frac{64}{45}\cosh x - \frac{16}{45}\cosh 2x = \frac{4}{189}x^6 + \frac{187}{9450}x^8 + \frac{607}{69300}x^{10}$$

and

$$\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\tan x}{x} - 1 - \frac{2}{5} + \frac{8}{15}\cosh x - \frac{2}{15}\cosh 2x = \frac{19}{1890}x^6 + \frac{29}{3600}x^8 + \frac{5911}{1663200}x^{10}.$$

Now we consider the following hyperbolic cosine polynomials expansions

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} - 3 = a + b\cosh x + c\cosh 2x + ...,$$

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 = a + b\cosh x + c\cosh 2x + ...,$$

$$\left(\frac{\sinh x}{x}\right)^2 \cdot \frac{\tanh x}{x} - 1 = a + b\cosh x + c\cosh 2x + ...$$

Using the same algorithm as above we find

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}-3-\frac{3}{5}\left(\cosh x-1\right)^2=-\frac{11}{140}x^6+\frac{1}{50}x^8-\frac{3307}{369600}x^{10},$$

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}-2-\frac{32}{45}\left(\cosh x-1\right)^2=-\frac{76}{945}x^6+\frac{187}{9450}x^8-\frac{16763}{1871100}x^{10},$$

$$\left(\frac{\sinh x}{x}\right)^2\cdot\frac{\tanh x}{x}-1-\frac{4}{15}\left(\cosh x-1\right)^2=-\frac{61}{1890}x^6+\frac{29}{3600}x^8-\frac{18107}{4989600}x^{10}.$$

Then, using a method developed and applied in [**9**], [**10**], [**11**], [**12**], we will express the above inequalities in terms of the double sided Taylor's approximations over the finite intervals.

For this, in the following, we will present an overview of the results related to double-sided Taylor's approximations.

Let us consider a real function $f:(a,b)\longrightarrow\mathbb{R}$, such that there exist finite limits $f^{(k)}(a+)=\lim_{x\to a+}f^{(k)}(x)$, for $k=0,1,...,n$.

TAYLOR's polynomial

$$T_n^{f,a+}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a+)}{k!}(x-a)^k,\ n\in\mathbb{N}_0,$$

and the polynomial

$$\mathbb{T}_n^{f;a+,b-}(x)=\begin{cases} T_{n-1}^{f,a+}(x)+\frac{1}{(b-a)^n}R_n^{f,a+}(b-)(x-a)^n, & n\geq 1\\ f(b-), & n=0,\end{cases}$$

are called the *first TAYLOR's approximation for the function f in the right neighborhood of* a, and the *second TAYLOR's approximation for the function f in the right neighborhood of* a, respectively.

Also, the following functions:

$$R_n^{f,a+}(x)=f(x)-T_{n-1}^{f,a+}(x),\ n\in\mathbb{N}$$

and

$$\mathbb{R}_n^{f;a+,b-}(x)=f(x)-\mathbb{T}_{n-1}^{f;a+,b-}(x),\ n\in\mathbb{N}$$

are called the *remainder of the first TAYLOR's approximation in the right neighborhood of* a, and the *remainder of the second TAYLOR's approximation in the right neighborhood of* a, respectively.

In our applications, of special interest is the following theorem:

Theorem 1. *Consider the real analytic functions* $f:(a,b)\longrightarrow\mathbb{R}$*:*

$$f\left(x\right)=\sum_{k=0}^{\infty}c_k\left(x-a\right)^k,$$

where $c_k \in \mathbb{R}$ *and* $c_k \geq 0$ *for all* $k \in \mathbb{N}_0$. *Then*

$$\begin{gathered} T_0^{f,a+}(x) \leq ... \leq T_n^{f,a+}(x) \leq T_{n+1}^{f,a+}(x) \leq ... \leq \\ ... \leq f(x) \leq ... \\ \leq \mathbb{T}_{n+1}^{f;a+,b-}(x) \leq \mathbb{T}_n^{f;a+,b-}(x) \leq ... \leq \mathbb{T}_0^{f;a+,b-}(x), \end{gathered}$$

for all $x \in (a,b)$. *If* $c_k \in \mathbb{R}$ *and* $c_k \leq 0$ *for all* $k \in \mathbb{N}_0$, *then the reversed inequalities hold.*

2. THE RESULTS

Now we can establish the first set of our main theorems.

Theorem 2. *(i) The following inequality*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 > \frac{3}{5}(\cosh x - 1)^2 \tag{4}$$

holds for all $x \in \left(0, \frac{\pi}{2}\right)$.

The constant $\frac{3}{5}$ *is the best possible.*

(ii) For every $x \in (0,b)$, $0 < b < \frac{\pi}{2}$ *and* $m \in \mathbb{N}$, $m \geq 3$, *the following inequalities hold:*

$$\begin{gathered} T_8^{f,0+}(x) \leq ... \leq T_{2m}^{f,0+}(x) \leq T_{2m+2}^{f,0+}(x) \leq \\ ... \leq f(x) \leq ... \\ \leq \mathbb{T}_{2m+2}^{f;0+,b-}(x) \leq \mathbb{T}_{2m}^{f;0+,b-}(x) \leq ... \leq \mathbb{T}_8^{f;0+,b-}(x), \end{gathered} \tag{5}$$

where

$$f(x) = 2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 - \frac{3}{5}(\cosh x - 1)^2,$$

$$T_{2m}^{f,0+}(x) = \sum_{k=3}^{m} A(k)\, x^{2k},$$

$$\mathbb{T}_{2m}^{f;0+,b-}(x) = \sum_{k=3}^{m-1} A(k)\, x^{2k} + \frac{1}{b^{2m}}\left(f(b) - \sum_{k=3}^{m-1} b^{2k} A(k)\right) \cdot x^{2m}$$

for

$$A(k) = \left(\frac{-2(-1)^{k+1}}{2k+1} + \frac{2^{2k+2}\left(2^{2k+2}-1\right)|B_{2k+2}|}{2k(k+1)} - \frac{3\left(2^{2k}-4\right)}{10}\right) \cdot \frac{1}{(2k)!}.$$

Remark 1. *According to the Lemma 1, the expression*

$$\frac{3}{5}(\cosh x - 1)^2 - \frac{3}{5}(1 - \cos x)^2 = \frac{3}{5}(\cosh x - \cos x)(\cosh x + \cos x - 2)$$

is positive for all $x \in \left(0, \frac{\pi}{2}\right)$.

It follows that we improve the inequality (1).

Theorem 3. *(i) The following inequality*

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2>\frac{32}{45}\left(\cosh x-1\right)^2 \tag{6}$$

holds for all $x\in\left(0,\frac{\pi}{2}\right)$.

The constant $\frac{32}{45}$ *is the best possible.*

(ii) For every $x\in(0,b)$, $0<b<\frac{\pi}{2}$ *and* $m\in\mathbb{N}$, $m\geq 3$, *the following inequalities hold:*

$$\begin{gathered}T_8^{f,0+}(x)\leq ...\leq T_{2m}^{f,0+}(x)\leq T_{2m+2}^{f,0+}(x)\leq\\ ...\leq g(x)\leq ...\\ \leq \mathbb{T}_{2m+2}^{f;0+,b-}(x)\leq \mathbb{T}_{2m}^{f;0+,b-}(x)\leq ...\leq \mathbb{T}_8^{f;0+,b-}(x),\end{gathered} \tag{7}$$

where

$$g(x)=\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2-\frac{32}{45}\left(\cosh x-1\right)^2,$$

$$T_{2m}^{f,0+}(x)=\sum_{k=3}^{m}B(k)\,x^{2k},$$

$$\mathbb{T}_{2m}^{f;0+,b-}(x)=\sum_{k=3}^{m-1}B(k)\,x^{2k}+\frac{1}{b^{2m}}\left(g(b)-\sum_{k=3}^{m-1}b^{2k}B(k)\right)\cdot x^{2m}$$

for

$$B(k)=\left(\frac{\left(2\left(2^{2k+2}-1\right)|B_{2k+2}|-(-1)^{k+1}\right)2^{2k+1}}{(2k+1)\,2\,(k+1)}-\frac{16\left(2^{2k+2}-4\right)}{45}\right)\cdot\frac{1}{(2k)!}.$$

Remark 2. *The inequality (6) improves the inequality (2).*

Theorem 4. *(i) The following inequality*

$$\left(\frac{\sin x}{x}\right)^2\cdot\frac{\tan x}{x}-1>\frac{4}{15}\left(\cosh x-1\right)^2 \tag{8}$$

holds for all $x\in\left(0,\frac{\pi}{2}\right)$.

The constant $\frac{4}{15}$ *is the best possible.*

(ii) For every $x\in(0,b)$, $0<b<\frac{\pi}{2}$ *and* $m\in\mathbb{N}$, $m\geq 3$, *the following inequalities hold:*

$$\begin{gathered}T_8^{f,0+}(x)\leq ...\leq T_{2m}^{f,0+}(x)\leq T_{2m+2}^{f,0+}(x)\leq\\ ...\leq h(x)\leq ...\\ \leq \mathbb{T}_{2m+2}^{f;0+,b-}(x)\leq \mathbb{T}_{2m}^{f;0+,b-}(x)\leq ...\leq \mathbb{T}_8^{f;0+,b-}(x),\end{gathered} \tag{9}$$

where

$$h(x)=\left(\frac{\sin x}{x}\right)^2\cdot\frac{\tan x}{x}-\frac{4}{15}\left(\cosh x-1\right)^2,$$

$$T_{2m}^{f,0+}(x)=\sum_{k=3}^{m}C(k)\,x^{2k},$$

$$\mathbb{T}_{2m}^{f;0+,b-}(x)=\sum_{k=3}^{m-1}C(k)\,x^{2k}+\frac{1}{b^{2m}}\left(h(b)-\sum_{k=3}^{m-1}b^{2k}C(k)\right)\cdot x^{2m}$$

for

$$C(k)=\left(\frac{\left(\frac{2\left(2^{2k+2}-1\right)}{k+2}\left|B_{2k+4}\right|+(-1)^k\right)2^{2k+2}}{(2k+1)(2k+2)(2k+3)}-\frac{2\left(2^{2k}-4\right)}{15}\right)\cdot\frac{1}{(2k)!}.$$

Remark 3. *The inequality (8) improves the inequality (3).*

The second set of our main theorems is the following.

Theorem 5. *(i) For all $x\neq 0$, one has:*

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}-3<\frac{3}{5}\left(\cosh x-1\right)^2. \tag{10}$$

The constant $\frac{3}{5}$ is the best possible.

(ii) For every $x\in(0,b)$, $b>0$ and $m\in\mathbb{N}$, $m\geq 3$, the following inequalities hold:

$$\begin{gathered}T_9^{f,0+}(x)\geq\ldots\geq T_{2m+1}^{f,0+}(x)\geq T_{2m+3}^{f,0+}(x)\geq\\ \ldots\geq t(x)\geq\ldots\\ \geq\mathbb{T}_{2m+3}^{f;0+,b-}(x)\geq\mathbb{T}_{2m+1}^{f;0+,b-}(x)\geq\ldots\geq\mathbb{T}_9^{f;0+,b-}(x),\end{gathered}\tag{11}$$

where

$$t(x)=\sinh 2x+\sinh x-3x\cosh x-\frac{3}{5}x\cosh x\left(\cosh x-1\right)^2,$$

$$T_{2m+1}^{f,0+}(x)=\sum_{k=3}^{m}D(k)\,x^{2k+1},$$

$$\mathbb{T}_{2m+1}^{f;0+,b-}(x)=\sum_{k=3}^{m-1}D(k)\,x^{2k+1}+\frac{1}{b^{2m+1}}\left(t(b)-\sum_{k=3}^{m-1}b^{2k+1}D(k)\right)\cdot x^{2m+1}$$

for

$$D(k)=\frac{4^k\left(24k+52\right)-(6k+3)\,9^k-162k-61}{20\,(2k+1)!}.$$

Remark 4. *From the inequalities (4) and (10) it follows that*

$$2\sin x+\tan x>2\sinh x+\tanh x \textit{ for all } x\in\left(0,\frac{\pi}{2}\right).$$

Theorem 6. *(i) For all $x\neq 0$, one has:*

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}-2<\frac{32}{45}\left(\cosh x-1\right)^2. \tag{12}$$

The constant $\frac{32}{45}$ is the best possible.

(ii) For every $x\in(0,b)$, $b>0$ and $m\in\mathbb{N}$, $m\geq 4$, the following inequalities hold:

$$\begin{gathered} T_{10}^{f,0+}(x)\geq ...\geq T_{2m}^{f,0+}(x)\geq T_{2m+2}^{f,0+}(x)\geq \\ ...\geq s(x)\geq ... \\ \geq \mathbb{T}_{2m+2}^{f;0+,b-}(x)\geq \mathbb{T}_{2m}^{f;0+,b-}(x)\geq ...\geq \mathbb{T}_{10}^{f;0+,b-}(x), \end{gathered} \tag{13}$$

where

$$s(x)=\cosh x\sinh^2 x+x\sinh x-2x^2\cosh x-\frac{32}{45}x^2\cosh x\left(\cosh x-1\right)^2,$$

$$T_{2m}^{f,0+}(x)=\sum_{k=4}^{m}E(k)\,x^{2k},$$

$$\mathbb{T}_{2m}^{f;0+,b-}(x)=\sum_{k=4}^{m-1}E(k)\,x^{2k}+\frac{1}{b^{2m}}\left(s(b)-\sum_{k=4}^{m-1}b^{2k}E(k)\right)\cdot x^{2m}$$

for

$$E(k)=\frac{E_1(k)}{180\,(2k)!},$$

where

$$E_1(k)=45(3^{2k}-1)+360k-64(2k-1)\cdot k\cdot 3^{2k-2}+256\,(2k-1)\cdot k\cdot 2^{2k-2}-1168\,(2k-1)\cdot k.$$

Remark 5. *From the inequalities (6) and (12) we find that*

$$\sin^2 x+x\tan x>\sinh^2 x+x\tanh x \textit{ for all } x\in\left(0,\frac{\pi}{2}\right).$$

Theorem 7. *(i) For all $x\neq 0$, one has:*

$$\left(\frac{\sinh x}{x}\right)^2\cdot\frac{\tanh x}{x}-1<\frac{4}{15}\left(\cosh x-1\right)^2. \tag{14}$$

The constant $\frac{4}{15}$ is the best possible.

(ii) For every $x \in (0, b)$, $b > 0$ and $m \in \mathbb{N}$, $m \geq 4$, the following inequalities hold:

$$\begin{aligned} T_{11}^{f,0+}(x) \geq ... \geq T_{2m+1}^{f,0+}(x) \geq T_{2m+3}^{f,0+}(x) \geq \\ ... \geq r(x) \geq ... \\ \geq \mathbb{T}_{2m+3}^{f;0+,b-}(x) \geq \mathbb{T}_{2m+1}^{f;0+,b-}(x) \geq ... \geq \mathbb{T}_{11}^{f;0+,b-}(x), \end{aligned} \tag{15}$$

where

$$r(x) = \sinh^3 x - x^3 \cosh x - \frac{4}{15} x^3 \cosh x (\cosh x - 1)^2,$$

$$T_{2m+1}^{f,0+}(x) = \sum_{k=4}^{m} F(k) x^{2k+1},$$

$$\mathbb{T}_{2m+1}^{f;0+,b-}(x) = \sum_{k=4}^{m-1} F(k) x^{2k+1} + \frac{1}{b^{2m+1}} \left(r(b) - \sum_{k=4}^{m-1} b^{2k+1} F(k) \right) \cdot x^{2m+1}$$

for

$$F(k) = \frac{(3^{2k+1} - 3) \cdot 15 + (2k-1)(2k)(2k+1) \left(-4 \cdot 3^{2k-2} + 16 \cdot 2^{2k-2} - 88 \right)}{60 \cdot (2k+1)!}.$$

Remark 6. *From the inequalities (8) and (14) we obtain that*

$$\sin^2 x \tan x > \sinh^2 x \tanh x \text{ for all } x \in \left(0, \frac{\pi}{2}\right).$$

3. THE PROOFS OF THE THEOREMS

The following elementary power series expansions are useful in our investigations:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad |x| < \infty, \tag{16}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad |x| < \infty, \tag{17}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n} \left(2^{2n} - 1\right) (-1)^{n-1} B_{2n}}{(2n)!} x^{2n-1}, \quad |x| < \frac{\pi}{2}, \tag{18}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad |x| < \infty, \tag{19}$$

(20) $$\cosh x=\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n},\quad |x|<\infty,$$

where B_n $(n=0,1,2,...)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

We first prove one lemma.

Lemma 1. *(i) For every* $x\in\left[0,\frac{\pi}{2}\right)$, *one has*

$$\cosh x+\cos x-2\geq 0.$$

(ii) For every $x\in[0,\infty)$, *one has*

$$x\cosh x-\sinh x>0.$$

Proof. (i) We consider the function $f_1:\left[0,\frac{\pi}{2}\right)\to\mathbb{R},\ f_1(x)=\cosh x+\cos x-2$. Elementary calculations reveal that

$$f_1'(x)=\sinh x-\sin x,$$
$$f_1^{(2)}(x)=\cosh x-\cos x,$$
$$f_1^{(3)}(x)=\sinh x+\sin x\geq 0 \text{ for all } x\in\left[0,\frac{\pi}{2}\right).$$

Then $f_1^{(2)}$ is strictly increasing on $\left[0,\frac{\pi}{2}\right)$. As $f_1^{(2)}(0)=0$, we get $f_1^{(2)}(0)\geq 0$ on $\left[0,\frac{\pi}{2}\right)$. Continuing the algorithm, finally we find $f_1(x)\geq 0$ for all $x\in\left[0,\frac{\pi}{2}\right)$.
(ii) The function $f_2:[0,\infty)\to\mathbb{R},\ f_2(x)=x\cosh x-\sinh x$ has the derivative $f_2'(x)=x\sinh x\geq 0$ for all $x\in[0,\infty)$. Then f_2 is strictly increasing on $[0,\infty)$. Since $f_2(0)=0$, it follows that $f_2(x)\geq 0$ for all $x\in[0,\infty)$. □

The proof of the theorem 2.

(i) Using power series expansions (16), (18) and (20), for $0<x<\frac{\pi}{2}$, we have

$$2\frac{\sin x}{x}+\frac{\tan x}{x}-3-\frac{3}{5}(\cosh x-1)^2=$$
$$2\frac{\sin x}{x}+\frac{\tan x}{x}-3-\frac{3}{10}(\cosh 2x-4\cosh x+3)$$
$$=\sum_{k=0}^{\infty}(-1)^k\cdot\frac{2x^{2k}}{(2k+1)!}+\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{(2k)!}x^{2k-2}-3-$$
$$-\frac{3}{10}\left(\sum_{k=0}^{\infty}\frac{2^{2k}x^{2k}}{(2k)!}-\sum_{k=0}^{\infty}\frac{4x^{2k}}{(2k)!}+3\right)$$
$$=\sum_{k=4}^{\infty}\left(\frac{-2(-1)^k}{2k-1}+\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-1)}-\frac{3\left(2^{2k-2}-4\right)}{10}\right)\frac{x^{2k-2}}{(2k-2)!}$$
$$=\frac{1}{35}x^6+\frac{1}{50}x^8+\frac{486}{554400}x^{10}+\frac{20159}{5616000}x^{12}+...$$

It is well known [**1**] that Bernoulli numbers with even indexes satisfy the following double inequality

$$\frac{2\,(2k)!}{(2\pi)^{2k}} < |B_{2k}| < \frac{2\,(2k)!}{(2\pi)^{2k}\left(1-2^{1-2k}\right)},\ k \geq 1. \tag{21}$$

By (21), we find that

$$\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-1)} > \frac{2^{2k}\left(2^{2k}-1\right)}{2k\,(2k-1)} \cdot \frac{2\,(2k)!}{(2\pi)^{2k}} > \frac{2}{2k-1} + \frac{3\left(2^{2k-2}-4\right)}{10}, \text{ for } k \geq 4.$$

Hence, we obtain

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 > \frac{3}{5}\left(\cosh x - 1\right)^2,\ \text{for } 0 < x < \frac{\pi}{2}.$$

To deduce that $\frac{3}{5}$ is the best possible constant in the inequality (4), we consider the inequality

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 > p\left(\cosh x - 1\right)^2,\ 0 < x < \frac{\pi}{2}.$$

This inequality holds for $p = \frac{3}{5}$. It also takes the equivalent form

$$p < \frac{2\frac{\sin x}{x} + \frac{\tan x}{x} - 3}{\left(\cosh x - 1\right)^2},\ 0 < x < \frac{\pi}{2}.$$

Let x tends to 0^+, we find that the ratio of the right-hand side tends to $\frac{3}{5}$.

Therefore, the inequality (4) holds for $0 < x < \frac{\pi}{2}$ with the best constant $p = \frac{3}{5}$.

The proof of (i) from the theorem 2 is complete.

(ii) Since all coefficients $A(k)$ are positive, by applying Theorem 1, we get the inequalities (5).

Example 1. *Taking $m = 4$ in inequalities (5) gives*

$$\frac{1}{35}x^6 + \frac{1}{50}x^8 < f\,(x) < \frac{1}{35}x^6 + \frac{1}{b^8}\left(f\,(b) - \frac{1}{35}b^6\right)x^8,$$

for $0 < x < b < \frac{\pi}{2}$.

The proof of the theorem 3.

(i) Using power series expansions (17), (18) and (20), for $0 < x < \frac{\pi}{2}$, we have

$$\begin{aligned}
&\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - \frac{32}{45}\left(\cosh x - 1\right)^2 \\
&= \frac{1}{2x^2} - \frac{1}{2x^2}\cos 2x + \frac{\tan x}{x} - \frac{16}{45}\left(\cosh 2x - 4\cosh x + 3\right) \\
&= 2 + \sum_{k=3}^{\infty} \frac{\left(2\left(2^{2k}-1\right)|B_{2k}| - (-1)^k\right)2^{2k-1}}{(2k)!} \cdot x^{2k-2} - \frac{16}{45}\left(\sum_{k=0}^{\infty} \frac{2^{2k}-4}{(2k)!} x^{2k}\right) - \frac{16}{45} \\
&= 2 + \sum_{k=4}^{\infty} \left(\frac{\left(2\left(2^{2k}-1\right)|B_{2k}| - (-1)^k\right)2^{2k-1}}{(2k-1)\,2k} - \frac{16\left(2^{2k}-4\right)}{45}\right) \cdot \frac{x^{2k-2}}{(2k-2)!} \\
&= 2 + \frac{4}{189}x^6 + \frac{187}{9450}x^8 + \frac{607}{69300}x^{10} + \frac{36667849}{10216206000} + \dots .
\end{aligned}$$

By (21), we may write

$$\frac{2\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-1)} > \frac{2^{2k}\left(2^{2k}-1\right)}{2k\,(2k-1)} \cdot \frac{2\,(2k)!}{(2\pi)^{2k}} > \frac{2^{2k-1}}{(2k-1)\,2k} + \frac{16\left(2^{2k-2}-4\right)}{45},$$

for all $k \geq 4$. Therefore, we deduce

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{32}{45}\left(\cosh x - 1\right)^2, \text{ for } 0 < x < \frac{\pi}{2}.$$

In order to prove that $\frac{32}{45}$ is the best possible constant in the inequality (6), we consider the inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > p\left(\cosh x - 1\right)^2, \ 0 < x < \frac{\pi}{2}.$$

This inequality holds for $p = \frac{32}{45}$. It also can be rewritten as

$$p < \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{\left(\cosh x - 1\right)^2}, \ 0 < x < \frac{\pi}{2}.$$

Using l'Hospital rules, it turns out that

$$\lim_{x \to 0^+} \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{\left(\cosh x - 1\right)^2} = \frac{32}{45}.$$

This means that the inequality (6) holds with the best possible constant $p = \frac{32}{45}$.

The proof of (i) from the theorem 3 is complete.

(ii) Since all corresponding coefficients $B(k)$ are positive, by using Theorem 1, we find the inequalities (7).

Example 2. *Taking $m = 4$ in inequalities (7) gives*

$$\frac{4}{189}x^6 + \frac{187}{9450}x^8 < g(x) < \frac{4}{189}x^6 + \frac{1}{b^8}\left(g(b) - \frac{4}{189}b^6\right)x^8,$$

for $0 < x < b < \frac{\pi}{2}$.

The proof of the theorem 4.

(i) Using power series expansions (17), (18) and (20), for $0 < x < \dfrac{\pi}{2}$, we have

$$\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\tan x}{x} - \frac{4}{15}(\cosh x - 1)^2 = \frac{2\tan x - \sin 2x}{2x^3} - \frac{2}{15}(\cosh 2x - 4\cosh x + 3)$$
$$= 1 + \sum_{k=3}^{\infty}\left(\frac{2\left(2^{2k}-1\right)}{k}|B_{2k}| + (-1)^k\right)\frac{2^{2k-2}}{(2k-1)!}\cdot x^{2k-4} - \frac{2}{15}\sum_{k=2}^{\infty}\frac{2^{2k}-4}{(2k)!}x^{2k}$$
$$= 1 + \sum_{k=4}^{\infty}\left(\frac{\left(\frac{2\left(2^{2k}-1\right)}{k+1}|B_{2k+2}| + (-1)^{k+1}\right)2^{2k}}{(2k-1)\,2k\,(2k+1)} - \frac{2\left(2^{2k-2}-4\right)}{15}\right)\frac{x^{2k-2}}{(2k-2)!}$$
$$= 1 + \frac{19}{1890}x^6 + \frac{29}{3600}x^8 + \frac{5911}{1663200}x^{10} + \frac{1200937}{825552000}x^{12} + \dots .$$

By (21), we deduce

$$\frac{\frac{2^{2k+1}\left(2^{2k+2}-1\right)}{k+1}|B_{2k+2}|}{(2k-1)\,2k\,(2k+1)} > \frac{2^{2k+1}\left(2^{2k+2}-1\right)}{(k+1)(2k-1)\,2k\,(2k+1)}\frac{2\,(2k+2)!}{(2\pi)^{2k+2}} >$$
$$> \frac{2^{2k}}{(2k-1)\,2k\,(2k+1)} + \frac{2\left(\left(2^{2k-2}-4\right)\right)}{15}, \text{ for all } k \geq 4.$$

Hence, we obtain

$$\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x} - 1 > \frac{4}{15}(\cosh x - 1)^2, \text{ for } 0 < x < \frac{\pi}{2}.$$

To deduce that $\frac{4}{15}$ is the best possible constant in the inequality (8), we consider the inequality

$$\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\tan x}{x} - 1 > p(\cosh x - 1)^2, \ 0 < x < \frac{\pi}{2}.$$

This inequality is true for $p = \frac{4}{15}$. It also takes the equivalent form

$$p < \frac{\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\tan x}{x} - 1}{(\cosh x - 1)^2}, \ 0 < x < \frac{\pi}{2}.$$

Since $\lim_{x\to 0^+}\frac{\left(\frac{\sin x}{x}\right)^2\cdot\frac{\tan x}{x}-1}{(\cosh x-1)^2}=\frac{4}{15}$, it follows that the inequality (8) is true with the best constant $p=\frac{4}{15}$.

This completes the proof of (i) from the theorem 4.

(ii) Since all coefficients $C(k)$ are positive, the assertions of Theorem 4 (ii) immediately follow from Theorem 1.

Example 3. *Taking $m=4$ in inequalities (9) gives*

$$\frac{19}{1890}x^6+\frac{29}{3600}x^8<h\left(x\right)<\frac{19}{1890}x^6+\frac{1}{b^8}\left(h\left(b\right)-\frac{19}{1890}b^6\right)x^8,$$

for $0<x<b<\frac{\pi}{2}$.

The proof of the theorem 5.

(i) Due to the form of the functions involved in the inequality (10), we can assume $x>0$.

Since the Taylor expansion of $\tanh x$ is only for $|x|<\frac{\pi}{2}$, we will rewrite the inequality (10) as follows:

$$\sinh 2x+\sinh x-3x\cosh x-\frac{3}{5}x\cosh x\left(\cosh x-1\right)^2<0.$$

Based on elementary trigonometric identities, we have

$$\cosh x\left(\cosh x-1\right)^2=\frac{1}{4}\left(\cosh 3x-4\cosh 2x+7\cosh x-4\right).$$

Using power series expansions (19) and (20) we find

$$\begin{aligned}
t\left(x\right) &= \sinh 2x+\sinh x-3x\cosh x-\frac{3}{5}x\cosh x\left(\cosh x-1\right)^2\\
&= \frac{3x}{5}+\sinh 2x+\sinh x-\frac{3x}{20}\cosh 2x+\frac{3x}{5}\cosh 2x-\frac{81}{20}\cosh x\\
&= \frac{3x}{5}+\sum_{k=0}^{\infty}\left(\frac{2^{2k+1}+1}{(2k+1)!}-\frac{3^{2k+1}}{20\,(2k)!}+\frac{3\cdot 2^k}{5\,(2k)!}-\frac{81}{20\,(2k)!}\right)x^{2k+1}\\
&= \sum_{k=3}^{\infty}\frac{4^k\left(24k+52\right)-\left(6k+3\right)9^k-162k-61}{20\left(2k+1\right)!}x^{2k+1}\\
&= -\frac{11}{140}x^7-\frac{27}{1400}x^9-\frac{821}{369600}x^{11}-\frac{34591}{216216000}x^{13}-\dots .
\end{aligned}$$

By induction it is easy to verify that $4^k\left(24k+52\right)<\left(6k+3\right)9^k+162k+61$ for all $k\in\mathbb{N}$, $k\geq 3$, therefore all corresponding coefficients are negative.

Hence we obtain the inequality (10).

To deduce that the inequality (10) is true with the best constant $\frac{3}{5}$, we consider the inequality

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}-3<p\left(\cosh x-1\right)^2,\ \text{for all } x>0.$$

This inequality holds for $p = \frac{3}{5}$. It also takes the equivalent form

$$p > \frac{2\frac{\sinh x}{x} + \frac{\tanh x}{x} - 3}{\left(\cosh x - 1\right)^2}, \text{ for all } x > 0.$$

Using l'Hospital rules, it turns out that

$$\lim_{x \to 0^+} \frac{2\frac{\sinh x}{x} + \frac{\tanh x}{x} - 3}{\left(\cosh x - 1\right)^2} = \frac{3}{5}.$$

This means that the inequality (10) holds with the best possible constant $p = \frac{3}{5}$.

This completes the proof of (i) from the theorem 5.

(ii) Since all coefficients $D(k)$ are negative, by using Theorem 1, we deduce the inequalities (11).

Example 4. *Taking $m = 4$ in inequalities (11) gives*

$$\begin{aligned} -\tfrac{11}{140}x^7 - \tfrac{27}{1400}x^9 &> \sinh 2x + \sinh x - 3x\cosh x - \tfrac{3}{5}x\cosh x\left(\cosh x - 1\right)^2 \\ &> -\tfrac{11}{140}x^7 + \tfrac{1}{b^4}\left(t\left(b\right) + \tfrac{11}{140}b^7\right)x^9 \end{aligned}$$

or equivalently

$$\begin{aligned} \left(-\tfrac{11}{140}x^6 - \tfrac{27}{1400}x^8\right)\tfrac{1}{\cosh x} &> 2\tfrac{\sinh x}{x} + \tfrac{\tanh x}{x} - 3 - \tfrac{3}{5}\left(\cosh x - 1\right)^2 \\ &> \left(-\tfrac{11}{140}x^6 + \tfrac{1}{b^4}\left(t\left(b\right) + \tfrac{11}{140}b^7\right)x^8\right)\tfrac{1}{\cosh x} \end{aligned}$$

for $0 < x < b$.

The proof of the theorem 6.

(i) Since the functions involved in the inequality (12) are even, we can assume $x > 0$.

Similarly, in order to avoid the Taylor expansion of $\tanh x$, we will rewrite the inequality (12) in an equivalent form:

$$\cosh x \sinh^2 x + x\sinh x - 2x^2\cosh x - \frac{32}{45}x^2\cosh x\left(\cosh x - 1\right)^2 < 0, \ x > 0.$$

After some elementary calculus, we have

$$\sinh^2 x \cosh x = \frac{\cosh 3x - \cosh x}{4}$$

and

$$\cosh x\left(\cosh x - 1\right)^2 = \frac{1}{4}\left(\cosh 3x - 4\cosh 2x + 7\cosh x - 4\right).$$

Using power series expansions (19) and (20) we find

$$\begin{aligned} s(x) &= \cosh x \sinh^2 x + x \sinh x - 2x^2 \cosh x - \frac{32}{45} x^2 \cosh x (\cosh x - 1)^2 \\ &= \frac{1}{4} \cosh 3x - \frac{1}{4} \cosh x + x \sinh x - \frac{8}{45} x^2 \cosh 3x + \\ & \quad \frac{32}{45} x^2 \cosh 2x - \frac{146}{45} x^2 \cosh x + \frac{32}{45} x^2 \\ &= \frac{32}{45} x^2 + \sum_{k=0}^{\infty} \frac{3^{2k} - 1}{4(2k)!} x^{2k} + \sum_{k=0}^{\infty} \left(\frac{1}{(2k+1)!} - \frac{8 \cdot 3^{2k} - 32 \cdot 2^{2k} + 145}{45(2k)!} \right) x^{2k+2} \\ &= \sum_{k=4}^{\infty} E(k) x^{2k} = -\frac{76}{945} x^8 - \frac{193}{9450} x^{10} - \frac{226}{93555} x^{12} - ..., \end{aligned}$$

where

$$E(k) = \frac{E_1(k)}{180(2k)!}$$

and

$$E_1(k) = 45(3^{2k}-1) + 360k - 64(2k-1) \cdot k \cdot 3^{2k-2} + 256(2k-1) \cdot k \cdot 2^{2k-2} - 1168(2k-1) \cdot k.$$

In order to prove that $E(k) < 0$ for all $k \in \mathbb{N}$, $k \geq 4$, we have to verify the inequality

$$4^{k-1}\left(512k^2 - 256k\right) < 9^{k-1}\left(128k^2 - 64k - 405\right) + 2336k^2 - 1528k + 45, \ k \in \mathbb{N}, k \geq 4.$$

By induction it is easy to demonstrate the above inequality.

Therefore we deduce the inequality (12).

In order to prove that $\frac{32}{45}$ is the best constant in the inequality (12), we consider the inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 < p(\cosh x - 1)^2, \text{ for all } x > 0.$$

This inequality holds for $p = \frac{32}{45}$. The above inequality can be rewritten as

$$p > \frac{\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2}{(\cosh x - 1)^2}, \ x > 0.$$

Let x tends to 0^+, we find that the ratio of the right-hand side tends to $\frac{32}{45}$.

This means that the inequality (12) holds for all $x \neq 0$ with the best constant $p = \frac{32}{45}$.

This completes the proof of (i) from the theorem 6.

(ii) All coefficients $E(k)$ are negative, therefore by applying Theorem 1, we get the inequalities (13).

Example 5. *Taking $m = 5$ in inequalities (13) gives*

$$-\tfrac{76}{945}x^8 - \tfrac{193}{9450}x^{10} > \cosh x \sinh^2 x + x \sinh x - 2x^2 \cosh x - \tfrac{32}{45}x^2 \cosh x \left(\cosh x - 1\right)^2$$
$$> -\tfrac{76}{945}x^8 + \tfrac{1}{b^{10}}\left(s\left(b\right) + \tfrac{76}{945}b^8\right)x^{10}$$

or equivalently

$$\left(-\tfrac{76}{945}x^6 - \tfrac{193}{9450}x^8\right)\tfrac{1}{\cosh x} > \left(\tfrac{\sinh x}{x}\right)^2 + \tfrac{\tanh x}{x} - 2 - \tfrac{32}{45}\left(\cosh x - 1\right)^2$$
$$> \left(-\tfrac{76}{945}x^6 + \tfrac{1}{b^{10}}\left(s\left(b\right) + \tfrac{76}{945}b^8\right)x^8\right)\tfrac{1}{\cosh x}$$

for $0 < x < b$.

The proof of the theorem 7.

(i) We also can assume $x > 0$.

We will rewrite the inequality (14) as follows:

$$\sinh^3 x - x^3 \cosh x - \frac{4}{15}x^3 \cosh x \left(\cosh x - 1\right)^2 < 0, \text{ for all } x > 0.$$

Since

$$\sinh^3 x = \frac{\sinh\left(3x\right) - 3\sinh x}{4}$$

and

$$\cosh x \left(\cosh x - 1\right) = \frac{1}{4}\left(\cosh 3x - 4\cosh 2x + 7\cosh x - 4\right),$$

the above inequality takes the equivalent form

$$\frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x - \frac{1}{15}x^3 \cosh 3x + \frac{4}{15}x^3 \cosh 2x - \frac{22}{15}x^3 \cosh x + \frac{4}{15}x^3 < 0, \text{ for all } x > 0.$$

The Taylor expansion of the function from the left-hand side is

$$\sum_{k=0}^{\infty}\frac{3^{2k+1} - 3}{4\left(2k+1\right)!}x^{2k+1} + \sum_{k=0}^{\infty}\left(\frac{-3^{2k}}{15 \cdot (2k)!} + \frac{4 \cdot 2^{2k}}{15 \cdot (2k)!} - \frac{22}{15 \cdot (2k)!}\right)x^{2k+3} + 16x^3$$
$$= \sum_{k=4}^{\infty}\frac{15\left(3^{2k+1} - 3\right) + (2k-1) \cdot 2k \cdot (2k+1)\left(-4 \cdot 3^{2k-2} + 16 \cdot 2^{2k-2} - 88\right)}{0 \cdot (2k+1)!}x^{2k+1}$$
$$= \sum_{k=4}^{\infty}F\left(k\right)x^{2k+1} = -\frac{61}{1890}x^9 - \frac{611}{75600}x^{11} - \frac{59}{62370}x^{13} - \cdots$$

The expression $F\left(k\right)$ can be rewrite as

$$F\left(k\right) = \frac{4^k\left(32k^3 - 8k\right) - 9^{k-1}\left(32k^3 - 8k - 405\right) - 704k^3 + 176k - 45}{60 \cdot (2k+1)!}.$$

By induction it is easy to prove that

$$4^k\left(32k^3 - 8k\right) < 9^{k-1}\left(32k^3 - 8k - 405\right) + 704k^3 - 176k + 45, \text{ for all } k \in \mathbb{N}, k \geq 4,$$

therefore all corresponding coefficients are negative.

Hence we obtain the inequality (14).

In order to prove that $\frac{4}{15}$ is the best constant in the inequality (14), we consider the inequality

$$\left(\frac{\sinh x}{x}\right)^2 \cdot \frac{\tanh x}{x} - 1 < p\left(\cosh x - 1\right)^2, \ x > 0, \, p \text{ real constant.}$$

From the first part of the proof, this inequality holds for $p = \frac{4}{15}$.

The above inequality takes the equivalent form

$$p > \frac{\left(\frac{\sinh x}{x}\right)^2 \cdot \frac{\tanh x}{x} - 1}{\left(\cosh x - 1\right)^2}, \text{ for all } x > 0.$$

Using l'Hospital rules, it turns out that

$$\lim_{x \to 0^+} \frac{\left(\frac{\sinh x}{x}\right)^2 \cdot \frac{\tanh x}{x} - 1}{\left(\cosh x - 1\right)^2} = \frac{4}{15}.$$

Therefore, the inequality (14) holds with the best possible constant $p = \frac{4}{15}$.

This completes the proof of (i) from the theorem 7.

(ii) All coefficients $F(k)$ are negative, hence by using Theorem 1, we deduce the inequalities (15).

Example 6. *Taking $m = 5$ in inequalities (15) gives*

$$\begin{aligned} -\tfrac{61}{1890}x^9 - \tfrac{611}{75600}x^{11} &> \sinh^3 x - x^3 \cosh x - \tfrac{4}{15}x^3 \cosh x \left(\cosh x - 1\right)^2 \\ &> -\tfrac{61}{1890}x^9 + \tfrac{1}{b^{11}}\left(r\left(b\right) + \tfrac{61}{1890}b^9\right)x^{11} \end{aligned}$$

or equivalently

$$\begin{aligned} \left(-\tfrac{61}{1890}x^6 - \tfrac{611}{75600}x^8\right)\tfrac{1}{\cosh x} &> \left(\tfrac{\sinh x}{x}\right)^2 \cdot \tfrac{\tanh x}{x} - 1 - \tfrac{4}{15}\left(\cosh x - 1\right)^2 \\ &> \left(-\tfrac{61}{1890}x^6 + \tfrac{1}{b^{11}}\left(r\left(b\right) + \tfrac{61}{1890}b^9\right)x^8\right)\tfrac{1}{\cosh x} \end{aligned}$$

for $0 < x < b$.

4. FINAL REMARKS

In this paper, Taylor expansion of the error function between the truncated sum of the first terms of the hyperbolic cosine series of the functions involved in Huygens - Wilker - Lazarović inequalities and the functions themselves is carried out. Then the best approximation of the functions which improve Huygens - Wilker - Lazarović inequalities is obtained.

We also find the inequalities between the left - hand sides of the Huygens - Wilker - Lazarović inequalities for the trigonometric functions and for the hyperbolic trigonometric functions respectively:

For all $x \in \left(0, \dfrac{\pi}{2}\right)$, one has:

i) $2\sin x + \tan x > 2\sinh x + \tanh x$

ii) $\sin^2 x + x\tan x > \sinh^2 x + x\tanh x$

iii) $\sin^2 x\tan x > \sinh^2 x\tanh x$.

We are convinced that the hyperbolic cosine polynomials series type method is suitable for refining many other analytical inequalities.

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