Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **16** (2022), 307–327.
https://doi.org/10.2298/AADM200120037M

UMBRAL OPERATORS FOR CAYLEY AND SYLVESTER CONTINUANTS

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We study the main umbral operators J, M and N associated with the Cayley continuants $U_n^{(\nu)}(x)$ and the generalized Sylvester continuants $H_n^{(\nu)}(x) = U_n^{(\nu+n)}(x)$. In particular, we obtain their representation in terms of the differential operator $\mathfrak{D}_x$ and the shift operator E. Then, by using these representations, we obtain some combinatorial and differential identities for the continuants $U_n^{(\nu)}(x)$ and $H_n^{(\nu)}(x)$.

1. INTRODUCTION

In 1854, James Joseph Sylvester [**24**] ([**12**, Vol. 2, pp. 425–426] [**25**]) considered the $n \times n$ tridiagonal determinants (or continuants)

$$H_n(\lambda) = \begin{vmatrix} \lambda & 1 & & & & & & \\ n-1 & \lambda & 2 & & & & & \\ & n-2 & \lambda & 3 & & & & \\ & & n-3 & \lambda & 4 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & \lambda & n-2 & \\ & & & & & 2 & \lambda & n-1 \\ & & & & & & 1 & \lambda \end{vmatrix}$$

stating, without proof, that

$$\begin{aligned} H_n(\lambda) &= \prod_{k=1}^{n} (\lambda - n + 2k - 1) \\ &= (\lambda - n + 1)(\lambda - n + 3) \cdots (\lambda + n - 3)(\lambda + n - 1)\,. \end{aligned}$$

2020 Mathematics Subject Classification. 05A40, 05A19, 05A15.
Keywords and Phrases. Umbral operators, Sheffer sequences, Tridiagonal determinants, Meixner polynomials, Central factorial polynomials.

This identity has been proved by Francesco Mazza in 1866 [**12**, vol. 2, p. 442] by means of elementary operations on the lines, and then by many others through various different techniques [**25, 8, 1, 10, 5**]. For $\lambda = 0$, the matrices underling Sylvester's determinants are also known as *Clement matrices* [**6**] or *Kac matrices* [**11**].

In 1858, Arthur Cayley [**4**] considered the $n \times n$ tridiagonal determinants

$$U_n^{(\nu)}(x) = \begin{vmatrix} x & 1 & & & & & & \\ \nu & x & 2 & & & & & \\ & \nu-1 & x & 3 & & & & \\ & & \nu-2 & x & 4 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & x & n-2 & \\ & & & & & \nu-n+3 & x & n-1 \\ & & & & & & \nu-n+2 & x \end{vmatrix}$$

and observed that the Sylvester continuants are given by

$$H_n(\lambda) = U_n^{(n-1)}(\lambda)\,.$$

Then, with a direct and rather laborious approach, he proved the identity

$$U_n^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{n-2k}\frac{(-1)^k}{2^k}k!\,p_k(\nu-n)\,H_{n-2k}(x) \tag{1}$$

where $p_n(x) = (x+1)(x+3)\cdots(x+2n-1)$. In [**16**], we obtained a simple proof of this identity by using the modern theory of formal series. In order to obtain such a proof, we considered the *generalized Sylvester continuants*

$$H_n^{(\nu)}(x) = U_n^{(\nu+n)}(x)\,,$$

that is the $n \times n$ tridiagonal determinants

$$H_n^{(\nu)}(x) = \begin{vmatrix} x & 1 & & & & & & \\ \nu+n & x & 2 & & & & & \\ & \nu+n-1 & x & 3 & & & & \\ & & \nu+n-2 & x & 4 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & x & n-2 & \\ & & & & & \nu+3 & x & n-1 \\ & & & & & & \nu+2 & x \end{vmatrix}.$$

Clearly, they generalize the original Sylvester continuants, since

$$H_n(x) = H_n^{(-1)}(x) = U_n^{(n-1)}(x)\,.$$

A relevant feature of the Cayley continuants and the generalized Sylvester continuants is that they belong to an important class of polynomials sequences, namely the class of Sheffer sequences. These sequences appear in various contexts, such as applied mathematics, physics, geometric probability, interpolation of functions, number theory, umbral calculus and combinatorics. They include several classical polynomials and possess numerous algebraic, analytic and combinatorial properties [**2, 18, 20, 21**] (see also [**13, 14, 15, 3, 9**]). Furthermore, several Sheffer polynomials can be expressed in terms of the Cayley continuants or the generalized Sylvester continuants, such as the Meixner polynomials of the first kind, the Mittag-Leffler polynomials, the Pidduck polynomials and the central factorial polynomials.

Given a polynomial sequence $\{p_n(x)\}_{n\geq 0}$, where each $p_n(x) \in \mathbb{Q}[x]$ has degree n, we can consider the linear operators $J, M, N : \mathbb{Q}[x] \to \mathbb{Q}[x]$ defined, for every $n \in \mathbb{N}$, by

$$\begin{aligned} Jp_n(x) &= np_{n-1}(x) \\ Mp_n(x) &= p_{n+1}(x) \\ Np_n(x) &= np_n(x)\,. \end{aligned}$$

The operator J is the *umbral derivative* (or *lowering operator*, or *annihilation operator*), the operator M is the *umbral shift* (or *raising operator*, or *creation operator*) and the operator N is the *umbral theta operator* associated with the sequence $\{p_n(x)\}_{n\geq 0}$. In particular, when $p_n(x) = x^n$, the operator J is the usual derivative $\mathfrak{D}_x$ with respect to x, while the operator N is the usual theta operator $\Theta_x = x\,\mathfrak{D}_x$.

By Pincherle's theorem [**17**], every linear operator $L : \mathbb{Q}[x] \to \mathbb{Q}[x]$ can be uniquely represented as an exponential series in the derivative $\mathfrak{D}_x : \mathbb{Q}[x] \to \mathbb{Q}[x]$, with coefficients in $\mathbb{Q}[x]$. More precisely, there exists a unique polynomial sequence $\{L_n(x)\}_{n\geq 0}$, where $L_n(x) \in \mathbb{Q}[x]$ for every $n \in \mathbb{N}$, such that

$$L = \sum_{k\geq 0} L_k(x)\,\frac{\mathfrak{D}_x^k}{k!}\,.$$

So, for every polynomial $p(x) \in \mathbb{Q}[x]$ of degree n, we have

$$Lp(x) = \sum_{k\geq 0} \frac{L_k(x)}{k!}\;\mathfrak{D}_x^k\,p(x) = \sum_{k=0}^{n} \frac{L_k(x)}{k!}\,p^{(k)}(x)\,.$$

For instance, the *shift operator* E^λ, defined by $E^\lambda p(x) = p(x+\lambda)$, is represented by the exponential series $\mathrm{e}^{\lambda\,\mathfrak{D}_x}$. Furthermore, the Sheffer sequences are characterized [**22, 18**] as the polynomial sequences for which the umbral operator J can be represented as

$$J = \sum_{k\geq 0} J_k\,\frac{\mathfrak{D}_x^k}{k!}$$

where all the coefficients J_k are constants.

In this paper we will study the umbral operators J, M and N for the Cayley continuants and the generalized Sylvester continuants. Specifically, in Section 2, we review the main properties of the continuants $U_n^{(\nu)}(x)$ and $H_n^{(\nu)}(x)$, and we prove that Cayley's identity (1) can be extended to the generalized Sylvester continuants. Then, in Section 3, we review the main definitions and properties of the Sheffer sequences and the associated umbral operators. Finally, in Sections 4 and 5, we obtain an explicit representation of the umbral operators J, M and N in terms of the differential operator $\mathfrak{D}_x$ and the shift operator E. Then, by using these representations, we obtain some combinatorial and differential identities involving the continuants $U_n^{(\nu)}(x)$ and $H_n^{(\nu)}(x)$. We also specialize some of these results to the Meixner polynomials of the first kind and to the central factorial polynomials.

2. CAYLEY AND SYLVESTER CONTINUANTS

The *Cayley continuants* $U_n^{(\nu)}(x)$ are polynomials of degree n in x. For the first values of n, we have

$$\begin{aligned}
U_0^{(\nu)}(x) &= 1\\
U_1^{(\nu)}(x) &= x\\
U_2^{(\nu)}(x) &= x^2 - \nu\\
U_3^{(\nu)}(x) &= x^3 - (3\nu - 2)x\\
U_4^{(\nu)}(x) &= x^4 - (6\nu - 8)x^2 + 3\nu^2 - 6\nu\\
U_5^{(\nu)}(x) &= x^5 - (10\nu - 20)x^3 + (15\nu^2 - 50\nu + 24)x\\
U_6^{(\nu)}(x) &= x^6 - (15\nu - 40)x^4 + (45\nu^2 - 210\nu + 184)x^2 - 15\nu^3 + 90\nu^2 - 120\nu\,.
\end{aligned}$$

Moreover, they satisfy the recurrence [**4, 16**]

$$U_{n+2}^{(\nu)}(x) = xU_{n+1}^{(\nu)}(x) - (n+1)(\nu - n)U_n^{(\nu)}(x)$$

from which it is straightforward to obtain the exponential generating series [**16**]

$$U^{(\nu)}(x;t) = \sum_{n\geq 0} U_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{(1+t)^{(x+\nu)/2}}{(1-t)^{(x-\nu)/2}} = (1-t^2)^{\nu/2}\left(\frac{1+t}{1-t}\right)^{x/2} \tag{2}$$

and, consequently, the expression

$$U_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x+\nu}{2}\right)^{\underline{k}}\left(\frac{x-\nu}{2}\right)^{\overline{n-k}}$$

where the polynomials $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ are the *falling factorials* and the polynomials $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$ are the *rising factorials.*

Also the *generalized Sylvester continuants* are polynomials of degree n in x. The first few are

$$H_0^{(\nu)}(x) = 1$$
$$H_1^{(\nu)}(x) = x$$
$$H_2^{(\nu)}(x) = x^2 - \nu - 2$$
$$H_3^{(\nu)}(x) = x^3 - (3\nu + 7)x$$
$$H_4^{(\nu)}(x) = x^4 - (6\nu + 16)x^2 + 3\nu^2 + 18\nu + 24$$
$$H_5^{(\nu)}(x) = x^5 - (10\nu + 30)x^3 + (15\nu^2 + 100\nu + 149)x$$
$$H_6^{(\nu)}(x) = x^6 - (15\nu + 50)x^4 + (45\nu^2 + 330\nu + 544)x^2 - 15\nu^3 - 180\nu^2 - 660\nu - 720\,.$$

Furthermore, their exponential generating series is given by

$$H^{(\nu)}(x;t) = \sum_{n\geq 0} H_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{(t+\sqrt{1+t^2})^x}{(1+t^2)^{\nu/2+1}}\,. \tag{3}$$

From this generating series we can derive a recurrence for the generalized Sylvester continuants. More precisely, we have the following theorem.

Theorem 1. *The generalized Sylvester continuants satisfy the recurrence*

$$H_{n+2}^{(\nu)}(x) = xH_{n+1}^{(\nu)}(x-1) + (n+1)(x-\nu-n-2)H_n^{(\nu)}(x)\,. \tag{4}$$

Proof. Differentiating series (3) with respect to t, we have

$$\frac{\partial}{\partial t}H^{(\nu)}(x;t) = \frac{x}{1+t^2}\,H^{(\nu)}(x-1;t) + \frac{(x-\nu-2)t}{1+t^2}\,H^{(\nu)}(x;t)\,.$$

Hence, we have the identity

$$(1+t^2)\frac{\partial}{\partial t}H^{(\nu)}(x;t) = x\,H^{(\nu)}(x-1;t) + (x-\nu-2)t\,H^{(\nu)}(x;t)$$

from which we immediately have

$$H_{n+1}^{(\nu)}(x) + n(n-1)H_{n-1}^{(\nu)}(x) = xH_n^{(\nu)}(x-1) + n(x-\nu-2)H_{n-1}^{(\nu)}(x)$$

that is

$$H_{n+1}^{(\nu)}(x) = xH_n^{(\nu)}(x-1) + n(x-\nu-n-1)H_{n-1}^{(\nu)}(x)\,.$$

Finally, replacing n by $n+1$, we obtain recurrence (4). □

Cayley's identity (1) can be generalized as follows.

Theorem 2. *We have the identities*

$$U_n^{(\mu-\nu-1)}(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k}\binom{n-k}{n-2k}\frac{k!}{2^k}\,p_k(\nu)\,U_{n-2k}^{(\mu)}(x) \tag{5}$$

$$H_n^{(\mu-\nu-1)}(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k}\binom{n-k}{n-2k} \frac{k!}{2^k}\, p_k(\nu-n)\, U_{n-2k}^{(\mu)}(x) \tag{6}$$

$$H_n^{(\mu+\nu+1)}(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k}\binom{n-k}{n-2k} \frac{(-1)^k}{2^k}\, k!\, p_k(\nu)\, H_{n-2k}^{(\mu)}(x) \tag{7}$$

$$U_n^{(\mu+\nu+1)}(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k}\binom{n-k}{n-2k} \frac{(-1)^k}{2^k}\, k!\, p_k(\nu-n)\, H_{n-2k}^{(\mu)}(x) \tag{8}$$

where $p_n(x) = (x+1)(x+3)\cdots(x+2n-1) = 2^n\left(\frac{x+1}{2}\right)^{\overline{n}}$.

Proof. From the generating series

$$p^{(\nu)}(t) = \sum_{n\geq 0} p_n(\nu)\,\frac{t^n}{n!} = \frac{1}{(1-2t)^{\frac{\nu+1}{2}}}$$

we have

$$p^{(\nu)}(t^2/2) = \sum_{n\geq 0} \binom{2n}{n} \frac{n!}{2^n}\, p_n(\nu)\, \frac{t^{2n}}{(2n)!} = \frac{1}{(1-t^2)^{\frac{\nu+1}{2}}}$$

and

$$p^{(\nu)}(-t^2/2) = \sum_{n\geq 0} \binom{2n}{n} \frac{(-1)^n}{2^n}\, n! p_n(\nu)\, \frac{t^{2n}}{(2n)!} = \frac{1}{(1+t^2)^{\frac{\nu+1}{2}}}\,.$$

Hence, from series (2) and (3), we have the relations

$$p^{(\nu)}(t^2/2)\, U^{(\mu)}(x;t) = U^{(\mu-\nu-1)}(x;t)$$

and

$$p^{(\nu)}(-t^2/2)\, H^{(\mu)}(x;t) = H^{(\mu+\nu+1)}(x;t)$$

from which we obtain identities (5) and (7), respectively. Finally, replacing ν by $\nu-n$ in (5) and (7), we obtain identities (6) and (8). □

Several classical polynomials can be expressed in terms of the Cayley continuants or the generalized Sylvester continuants. For instance, the *Meixner polynomials of the first kind* $M_n^{(\alpha)}(x)$ [**20**, p. 125] have exponential generating series

$$\sum_{n\geq 0} M_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^\alpha}\left(\frac{1+t}{1-t}\right)^x = U^{(-\alpha)}(2x+\alpha;t) \tag{9}$$

and consequently

$$M_n^{(\alpha)}(x) = U_n^{(-\alpha)}(2x+\alpha)\,. \tag{10}$$

For $\alpha = 0$, we have the *Mittag-Leffler polynomials* $M_n(x) = U_n^{(0)}(2x)$ [**20**, p. 75]. For $\alpha = 1$, we have the *Pidduck polynomials* $P_n(x) = U_n^{(-1)}(2x+1)$ [**20**, p. 126].

Notice that, with the substitution $x \to 2x+\alpha$, $\mu \to \nu$ and $\nu \to \nu+\alpha-1$ in (5), we have

$$M_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{n-2k} k!((\nu+\alpha)/2)^{\overline{k}}\, U_{n-2k}^{(\nu)}(2x+\alpha)\,,$$

while, with the substitution $\mu \to -\alpha$ and $\nu \to -\nu-\alpha-1$ always in (5), we have

$$U_n^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{n-2k} (-1)^k k!((\nu+\alpha)/2)^{\underline{k}}\, M_{n-2k}^{(\alpha)}((x-\alpha)/2)\,.$$

Furthermore, the *central factorial polynomials* [**23**] [**19**, pp. 212-217] [**20**, p. 68], defined by

$$x^{[0]} = 1\,, \qquad x^{[n]} = x\prod_{k=1}^{n-1}(x+n/2-k) = x(x+n/2-1)^{\underline{n-1}} \qquad n \geq 1\,,$$

have exponential generating series

$$\sum_{n\geq 0} x^{[n]}\,\frac{t^n}{n!} = \mathrm{e}^{2x\,\mathrm{arcsinh}(t/2)} = \left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)^{2x} = H^{(-2)}(2x;t/2). \tag{11}$$

Hence, we have

$$x^{[n]} = \frac{1}{2^n}\,H_n^{(-2)}(2x) \qquad \text{or} \qquad x^{[n]} = \frac{x}{2^{n-1}}\,H_{n-1}(2x) \qquad \text{for } n \geq 1\,. \tag{12}$$

In this case, with the substitution $x \to 2x$, $\mu \to -2$ and $\nu \to 0$ in (7), we have

$$\frac{1}{2^n}\,H_n(2x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{n-2k} \frac{(-1)^k}{2^{4k}}\,(2k)!\,x^{[n-2k]}\,,$$

while, with the substitution $x \to 2x$, $\mu \to -1$ and $\nu \to -2$ always in (7), we have

$$x^{[n]} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{n-2k} \frac{(-1)^{k+1}}{2^{n+2k}}\,\frac{(2k)!}{2k-1}\,H_{n-2k}(2x)\,.$$

3. SHEFFER SEQUENCES AND UMBRAL OPERATORS

A *Sheffer sequence* [**22**] is a polynomial sequence $\{s_n(x)\}_{n\geq 0}$ having exponential generating series

$$s(x;t) = \sum_{n\geq 0} s_n(x)\,\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xf(t)}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ are two exponential series with $g_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$. We say that the Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ has *spectrum* $(g(t), f(t))$.

A Sheffer sequence $\{s_n(x)\}_{n\geq 0}$, where $s_n(x) = \sum_{k=0}^{n} s_{n,k} x^k$, is equivalent to a lower triangular matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$, called the associated Sheffer matrix, where the columns have exponential generating series

$$\sum_{n\geq k} s_{n,k} \frac{t^n}{n!} = g(t)\, \frac{f(t)^k}{k!}\,.$$

These matrices form a group with respect to the ordinary matrix multiplication. Specifically, the product of two Sheffer matrices $(g_1(t), f_1(t))$ and $(g_2(t), f_2(t))$ is defined by

$$(g_1(t), f_1(t))\,(g_2(t),\ f_2(t)) = (g_1(t)\, g_2(f_1(t)),\ f_2(f_1(t)))\,,$$

the neutral element is the identity matrix $I = (1, t)$, and the inverse of a Sheffer matrix $(g(t), f(t))$ is given by

$$(g(t), f(t))^{-1} = \left(g(\widehat{f}(t))^{-1}, \widehat{f}(t) \right)$$

where $\widehat{f}(t)$ is the compositional inverse of $f(t)$.

The umbral operators J, M and N associated with a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ can be expressed [**20**, pp. 49-50] in terms of its spectrum $(g(t), f(t))$ as follows

$$J = \widehat{f}(\mathfrak{D}_x) \tag{13}$$

$$M = \frac{g'(\widehat{f}(\mathfrak{D}_x))}{g(\widehat{f}(\mathfrak{D}_x))} + x f'(\widehat{f}(\mathfrak{D}_x)) \tag{14}$$

$$N = MJ = \left(\frac{g'(\widehat{f}(\mathfrak{D}_x))}{g(\widehat{f}(\mathfrak{D}_x))} + x f'(\widehat{f}(\mathfrak{D}_x)) \right) \widehat{f}(\mathfrak{D}_x)\,. \tag{15}$$

In this case, the operator M is also called *Sheffer shift* [**20**, p. 45].

Sheffer sequences are preserved by several transformations. For instance, if $p, q, \alpha \in \mathbb{Q}$, $p, q \neq 0$, and $\{s_n(x)\}_{n\geq 0}$ is a Sheffer sequence with spectrum $(g(t), f(t))$, then $\{p^n s_n(qx + \alpha)\}_{n\geq 0}$ is a Sheffer sequence with spectrum $(g(pt)\, e^{\alpha f(pt)}, q f(pt))$. Hence, if the sequence $\{s_n(x)\}_{n\geq 0}$ has umbral operators $J(\mathfrak{D}_x)$, $M(x, \mathfrak{D}_x)$ and $N(x, \mathfrak{D}_x)$, then the sequence $\{p^n s_n(qx)\}_{n\geq 0}$ has umbral operators

$$J^{(p,q,\alpha)}(\mathfrak{D}_x) = \frac{1}{p} J(\mathfrak{D}_x / q) \tag{16}$$

$$M^{(p,q,\alpha)}(x, \mathfrak{D}_x) = pM(qx + \alpha, \mathfrak{D}_x / q) \tag{17}$$

$$N^{(p,q,\alpha)}(x, \mathfrak{D}_x) = N(qx + \alpha, \mathfrak{D}_x / q)\,. \tag{18}$$

From the exponential generating series (2), (9), (3) and (11), we have that the Cayley continuants $U_n^{(\nu)}(x)$, the Meixner polynomials $M_n^{(\alpha)}(x)$, the generalized Sylvester continuants $H_n^{(\nu)}(x)$ and the central factorials $x^{[n]}$ form a Sheffer sequence, respectively with spectrum

$$U^{(\nu)} = \left((1-t^2)^{\nu/2}, \frac{1}{2}\ln\frac{1+t}{1-t}\right) = \left((1-t^2)^{\nu/2}, \operatorname{arctanh} t\right) \tag{19}$$

$$M^{(\alpha)} = \left(\frac{1}{(1-t)^{\alpha}}, \ln\frac{1+t}{1-t}\right) = \left(\frac{1}{(1-t)^{\alpha}}, 2\operatorname{arctanh} t\right)$$

$$H^{(\nu)} = \left(\frac{1}{(1+t^2)^{\nu/2+1}}, \ln(t+\sqrt{1+t^2})\right) = \left(\frac{1}{(1+t^2)^{\nu/2+1}}, \operatorname{arcsinh} t\right) \tag{20}$$

$$C = \left(1, 2\ln\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right) = (1, 2\operatorname{arcsinh}(t/2))\,.$$

Notice that the inverses of the associated Sheffer matrices are

$$(U^{(\nu)})^{-1} = (\cosh^{\nu} t, \tanh t) \qquad (M^{(\alpha)})^{-1} = \left(\left(\frac{2}{\mathrm{e}^t+1}\right)^{\alpha}, \tanh(t/2)\right)$$

$$(H^{(\nu)})^{-1} = (\cosh^{\nu+2} t, \sinh t) \qquad C^{-1} = (1, 2\sinh(t/2)).$$

4. UMBRAL OPERATORS FOR THE CAYLEY CONTINUANTS

4.1 Operator J

The umbral derivative admits the following representations.

Theorem 3. *The operator* J *is given by*

$$J = \tanh \mathfrak{D}_x \tag{21}$$

$$J = \frac{E - E^{-1}}{E + E^{-1}}\,. \tag{22}$$

Proof. By formula (13), we have $J = \widehat{f}(\mathfrak{D}_x)$. By spectrum (19), we have

$$f(t) = \frac{1}{2}\ln\frac{1+t}{1-t} = \operatorname{arctanh} t \tag{23}$$

and consequently

$$\widehat{f}(t) = \tanh t\,. \tag{24}$$

This immediately implies formula (21). Then, since the operator $\mathrm{e}^{\lambda\,\mathfrak{D}_x}$ corresponds to the shift operator E^{λ}, formula (21) can be rewritten as

$$J = \tanh \mathfrak{D}_x = \frac{\mathrm{e}^{\mathfrak{D}_x} - \mathrm{e}^{-\mathfrak{D}_x}}{\mathrm{e}^{\mathfrak{D}_x} + \mathrm{e}^{-\mathfrak{D}_x}} = \frac{E - E^{-1}}{E + E^{-1}}$$

and this is formula (22). □

Theorem 3 implies the following two results.

Theorem 4. *The Cayley continuants satisfy the differential identity*

$$\sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{2^{2k}(2^{2k}-1)}{(2k)!} B_{2k}\ \mathfrak{D}_x^{2k-1}\ U_n^{(\nu)}(x) = n\,U_{n-1}^{(\nu)}(x) \tag{25}$$

where the coefficients B_n are the Bernoulli numbers.

Proof. We have $JU_n^{(\nu)}(x) = nU_{n-1}^{(\nu)}(x)$, where (by formula (21))

$$J = \tanh \mathfrak{D}_x = \sum_{k\geq 1} \frac{2^{2k}(2^{2k}-1)}{(2k)!} B_{2k}\ \mathfrak{D}_x^{2k-1}\ .$$

Since $U_n^{(\nu)}(x)$ is a polynomial of degree n, we have identity (25). □

Theorem 5. *The Cayley continuants satisfy the identity*

$$U_n^{(\nu)}(x+1) - U_n^{(\nu)}(x-1) = nU_{n-1}^{(\nu)}(x+1) + nU_{n-1}^{(\nu)}(x-1)\,. \tag{26}$$

Proof. Since $JU_n^{(\nu)}(x) = nU_{n-1}^{(\nu)}(x)$, we have (by formula (22))

$$\frac{E-E^{-1}}{E+E^{-1}}\,U_n^{(\nu)}(x) = nU_{n-1}^{(\nu)}(x)$$

or

$$(E-E^{-1})U_n^{(\nu)}(x) = n(E+E^{-1})U_{n-1}^{(\nu)}(x)\,.$$

This is identity (26). □

Finally, we can express the derivative of a Cayley continuant as a linear combination of the same Cayley continuants.

Theorem 6. *The Cayley continuants satisfy the identity*

$$\mathfrak{D}_x\, U_{n+1}^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+1}{2k+1} (2k)!\, U_{n-2k}^{(\nu)}(x)\,. \tag{27}$$

Proof. From formula (21), we have $J = \tanh \mathfrak{D}_x$. Consequently, we have

$$\mathfrak{D}_x = \operatorname{arctanh} J = \sum_{k\geq 0} \frac{J^{2k+1}}{2k+1}\,.$$

Since $JU_n^{(\nu)}(x) = nU_{n-1}^{(\nu)}(x)$, we have

$$J^k U_n^{(\nu)}(x) = n(n-1)\cdots(n-k+1)\,U_{n-k}^{(\nu)}(x) = \binom{n}{k} k!\, U_{n-k}^{(\nu)}(x)\,.$$

Hence, we have

$$\mathfrak{D}_x\, U_{n+1}^{(\nu)}(x) = \sum_{k\geq 0} \frac{1}{2k+1}\, J^{2k+1}\, U_{n+1}^{(\nu)}(x) = \sum_{k\geq 0} \binom{n+1}{2k+1} \frac{(2k+1)!}{2k+1}\, U_{n-2k}^{(\nu)}(x)$$

which simplifies in identity (27). □

4.2 Operator M

The umbral shift admits the following representations.

Theorem 7. *The operator M is given by*

$$M = \frac{x}{2} + \frac{x}{2}\cosh 2\,\mathfrak{D}_x - \frac{\nu}{2}\sinh 2\,\mathfrak{D}_x \tag{28}$$

$$M = \frac{x-\nu}{4}\,E^2 + \frac{x}{2} + \frac{x+\nu}{4}\,E^{-2} \tag{29}$$

Proof. The operator M is given by formula (14). By spectrum (19), we have

$$g(t) = (1-t^2)^{\nu/2}\,, \qquad g'(t) = -\frac{\nu t}{1-t^2}\,g(t) \qquad \text{and} \qquad \frac{g'(t)}{g(t)} = -\frac{\nu t}{1-t^2}\,.$$

Then, by identity (24), we have

$$\frac{g'(\widehat{f}(t))}{g(\widehat{f}(t))} = -\frac{\nu\,\tanh t}{1-\tanh^2 t} = -\frac{\nu}{2}\,\sinh 2t\,.$$

Moreover, by identity (23), we have

$$f'(t) = \frac{1}{1-t^2} \qquad \text{and} \qquad f'(\widehat{f}(t)) = \frac{1}{1-\tanh^2 t} = \cosh^2 t = \frac{\cosh 2t + 1}{2}\,.$$

Hence

$$M = -\frac{\nu}{2}\,\sinh 2\,\mathfrak{D}_x + x\,\frac{\cosh 2\,\mathfrak{D}_x + 1}{2}$$

and this implies formula (28). Using such a formula and the identity $\mathrm{e}^{\lambda\,\mathfrak{D}_x} = E^{\lambda}$, we have

$$\begin{aligned} M &= -\frac{\nu}{2}\,\frac{\mathrm{e}^{2\,\mathfrak{D}_x} - \mathrm{e}^{-2\,\mathfrak{D}_x}}{2} + \frac{x}{2}\,\frac{\mathrm{e}^{2\,\mathfrak{D}_x} + \mathrm{e}^{-2\,\mathfrak{D}_x}}{2} + \frac{x}{2} \\ &= -\frac{\nu}{4}\,(E^2 - E^{-2}) + \frac{x}{4}\,(E^2 + E^{-2}) + \frac{x}{2} \end{aligned}$$

and this simplifies in formula (29). □

By Theorem 7 we can obtain the following two recurrences for the Cayley continuants.

Theorem 8. *The Cayley continuants satisfy the recurrence*

$$\begin{aligned} U_{n+1}^{(\nu)}(x) = \frac{x}{2}\,U_n^{(\nu)}(x) + \frac{x}{2}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{2^{2k}}{(2k)!}\,\mathfrak{D}_x^{2k}\,U_n^{(\nu)}(x) + \\ -\frac{\nu}{2}\sum_{k=0}^{\lfloor (n-1)/2\rfloor} \frac{2^{2k+1}}{(2k+1)!}\,\mathfrak{D}_x^{2k+1}\,U_n^{(\nu)}(x)\,. \end{aligned} \tag{30}$$

Proof. We have $MU_n^{(\nu)}(x) = U_{n+1}^{(\nu)}(x)$, where (by formula (28))

$$\begin{aligned} M &= \frac{x}{2} + \frac{x}{2}\cosh 2\mathfrak{D}_x - \frac{\nu}{2}\sinh 2\mathfrak{D}_x \\ &= \frac{x}{2} + \frac{x}{2}\sum_{k\geq 0}\frac{2^{2k}}{(2k)!}\mathfrak{D}_x^{2k} - \frac{\nu}{2}\sum_{k\geq 0}\frac{2^{2k+1}}{(2k+1)!}\mathfrak{D}_x^{2k+1}\,. \end{aligned}$$

Since $U_n^{(\nu)}(x)$ is a polynomial of degree n, we have identity (30). □

Theorem 9. *The Cayley continuants satisfy the recurrence*

$$U_{n+1}^{(\nu)}(x) = \frac{x-\nu}{4}U_n^{(\nu)}(x+2) + \frac{x}{2}U_n^{(\nu)}(x) + \frac{x+\nu}{4}U_n^{(\nu)}(x-2)\,. \tag{31}$$

Proof. We have $MU_n^{(\nu)}(x) = U_{n+1}^{(\nu)}(x)$, where the operator M is given by formula (29). This implies recurrence (31). □

4.3 Operator N

The umbral theta operator admits the following representations.

Theorem 10. *The operator N is given by*

$$N = \frac{\nu}{2} - \frac{\nu}{2}\cosh 2\mathfrak{D}_x + \frac{x}{2}\sinh 2\mathfrak{D}_x \tag{32}$$

$$N = \frac{x-\nu}{4}E^2 + \frac{\nu}{2} - \frac{x+\nu}{4}E^{-2}\,. \tag{33}$$

Proof. The operator N is given by formula (15). Hence, by identities (28) and (21), we have

$$\begin{aligned} N &= \left(x\cosh^2\mathfrak{D}_x - \nu\sinh\mathfrak{D}_x\cosh\mathfrak{D}_x\right)\frac{\sinh\mathfrak{D}_x}{\cosh\mathfrak{D}_x} \\ &= x\sinh\mathfrak{D}_x\cosh\mathfrak{D}_x - \nu\sinh^2\mathfrak{D}_x \\ &= \frac{x}{2}\sinh 2\mathfrak{D}_x - \nu\frac{\cosh 2\mathfrak{D}_x - 1}{2}\,, \end{aligned}$$

from which we obtain formula (32). Finally, from this formula and the identity $e^{\lambda\mathfrak{D}_x} = E^\lambda$, we have

$$\begin{aligned} N &= \frac{\nu}{2} - \frac{\nu}{2}\frac{e^{2\mathfrak{D}_x} + e^{-2\mathfrak{D}_x}}{2} + \frac{x}{2}\frac{e^{2\mathfrak{D}_x} - e^{-2\mathfrak{D}_x}}{2} \\ &= \frac{\nu}{2} - \frac{\nu}{4}(E^2 + E^{-2}) + \frac{x}{4}(E^2 - E^{-2}) \end{aligned}$$

and this simplifies in formula (33). □

From Theorem 10, we have the following relation.

Theorem 11. *The Cayley continuants satisfy the differential equation*

(34)
$$-\nu \sum_{k=0}^{\lfloor n/2\rfloor} \frac{2^{2k}}{(2k)!}\, \mathfrak{D}_x^{2k}\, U_n^{(\nu)}(x) + x \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \frac{2^{2k+1}}{(2k+1)!}\, \mathfrak{D}_x^{2k+1}\, U_n^{(\nu)}(x) = (2n-\nu)\, U_n^{(\nu)}(x)\,.$$

Proof. We have $NU_n^{(\nu)}(x) = nU_n^{(\nu)}(x)$, where (by formula (32))

$$\begin{aligned} N &= \frac{\nu}{2} - \frac{\nu}{2}\cosh 2\,\mathfrak{D}_x + \frac{x}{2}\sinh 2\,\mathfrak{D}_x \\ &= \frac{\nu}{2} - \frac{\nu}{2}\sum_{k\ge 0}\frac{2^{2k}}{(2k)!}\,\mathfrak{D}_x^{2k} + \frac{x}{2}\sum_{k\ge 0}\frac{2^{2k+1}}{(2k+1)!}\,\mathfrak{D}_x^{2k+1}\,. \end{aligned}$$

Since $U_n^{(\nu)}(x)$ is a polynomial of degree n, we obtain the differential equation (34). □

Adding the two identities (30) and (34), we obtain the following relation.

Theorem 12. *The Cayley continuants satisfy the differential identity*

$$(x-\nu)\sum_{k=0}^{n}\frac{2^k}{k!}\,\mathfrak{D}_x^k\, U_n^{(\nu)}(x) = 2U_{n+1}^{(\nu)}(x) - (x-2n+\nu)\, U_n^{(\nu)}(x)$$

or, equivalently,

(35)
$$(x-\nu)\, U_n^{(\nu)}(x+2) = 2U_{n+1}^{(\nu)}(x) - (x-2n+\nu)\, U_n^{(\nu)}(x)\,.$$

4.4 Meixner polynomials

By formula (10), the Meixner polynomials can be expressed in terms of the Cayley continuants, namely $M_n^{(\alpha)}(x) = U_n^{(-\alpha)}(2x+\alpha)$. Hence, using formulas (16), (17) and (18) with $p=1$ and $q=2$, and formulas (21), (22), (28), (29) and (32), (33) with $\nu=-\alpha$, the umbral operators associated with the Meixner polynomials are $J' = J(\mathfrak{D}_x/2)$, $M' = M(2x+\alpha, \mathfrak{D}_x/2)$ and $N' = N(2x+\alpha, \mathfrak{D}_x/2)$, that is

$$\begin{aligned} J' &= \tanh\frac{\mathfrak{D}_x}{2} = \frac{E^{1/2}-E^{-1/2}}{E^{1/2}+E^{-1/2}} = \frac{E-1}{E+1} \\ M' &= x(1+\cosh\mathfrak{D}_x) + \frac{\alpha}{2}(1+\mathrm{e}^{\mathfrak{D}_x}) = \frac{x+\alpha}{2}\,E + \frac{2x+\alpha}{2} + \frac{x}{2}\,E^{-1} \\ N' &= -\frac{\alpha}{2} + \frac{\alpha}{2}\,\mathrm{e}^{\mathfrak{D}_x} + x\sinh\mathfrak{D}_x = \frac{x+\alpha}{2}\,E - \frac{\alpha}{2} - \frac{x}{2}\,E^{-1}\,. \end{aligned}$$

Moreover, all the identities obtained for the Cayley continuants can be specialized to the Meixner polynomials. For instance, with the substitution $x \to 2x+\alpha$ and $\nu \to -\alpha$, identities (26), (31) and (35) become

$$M_n^{(\alpha)}(x+1/2) - M_n^{(\alpha)}(x-1/2) = nM_{n-1}^{(\alpha)}(x+1/2) + nM_{n-1}^{(\alpha)}(x-1/2)$$

$$M_{n+1}^{(\alpha)}(x) = \frac{x+\alpha}{2}\, M_n^{(\alpha)}(x+1) + \frac{2x+\alpha}{2}\, M_n^{(\alpha)}(x) + \frac{x}{2}\, M_n^{(\alpha)}(x-1)$$

$$(x+\alpha) M_n^{(\alpha)}(x+1) = M_{n+1}^{(\alpha)}(x) - (x-n) M_n^{(\alpha)}(x)\,.$$

5. UMBRAL OPERATORS FOR THE GENERALIZED SYLVESTER CONTINUANTS

5.5 Operator J

The umbral derivative admits the following representations.

Theorem 13. *The operator J is given by*

$$J = \sinh \mathfrak{D}_x \tag{36}$$

$$J = \frac{E - E^{-1}}{2}\,. \tag{37}$$

Proof. By formula (13), we have $J = \widehat{f}(\mathfrak{D}_x)$. By spectrum (20), we have

$$f(t) = \ln(t + \sqrt{1+t^2}) = \operatorname{arcsinh} t \tag{38}$$

and consequently

$$\widehat{f}(t) = \sinh t\,. \tag{39}$$

This implies formula (36), which can be rewritten as formula (37), being $\mathrm{e}^{\lambda\,\mathfrak{D}_x} = E^{\lambda}$. □

Theorem 13 implies the following two results.

Theorem 14. *The generalized Sylvester continuants satisfy the differential identity*

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{(2k+1)!}\; \mathfrak{D}_x^{2k+1}\, H_n^{(\nu)}(x) = n\, H_{n-1}^{(\nu)}(x)\,.$$

Proof. From the identity $JH_n^{(\nu)}(x) = nH_{n-1}^{(\nu)}(x)$ and formula (36), recalling that the continuants $H_n^{(\nu)}(x)$ are polynomials of degree n in x. □

Theorem 15. *The generalized Sylvester continuants satisfy the identity*

$$H_n^{(\nu)}(x+1) - H_n^{(\nu)}(x-1) = 2nH_{n-1}^{(\nu)}(x)\,. \tag{40}$$

Proof. From the identity $JH_n^{(\nu)}(x) = nH_{n-1}^{(\nu)}(x)$ and formula (37). □

We also have the following result.

Theorem 16. *The generalized Sylvester continuants satisfy the identity*

$$\mathfrak{D}_x\, H_{n+1}^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} \binom{2k}{k} \frac{(-1)^k}{2^{2k}}\,(2k)!\, H_{n-2k}^{(\nu)}(x) \tag{41}$$

or, equivalently

$$\mathfrak{D}_x\, H_{n+1}^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{n-2k} \frac{n+1}{2k+1}\, \frac{(-1)^k}{2^{2k}}\,(2k)!\, H_{n-2k}^{(\nu)}(x)\,.$$

Proof. From formula (36), we have $J = \sinh \mathfrak{D}_x$. Consequently, we have

$$\mathfrak{D}_x = \operatorname{arcsinh} J = \sum_{k \ge 0} \binom{2k}{k} \frac{(-1)^k}{2^{2k}}\, \frac{J^{2k+1}}{2k+1}$$

and

$$\begin{aligned} \mathfrak{D}_x\, H_{n+1}^{(\nu)}(x) &= \sum_{k\ge 0} \binom{2k}{k} \frac{(-1)^k}{2^{2k}}\, \frac{1}{2k+1}\, J^{2k+1} H_{n+1}^{(\nu)}(x) \\ &= \sum_{k\ge 0} \binom{n+1}{2k+1} \binom{2k}{k} \frac{(-1)^k}{2^{2k}}\, \frac{(2k+1)!}{2k+1}\, H_{n-2k}^{(\nu)}(x) \end{aligned}$$

which simplifies in identity (41). □

5.6 Operator M

The umbral shift admits the following representations, where

$$e(t) = \sum_{n\ge 0} E_n \frac{t^n}{n!} = \frac{1}{\cosh t} \tag{42}$$

$$e'(t) = \sum_{n\ge 0} E_{n+1} \frac{t^n}{n!} = -\frac{\sinh t}{\cosh^2 t} \tag{43}$$

are the exponential generating series of the *Euler numbers* E_n, [**7**, p. 49].

Theorem 17. *The operator* M *is given by*

$$M = -(\nu+2)\, \frac{\sinh \mathfrak{D}_x}{\cosh^2 \mathfrak{D}_x} + x\, \frac{1}{\cosh \mathfrak{D}_x} \tag{44}$$

$$M = (\nu+2)\, e'(\mathfrak{D}_x) + x\, e(\mathfrak{D}_x) \tag{45}$$

$$M = -2(\nu+2)\, \frac{E - E^{-1}}{(E+E^{-1})^2} + x\, \frac{2}{E+E^{-1}} \tag{46}$$

Proof. The operator M is given by formula (14). By spectrum (20), we have

$$g(t) = \frac{1}{(1+t^2)^{\nu/2+1}}\,, \quad g'(t) = -\frac{(\nu+2)\,t}{1+t^2}\, g(t) \quad \text{and} \quad \frac{g'(t)}{g(t)} = -\frac{(\nu+2)\,t}{1+t^2}\,.$$

Hence, by identity (39), we have

$$\frac{g'(\widehat{f}(t))}{g(\widehat{f}(t))} = -\frac{(\nu+2)\sinh t}{1+\sinh^2 t} = -(\nu+2)\,\frac{\sinh t}{\cosh^2 t}\,.$$

Moreover, by identity (38), we have

$$f'(t) = \frac{1}{\sqrt{1+t^2}} \qquad \text{and} \qquad f'(\widehat{f}(t)) = \frac{1}{\sqrt{1+\sinh^2}} = \frac{1}{\cosh t}\,.$$

This yields formula (44). By series (42) and (43), this formula can be rewritten as formula (45). Finally, by formula (44) and the identity $\mathrm{e}^{\lambda\,\mathfrak{D}_x} = E^\lambda$, we have formula (46). □

Theorem 17 implies the following identities.

Theorem 18. *The generalized Sylvester continuants satisfy the recurrence*

$$H_{n+1}^{(\nu)}(x) = (\nu+2)\sum_{k=0}^{n} \frac{E_{k+1}}{k!}\,\mathfrak{D}_x^k\, H_n^{(\nu)}(x) + x\sum_{k=0}^{n}\frac{E_k}{k!}\,\mathfrak{D}_x^k\, H_n^{(\nu)}(x)\,.$$

Proof. From the identity $MH_n^{(\nu)}(x) = H_{n+1}^{(\nu)}(x)$ and formula (45). □

Theorem 19. *The generalized Sylvester continuants satisfy the identity*

$$\begin{aligned}&\sum_{k=0}^{\lfloor (n+1)/2\rfloor} \frac{1}{(2k)!}\,\mathfrak{D}_x^{2k}\, H_{n+1}^{(\nu)}(x) + \\ &\quad + (\nu+1)\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\frac{2^{2k}(2^{2k}-1)}{(2k)!}\, B_{2k}\,\mathfrak{D}_x^{2k-1}\, H_n^{(\nu)}(x) = xH_n^{(\nu)}(x)\,.\end{aligned} \tag{47}$$

Proof. From the identity $MH_n^{(\nu)}(x) = H_{n+1}^{(\nu)}(x)$ and formula (44), we have

$$H_{n+1}^{(\nu)}(x) = -(\nu+2)\,\frac{\sinh\mathfrak{D}_x}{\cosh^2\mathfrak{D}_x}\, H_n^{(\nu)}(x) + x\,\frac{1}{\cosh\mathfrak{D}_x}\, H_n^{(\nu)}(x)$$

or

$$\cosh\mathfrak{D}_x\, H_{n+1}^{(\nu)}(x) = -(\nu+2)\,\tanh\mathfrak{D}_x\, H_n^{(\nu)}(x) + \cosh\mathfrak{D}_x\left(x\,\frac{1}{\cosh\mathfrak{D}_x}\, H_n^{(\nu)}(x)\right).$$

Now, we have the commutation property

$$(\cosh\mathfrak{D}_x)\,x = x\,\cosh\mathfrak{D}_x + \sinh\mathfrak{D}_x\,.$$

Indeed, we have

$$(\cosh \mathfrak{D}_x)\, x = \frac{E+E^{-1}}{2}\, x = \frac{1}{2}\left((x+1)E + (x-1)E^{-1}\right)$$
$$= x\,\frac{E+E^{-1}}{2} + \frac{E-E^{-1}}{2} = x\,\cosh \mathfrak{D}_x + \sinh \mathfrak{D}_x\ .$$

So, by this property, our equation becomes

$$\cosh \mathfrak{D}_x\, H^{(\nu)}_{n+1}(x) = -(\nu+2)\,\tanh \mathfrak{D}_x\ H^{(\nu)}_n(x) + xH^{(\nu)}_n(x) + \tanh \mathfrak{D}_x\ H^{(\nu)}_n(x)$$

or

$$\cosh \mathfrak{D}_x\, H^{(\nu)}_{n+1}(x) + (\nu+1)\,\tanh \mathfrak{D}_x\ H^{(\nu)}_n(x) = xH^{(\nu)}_n(x)\,.$$

Expanding the hyperbolic cosine and the hyperbolic tangent, we have identity (47). □

Theorem 20. *The generalized Sylvester continuants satisfy the identity*

(48)
$$H^{(\nu)}_{n+1}(x+2) + 2H^{(\nu)}_{n+1}(x) + H^{(\nu)}_{n+1}(x-2) =$$
$$= 2(x-\nu)\,H^{(\nu)}_n(x+1) + 2(x+\nu)\,H^{(\nu)}_n(x-1)\,.$$

Proof. From the identity $MH^{(\nu)}_n(x) = H^{(\nu)}_{n+1}(x)$ and formula (46), we have

$$H^{(\nu)}_{n+1}(x) = -2(\nu+2)\,\frac{E-E^{-1}}{(E+E^{-1})^2}\,H^{(\nu)}_n(x) + x\frac{2}{E+E^{-1}}\,H^{(\nu)}_n(x)$$

or

$$(E{+}E^{-1})^2 H^{(\nu)}_{n+1}(x) = -2(\nu{+}2)\,(E{-}E^{-1})\,H^{(\nu)}_n(x){+}(E{+}E^{-1})^2 x\frac{2}{E+E^{-1}}\,H^{(\nu)}_n(x)\,.$$

By the commutation identity

(49)
$$(E+E^{-1})^2 x = x(E+E^{-1})^2 + 2(E-E^{-1})(E+E^{-1})\,,$$

we have

$$(E^2+2+E^{-2})H^{(\nu)}_{n+1}(x) =$$
$$= -2(\nu+2)\,(E-E^{-1})\,H^{(\nu)}_n(x) + 2x(E+E^{-1})\,H^{(\nu)}_n(x) + 4(E-E^{-1})\,H^{(\nu)}_n(x)$$
$$= -2\nu\,(E-E^{-1})\,H^{(\nu)}_n(x) + 2x(E+E^{-1})\,H^{(\nu)}_n(x)$$

or

$$E^2 H^{(\nu)}_{n+1}(x) + 2H^{(\nu)}_{n+1}(x) + E^{-2}H^{(\nu)}_{n+1}(x) =$$
$$= 2(x-\nu)\,E\,H^{(\nu)}_n(x) + 2(x+\nu)\,E^{-1}\,H^{(\nu)}_n(x)$$

This is identity (48). □

5.7 Operator N

The umbral theta operator admits the following representations.

Theorem 21. *The operator* N *is given by*

$$N = -(\nu+2)\tanh^2 \mathfrak{D}_x + x \tanh \mathfrak{D}_x \tag{50}$$

$$N = -(\nu+2)\left(\frac{E-E^{-1}}{E+E^{-1}}\right)^2 + x\,\frac{E-E^{-1}}{E+E^{-1}}\,. \tag{51}$$

Proof. From definition (15) and formulas (44) and (36). □

Theorem 21 implies the following identities.

Theorem 22. *The generalized Sylvester continuants satisfy the identity*

$$\begin{aligned} &-(\nu+2)\sum_{k=2}^{\lfloor (n+2)/2\rfloor} \frac{2^{2k}}{(2k)!}\, C_k\ \mathfrak{D}_x^{2k-2}\, H_n^{(\nu)}(x) + \\ &\quad + x\sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{2^{2k}(2^{2k}-1)}{(2k)!}\, B_{2k}\ \mathfrak{D}_x^{2k-1}\, H_n^{(\nu)}(x) = n\, H_n^{(\nu)}(x) \end{aligned}$$

where the coefficients B_n *are the Bernoulli numbers and*

$$C_n = \sum_{k=1}^{n-1} \binom{2n}{2k}(2^{2k}-1)(2^{2n-2k}-1)B_{2k}B_{2n-2k}\,.$$

Proof. From the identity $NH_n^{(\nu)}(x) = nH_n^{(\nu)}(x)$ and formula (50). □

Theorem 23. *The generalized Sylvester continuants satisfy the identity*

$$(x-\nu-n)\,H_n^{(\nu)}(x+2) - 2(n-\nu)H_n^{(\nu)}(x) - (x+\nu+n)\,H_n^{(\nu)}(x-2) = 0\,. \tag{52}$$

Proof. From the identity $NH_n^{(\nu)}(x) = nH_n^{(\nu)}(x)$ and formula (51), we have

$$nH_n^{(\nu)}(x) = -(\nu+2)\left(\frac{E-E^{-1}}{E+E^{-1}}\right)^2 H_n^{(\nu)}(x) + x\,\frac{E-E^{-1}}{E+E^{-1}}\,H_n^{(\nu)}(x)$$

that is

$$n(E+E^{-1})^2 H_n^{(\nu)}(x) = -(\nu+2)(E-E^{-1})^2\, H_n^{(\nu)}(x) + (E+E^{-1})^2 x\,\frac{E-E^{-1}}{E+E^{-1}}\,H_n^{(\nu)}(x)$$

or, by the commutation identity (49),

$$\begin{aligned} &n(E^2+2+E^{-2})H_n^{(\nu)}(x) = \\ &\quad = -(\nu+2)(E-E^{-1})^2\, H_n^{(\nu)}(x) + (E+E^{-1})^2\, x\,\frac{E-E^{-1}}{E+E^{-1}}\,H_n^{(\nu)}(x) \\ &\quad = -(\nu+2)(E-E^{-1})^2\, H_n^{(\nu)}(x) + x\,(E^2-E^{-2})\,H_n^{(\nu)}(x) + 2(E-E^{-1})^2\, H_n^{(\nu)}(x) \\ &\quad = -\nu(E-E^{-1})^2\, H_n^{(\nu)}(x) + x\,(E^2-E^{-2})\,H_n^{(\nu)}(x) \end{aligned}$$

that is

$$(x-\nu-n)\,E^2 H_n^{(\nu)}(x) - 2(n-\nu)H_n^{(\nu)}(x) - (x+\nu+n)\,E^{-2} H_n^{(\nu)}(x) = 0\,.$$

This is identity (52). □

5.8 Central factorials

By the first formula in (12), the central factorials are a specialization of the generalized Sylvester continuants, namely $x^{[n]} = \frac{1}{2^n}\, H_n^{(-2)}(2x)$. Hence, from formulas (16), (17) and (18) with $p = 1/2$ and $q = 2$, and formulas (36), (37), (44), (46) and (50), (51) with $\nu = -2$ and $\alpha = 0$, the umbral operators J', M' and N' associated with the central factorials are given by

$$\begin{aligned} J' &= 2J(\mathfrak{D}_x/2) = 2\sinh\frac{\mathfrak{D}_x}{2} = E^{1/2} - E^{-1/2} \\ M' &= \frac{1}{2}\, M(2x, \mathfrak{D}_x/2) = x\,\frac{1}{\cosh\frac{\mathfrak{D}_x}{2}} = 2x\,\frac{1}{E^{1/2}+E^{-1/2}} \\ N' &= N(2x, \mathfrak{D}_x/2) = 2x\tanh\frac{\mathfrak{D}_x}{2} = 2x\,\frac{E^{1/2}-E^{-1/2}}{E^{1/2}+E^{-1/2}}\,. \end{aligned}$$

Finally, all the identities obtained for the generalized Sylvester continuants can be specialized to the central factorials. For instance, with the substitution $x \to 2x$ and $\nu \to -2$, identities (40), (48) and (52) become

$$(x+1/2)^{[n]} - (x-1/2)^{[n]} = nx^{[n-1]}$$

$$(x+1)^{[n+1]} + 2x^{[n+1]} + (x-1)^{[n+1]} = 2(x+1)\,(x+1/2)^{[n]} - 2(x-1)\,(x-1/2)^{[n]}$$

$$(2x-n+2)\,(x+1)^{[n]} - 2(n+2)\,x^{[n]} - (2x+n-2)\,(x-1)^{[n]} = 0\,.$$

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(Received 22. 01. 2020.)
(Revised 20. 10. 2022.)