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ASYMMETRIC EXTENSION OF PASCAL-DELANNOY TRIANGLES

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In this paper, we give a generalization of the Pascal triangle called the quasi s-Pascal triangle. For this, consider a set of lattice path, which is a dual approach to the definition of Ramirez and Sirvent: A Generalization of the k-bonacci Sequence from Riordan Arrays. The electronic journal of combinatorics, 22(1) (2015), 1–38. We give the recurrence relation for the sum of elements lying over finite ray of the quasi s-Pascal triangle, then, we establish a q-analogue of the coefficient of this triangle. Some identities are also given.

1. INTRODUCTION

A lattice path in the plane-(x, y) is a set of edges $\{p_0, p_1, \ldots, p_n\}$ in $\mathbb{Z}^2$, such that two edges are related by one vertex, the set of vertices connecting p_0 to p_n is called a lattice path. Several authors have studied and enumerated the lattice path. For example, Mohanty and Handa [**18**] enumerate the unrestricted lattice paths from $(0, 0)$ to (n, k) where the diagonal steps are allowed at each position, Dziemianczuk [**16**] counts the lattice path with four steps: horizontal $H = (1, 0)$, vertical $V = (0, 1)$, diagonal $D = (1, 1)$, and sloping $L = (-1, 1)$, Fray and Roselle [**17**] determine the number of unrestricted weighted lattice paths from $(0, 0)$ to (n, k) with horizontal, vertical, and diagonal steps, Rohatgi [**22**] enumerates the paths which must remain below the line $y = ax + b$ where the diagonal steps are allowed in addition to the usual horizontal and vertical steps. In a Pascal triangle, the binomial coefficients $\binom{n}{k}$ count the number of lattice paths from $(0, 0)$ to (n, k) using the steps $\{H = (1, 0) \rightarrow, D = (1, 1) \nearrow\}$.

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It is well known that the terms of Fibonacci sequence $(F_n)_n$ are obtained by summing the elements crossing the principal diagonal rays in the Pascal triangle,

$$F_{n+1} = \sum_k \binom{n-k}{k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $n \geq k \geq 0$ and $\binom{n}{k} = 0$ otherwise.
The generating function of the binomial coefficients is given by

$$\sum_{n\geq 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$$

Alladi and Hoggat [**1**] extended the Pascal triangle; they established the Tribonacci triangle and proved that the sum of elements lying over the principal diagonal rays in the Tribonacci triangle gives the Tribonacci sequence

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}, \qquad n \geq 2,$$

with $T_0 = 0, T_1 = 1, T_2 = 1$.
Denote by $\binom{n}{k}_{[2]}$ the element in the n^{th} row and k^{th} column of the Tribonacci triangle. The triangle is produced by the recurrence relation

$$\binom{n}{k}_{[2]} = \binom{n-1}{k}_{[2]} + \binom{n-1}{k-1}_{[2]} + \binom{n-2}{k-1}_{[2]},$$

where $\binom{n}{0}_{[2]} = \binom{n}{n}_{[2]} = 1$. We use the convention $\binom{n}{k}_{[2]} = 0$ for $k \notin \{0, \ldots, n\}$.
Moreover, Barry [**5**] has shown that for $0 \leq k \leq n$, these coefficients satisfy the relation

$$\binom{n}{k}_{[2]} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k}.$$

Table 1. Tribonacci triangle.

$n \setminus k$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1									
1	1	1								
2	1	3	1							
3	1	5	5	1						
4	1	7	13	7	1					
5	1	9	25	25	9	1				
6	1	11	41	63	41	11	1			
7	1	13	61	129	129	61	13	1		
8	1	15	85	231	321	231	85	15	1	
9	1	17	113	377	681	681	377	113	17	1

The coefficient $\binom{n}{k}_{[2]}$: counts the number of lattice paths from $(0,0)$ to (n,k) using the steps $\{H = (1,0), D = (1,1), L = (2,1)\}$.
In what follows, s is a positive integer.

1.1 The s-Pascal triangle

Let $k \in \{0, 1, \dots, sn\}$, the bi^{s}nomial coefficient $\binom{n}{k}_s$ is defined as the k^{th} coefficient in the expansion

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k}_s x^k,$$

with $\binom{n}{k}_s = 0$ for $k > sn$ or $k < 0$.
Using the classical binomial coefficient, see [**4, 7, 11**], one has

$$\binom{n}{k}_s = \sum_{j_1 + j_2 + \cdots + j_s = k} \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{s-1}}{j_s}.$$

Some other readily well known established properties are:
the symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s,$$

the longitudinal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s,$$

the diagonal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{n} \binom{n}{j}\binom{j}{k-j}_{s-1},$$

and de Moivre expression see ([**19, 20**])

$$\binom{n}{k}_s = \sum_{j=0} (-1)^j \binom{n}{j}\binom{k - j(s+1) + n - 1}{n-1}. \tag{1}$$

Belbachir and Benmezai [**6**] have given the following relation

$$\binom{n}{k}_s = (-1)^k \sum_{j_1 + j_2 + \cdots + j_s = k} \binom{n}{j_1}\binom{n}{j_2} \cdots \binom{n}{j_s} a^{-\sum_{r=1}^{s} r j_r}, \tag{2}$$

where $a = \exp(2i\pi/(s+1))$.
These coefficients, as for the usual binomial coefficients, make, as for the Pascal triangle, the called s-Pascal triangle. For $s = 3$ we have the Biquadranomial triangle:

Table 2. Biquadranomial triangle ($s = 3$).

n\k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1	1	1	1										
2	1	2	3	4	3	2	1							
3	1	3	6	10	12	12	10	6	3	1				
4	1	4	10	20	31	40	44	40	31	20	10	4	1	
5	1	5	15	35	65	101	135	155	155	135	101	65	35	15

1.2 The q-binomial coefficient

The q-analogue of binomial coefficient or the q-binomial coefficient $\left[{n \atop k}\right]$ generalizes the binomial coefficient $\binom{n}{k}$ [**10, 14**]. It is defined as follows

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q![n-k]_q!} q^{\binom{k}{2}},$$

with $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$, we use the convention $\left[{n \atop k}\right] = 0$ for $k \notin \{0, \ldots, n\}$.

The q-binomial coefficient satisfies the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \tag{3}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{4}$$

The generating functions are

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = (1+x)(1+qx)(1+q^2x)\cdots(1+q^{n-1}x), \tag{5}$$

and

$$\sum_{n \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} x^n = \frac{x^k q^{\binom{k}{2}}}{(1-x)(1-qx)\cdots(1-q^kx)}. \tag{6}$$

Belbachir and Benmezai [**6**] proposed the q-bi^{s}nomial coefficient, denoted by $\left[{n \atop k}\right]^{(s)}$, as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(s)} := (-1)^k \sum_{j_1+j_2+\cdots+j_s=k} \begin{bmatrix} n \\ j_1 \end{bmatrix} \begin{bmatrix} n \\ j_2 \end{bmatrix} \cdots \begin{bmatrix} n \\ j_s \end{bmatrix} a^{-\sum_{r=1}^{s} r j_r}.$$

The q-bi^{s}nomial coefficient satisfies the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(s)} = \sum_{j=0}^{s} q^{k-j} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}^{(s)},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(s)} = \sum_{j=0}^{s} q^{(n-1)j} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}^{(s)}.$$

According to (5), the generating function is given by

$$\sum_{k=0}^{ns} \begin{bmatrix} n \\ k \end{bmatrix}^{(s)} x^k = \prod_{j=0}^{n-1} (1 + q^j x + (q^j x)^2 + \cdots + (q^j x)^s). \tag{7}$$

In the first section, we introduce the quasi s-Pascal triangle by using a family of lattice paths, we establish an explicit formula for the elements of the quasi s-Pascal triangle, and we prove that the sums of the elements crossing the diagonal rays yield the terms of s-bonacci sequence; we close this section by giving a relation between s-Pascal triangle and quasi s-Pascal triangle. The second section is devoted to the linear recurrence relation obtained by summing the elements lying over any finite rays of the quasi s-Pascal triangle and we give the corresponding generating function. In the third section, we give the de Moivre like summation with some other identities. In section four, we establish the q-analogue of the elements of the quasi s-Pascal triangle.

2. THE QUASI S-PASCAL TRIANGLE

In this section, we define the quasi s-Pascal triangle. We denote by $\binom{n}{k}_{[s]}$ the coefficient in the n^{th} row and k^{th} column of the quasi s-Pascal triangle. It should be noted that most properties proposed in this section are already obtained by Ramirez and Sirvent [**21**]. We give a dual approach to these properties with new proofs and by using Definition 1 in order to make easy the comprehension of the remaining of this paper.

Definition 1. The quasi-bi^{s}nomial coefficient $\binom{n}{k}_{[s]}$ is defined by the number of lattice path from $(0,0)$ to (n,k) with steps in $\{L = (1,0), L_1 = (1,1), L_2 = (2,1), \ldots, L_s = (s,1)\}$, with $\binom{n}{0}_{[s]} = \binom{n}{n}_{[s]} = 1$ and the convention $\binom{n}{k}_{[s]} = 0$ for $k > n$ or $k < 0$.

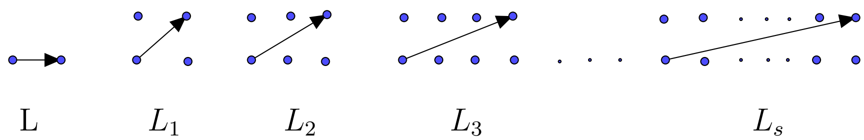

Figure 1. Ilustration of possible steps for the coefficient $\binom{n}{k}_{[s]}$.

Lemma 2. *The quasi-bi^{s}nomial coefficient* $\binom{n}{k}_{[s]}$ *satisfies the following recurrence relation*

$$\binom{n}{k}_{[s]} = \binom{n-1}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + \binom{n-2}{k-1}_{[s]} + \cdots + \binom{n-s}{k-1}_{[s]}, \tag{8}$$

Proof. By Definition 1, the last step of any path is one of $\{L = (1,0), L_1 = (1,1), L_2 = (2,1), \ldots, L_s = (s,1)\}$, then if the last step is $L = (1,0)$, it remains to enumerate the number of lattice paths from $(0,0)$ to $(n-1,k)$, which is $\binom{n-1}{k}_{[s]}$, or if, the last one is $L_1 = (1,1)$, it remains to enumerate the number of lattice paths from $(0,0)$ to $(n-1,k-1)$, which is $\binom{n-1}{k-1}_{[s]}, \ldots$. If the last step is $L_s = (s,1)$, it remains to enumerate the number of lattice paths from $(0,0)$ to $(n-s,k-1)$, which is $\binom{n-s}{k-1}_{[s]}$. Considering all possibilities, we construct our recurrence. □

2.3 Generating function

Here is given the generating function of $\{\binom{n}{k}_{[s]}\}_n$;

Theorem 3. *Let* $F_k(x) := \sum_{n\geq 0} \binom{n}{k}_{[s]} x^n$, *then*

$$F_k(x) = (1 + x + x^2 + \cdots + x^{s-1})^k \frac{x^k}{(1-x)^{k+1}}.$$

Proof. It follows from Relation (8) that

$$F_k(x) = xF_k(x) + xF_{k-1}(x) + x^2 F_{k-1}(x) + \cdots + x^s F_{k-1}(x),$$

repeated applications of this recurrence give the result. □

2.4 Binomial coefficients explicit formula

The following result gives an explicit formula for the coefficients of the quasi s-Pascal triangle in terms of binomial coefficients and a variant with multinomial coefficients.

Theorem 4. *The explicit formula for the quasi-bi^{s}nomial coefficient is given by*

$$\binom{n}{k}_{[s]}=\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-\sum_{i=1}^{s-1}j_i}{k}, \tag{9}$$

the multinomial version is

$$\binom{n}{k}_{[s]}=\sum_{k_1,k_2,\ldots,k_s}\binom{k}{k_1,k_2,\ldots,k_s}\binom{n+k-\sum_{i=1}^{s}ik_i}{k}, \tag{10}$$

where $\binom{k}{k_1,k_2,\ldots,k_s}=\frac{k!}{k_1!k_2!\cdots k_s!}$ *for* $k_1+k_2+\cdots+k_s=k$ *and* $\binom{k}{k_1,k_2,\ldots,k_s}=0$*, else.*

Proof. For Relation (9), we need to prove that

$$\sum_{n=0}^{\infty}\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-\sum_{i=1}^{s-1}j_i}{k}x^n$$
$$=\left(\frac{x+x^2+\cdots+x^s}{1-x}\right)^k\frac{1}{1-x}.$$

So

$$\sum_{n=0}^{\infty}\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-\sum_{i=1}^{s-1}j_i}{k}x^n$$
$$=\sum_{j_1}\binom{k}{j_1}x^{j_1}\sum_{j_2}\binom{j_1}{j_2}x^{j_2}\cdots\sum_{j_{s-1}}\binom{j_{s-2}}{j_{s-1}}x^{j_{s-1}}\sum_{n=0}^{\infty}\binom{n-\sum_{i=1}^{s-1}j_i}{k}x^{n-\sum_{i=1}^{s-1}j_i}$$
$$=\frac{x^k}{(1-x)^{k+1}}\sum_{j_1}\binom{k}{j_1}x^{j_1}\sum_{j_2}\binom{j_1}{j_2}x^{j_2}\cdots\sum_{j_{s-2}}\binom{j_{s-3}}{j_{s-2}}x^{j_{s-2}}\sum_{j_{s-1}}\binom{j_{s-2}}{j_{s-1}}x^{j_{s-1}}$$
$$=\frac{x^k}{(1-x)^{k+1}}\sum_{j_1}\binom{k}{j_1}x^{j_1}\sum_{j_2}\binom{j_1}{j_2}x^{j_2}\cdots\sum_{j_{s-2}}\binom{j_{s-3}}{j_{s-2}}(x+x^2)^{j_{s-2}}$$
$$\vdots$$
$$=\frac{x^k}{(1-x)^{k+1}}\sum_{j_1}\binom{k}{j_1}(x+x^2+\cdots+x^{s-1})^{j_1}=\left(\frac{x+x^2+\cdots+x^s}{1-x}\right)^k\frac{1}{1-x}.$$

For Relation (10), we have

$$\sum_{n\geq 0}\sum_{k_1,k_2,\ldots,k_s}\binom{k}{k_1,k_2,\ldots,k_s}\binom{n-\sum_{i=1}^{s}ik_i+k}{k}x^n$$
$$=\frac{1}{x^k}\sum_{k_1,k_2,\ldots,k_s}\binom{k}{k_1,k_2,\ldots,k_s}x^{\sum_{i=1}^{s}ik_i}\sum_{n\geq 0}\binom{n-\sum_{i=1}^{s}ik_i+k}{k}x^{n-\sum_{i=1}^{s}ik_i+k}$$

$$= \frac{1}{(1-x)^{k+1}} \sum_{k_1,k_2,\ldots,k_s} \binom{k}{k_1, k_2, \ldots, k_s} x^{\sum_{i=1}^{s} ik_i}$$
$$= \left(\frac{x + x^2 + \cdots + x^s}{1-x}\right)^k \frac{1}{1-x}.$$

□

2.5 Link with generalized Delannoy matrix

Ramirez and Sirvent [**21**] propose a generalization of Delannoy and Pascal Riordan arrays. They denoted by $\mathcal{D}_m(n,k)$ the element in the n^{th} row and k^{th} column of the generalized Delannoy matrix, such that $\mathcal{D}_m(n,k)$ satisfies the following recurrence relation

$$\mathcal{D}_m(n+1,k) = a_1\mathcal{D}_m(n,k) + a_2\mathcal{D}_m(n+1,k-1) + \sum_{i=1}^{m-1} a_{i+2}\mathcal{D}_m(n,k-i),$$

with $k \geq m-1, n \geq 1$ and initial conditions $\mathcal{D}_m(0,k) = a_2^k$ and $\mathcal{D}_m(n,0) = a_1^n$. The coefficient $\mathcal{D}_m(n,k)$ is given by the following explicit formula

$$\mathcal{D}_m(n,k) = \sum_{j_1,j_2,\ldots,j_{m-1}} \binom{n}{j_1}\binom{n-j_1}{j_2}\cdots\binom{n-j_1-\cdots-j_{m-2}}{j_{m-1}}\binom{n+k-u}{n}\times$$
$$\times a_1^{j_1}a_3^{j_2}\cdots a_m^{j_{m-1}}a_{m+1}^{n-\sum_{i=1}^{m-1} j_i}a_2^{k-u},$$

where $u = (m-1)(n-j_1) + \sum_{i=2}^{m-1}(i-m)j_i$.

The following result gives a relation between $\mathcal{D}_m(n,k)$ and $\binom{n}{k}_{[s]}$.

Theorem 5. *For $m = s$, $a = 1$ and $a_i = 1$, $i \in \{1,\ldots,s\}$*

$$\mathcal{D}_s(k,n-k) = \binom{n}{k}_{[s]}.$$

Proof. For $a = 1$ and $a_i = 1$, $i \in \{1,\ldots,s\}$, we have

$$\mathcal{D}_s(n,k) = \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{n}{j_1}\binom{n-j_1}{j_2}\cdots\binom{n-j_1-\cdots-j_{s-2}}{j_{s-1}}\binom{n+k-u}{n},$$

where $u = (s-1)(n-j_1) + \sum_{i=2}^{s-1}(i-s)j_i$, then

$$\mathcal{D}_s(n,k) = \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{n}{n-j_1}\binom{n-j_1}{n-j_1-j_2}\cdots\binom{n-\sum_{i=1}^{s-2} j_i}{n-\sum_{i=1}^{s-1} j_i}\binom{n+k-u}{n},$$

we put $j'_v \to n - \sum_{l=1}^{v} j_l,\ v = \{1, \dots, s-1\}$, then $j'_1 + j'_2 + \cdots + j'_{s-1} = u$,

$$\mathcal{D}_s(n,k) = \sum_{j'_1, j'_2, \dots, j'_{s-1}} \binom{n}{j'_1}\binom{j'_1}{j'_2}\cdots\binom{j'_{s-2}}{j'_{s-1}}\binom{n+k-\sum_{i=1}^{s-1} j'_i}{n}.$$

Thus $\mathcal{D}_s(k, n-k) = \binom{n}{k}_{[s]}$.

□

2.6 Recurrence relation or s-bonacci sequence

Now, we establish the recurrence relation for the s-bonacci sequence, which is a generalization of Fibonacci sequence. Let $(T_{n,s})_n$ be the terms of the s-bonacci sequence obtained by summing the elements lying over the principal diagonal rays in the quasi s-Pascal triangle.
Let $(T_{n+1,s})_n$ be the sequence defined by

$$T_{n+1,s} := \sum_k \binom{n-k}{k}_{[s]},$$

with $T_{0,s} = 0$.

Theorem 6. *For $n \geq 0$, $(T_{n,s})_n$ satisfies the following recurrence relation*

$$T_{n+1,s} = T_{n,s} + T_{n-1,s} + \cdots + T_{n-s,s},$$

with $T_{1,s} = 1, T_{-i,s} = 0$ for $i \in \{0, -1, \dots, -(s-1)\}$.

It is not else s-bonacci sequence.

Proof. We have $T_{n+1,s} = \sum_k \binom{n-k}{k}_{[s]}$ and by Relation (8), we obtain

$$\begin{aligned} T_{n+1,s} &= \sum_k \binom{n-k-1}{k}_{[s]} + \sum_k \binom{n-k-1}{k-1}_{[s]} + \cdots + \sum_k \binom{n-k-s}{k-1}_{[s]} \\ &\qquad\qquad k' \to k-1 \\ &= \sum_k \binom{n-k-1}{k}_{[s]} + \sum_{k'} \binom{n-k'-2}{k'}_{[s]} + \cdots + \sum_{k'} \binom{n-k'-s-1}{k'}_{[s]} \\ &= T_{n,s} + T_{n-1,s} + \cdots + T_{n-s,s}. \end{aligned}$$

□

For $s = 1$ and $s = 2$, we obtain the terms of Fibonacci and Tribonacci sequences respectively.

Example 7. For $s = 3$ we have the quadrabonacci triangle

Table 3. The quadrabonacci triangle.

n\k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1									
1	1	1								
2	1	3	1							
3	1	6	5	1						
4	1	9	15	7	1					
5	1	12	33	28	9	1				
6	1	15	60	81	45	11	1			
7	1	18	96	189	66	33	13	1		
8	1	21	141	378	459	281	91	15	1	
9	1	24	195	675	1107	946	449	120	17	1

By Relation (8), the elements of the quadrabonacci triangle ($s = 3$) are given by $\binom{n}{0}_{[3]} = \binom{n}{n}_{[3]} = 1$, and

$$\binom{n}{k}_{[3]} = \binom{n-1}{k}_{[3]} + \binom{n-1}{k-1}_{[3]} + \binom{n-2}{k-1}_{[3]} + \binom{n-3}{k-1}_{[3]}.$$

For $s = 3$, $\binom{n}{k}_{[3]}$ counts the number of lattice paths with steps in $(1, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$ from $(0, 0)$ to (n, k). For example, for the value 6 in the quadrabonacci triangle, we have the lattice path

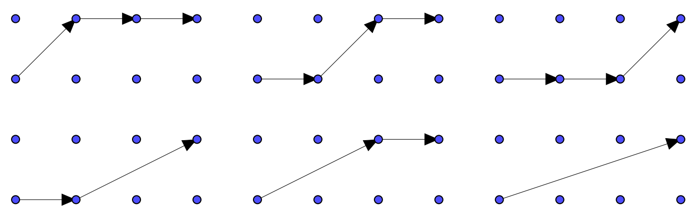

Figure 2. Illustration of possible paths from $(0, 0)$ to $(3, 1)$ using the steps $(1, 0), (1, 1), (2, 1), (3, 1)$.

Notice that for $s = 1$ and $s = 2$, the obtained triangles are symmetric, unlike the cases where $s > 2$.

2.7 s-Pascal triangle versus quasi-s-Pascal triangle

The following result establishes the relation between the quasi s-Pascal triangle and s-Pascal triangle.

Theorem 8. *For fixed non negative integers n, k and s, we have*

$$\binom{n}{k}_{[s]} = \sum_{i} \binom{n-i}{k} \binom{k}{i}_{s-1}.$$

Proof. We have

$$\binom{n}{k}_{[s]} = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-j_2-\cdots-j_{s-1}}{k},$$

considering the summations by blocks $j_1+j_2+\cdots+j_{s-1}=i,$ we get

$$\binom{n}{k}_{[s]} = \sum_{i}\binom{n-i}{k}\sum_{j_1+j_2+\cdots+j_{s-1}=i}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}} = \sum_{i}\binom{n-i}{k}\binom{k}{i}_{s-1}.$$

□

3. LINEAR RECURRENCE RELATION AND GENERATING FUNCTION ASSOCIATED TO FINITE TRANSVERSALS OF THE QUASI S-PASCAL TRIANGLE

This section is devoted to establish a recurrence relation associated to the sums of the elements lying over the transversals of direction (α, r) in the quasi s-Pascal triangle. In [**8**], we find the details about the concept of direction in Pascal triangle. The study was extended for the arithmetic triangle, see [**3, 9**]. We generalize the concept to our triangle as an extension of Theorem 6 to the case where $r \in \mathbb{Z},\ \alpha \in \mathbb{N},\ \beta \in \mathbb{Z}^+$ with $0 \le \beta < \alpha$ and $r + \alpha > 0$. This corresponds to the finite sequences lying over finite transversals of the quasi s-Pascal triangle.
Let be the sequence

$$T_{n+1,s}^{(\alpha,\beta,r)} := \sum_{k}\binom{n-rk}{\beta+\alpha k}_{[s]},\ \text{with}\ T_{0,s}^{(\alpha,\beta,r)} = 0.$$

The following figure illustrates the direction $(\alpha, r) = (2, 1)$ and $\beta = 0$ on the Tribonacci triangle.

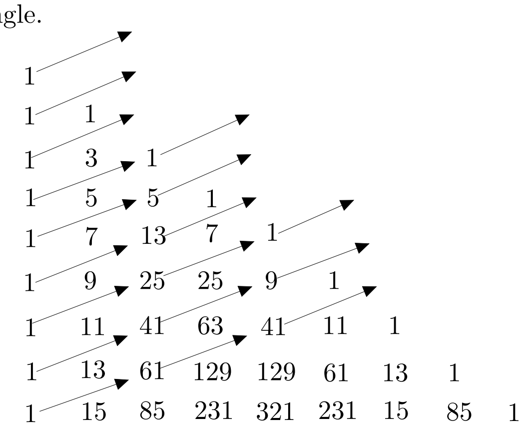

Figure 3: Tribonacci triangle.

Theorem 9. *For $n \geq \alpha s + r$, $(T^{(\alpha,\beta,r)}_{n+1,s})_n$ satisfies the following linear recurrence relation*

$$\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}T^{(\alpha,\beta,r)}_{n-i,s} = \sum_{i=0}^{\alpha(s-1)}\binom{\alpha}{i}_{s-1}T^{(\alpha,\beta,r)}_{n-\alpha-r-i,s},$$

we can recover the initial conditions $\mathcal{T}^{(\alpha,\beta,r)}_{1,s}, \ldots, \mathcal{T}^{(\alpha,\beta,r)}_{\alpha s+r-1,s}$ by $\sum_k \binom{n-rk}{\beta+\alpha k}_{[s]}$.

For the proof, we need the following Lemma.

Lemma 10 ([**8**]). *Let a,b and α be non negative integers satisfying the conditions $\alpha \leq a$, then*

$$\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\binom{a-i}{b} = \binom{a-\alpha}{b-\alpha}. \tag{11}$$

Proof of Theorem 9, from Relations (9) and (11), we get

$$\begin{aligned}
&\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}T^{(\alpha,\beta,r)}_{n-i,s}\\
&= \sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_k\binom{n-rk-i-1}{\beta+\alpha k}_{[s]}\\
&= \sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_k\sum_{j_1,\ldots,j_{s-1}}\binom{\beta+\alpha k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-rk-i-\sum_{i=1}^{s-1}j_i-1}{\beta+\alpha k}\\
&= \sum_k\sum_{j_1,\ldots,j_{s-1}}\binom{\beta+\alpha k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\binom{n-rk-i-\sum_{i=1}^{s-1}j_i-1}{\beta+\alpha k}\\
&= \sum_k\sum_{j_1,j_2,\ldots,j_{s-1}}\binom{\beta+\alpha k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-rk-\alpha-\sum_{i=1}^{s-1}j_i-1}{\beta+\alpha(k-1)}\\
&\quad (k^{'}\to k-1)\\
&= \sum_{k'}\sum_{j_1,j_2,\ldots,j_{s-1}}\binom{\beta+\alpha k^{'}+\alpha}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-rk^{'}-r-\alpha-\sum_{i=1}^{s-1}j_i-1}{\beta+\alpha k'}\\
&= \sum_{k'}\sum_{j_1,j_2,\ldots,j_{s-1}}\binom{\beta+\alpha k^{'}+\alpha}{j_1}\binom{j_1-i_1+i_1}{j_2}\cdots\binom{j_{s-2}-i_{s-2}+i_{s-2}}{j_{s-1}}\times\\
&\times\binom{n-rk^{'}-r-\alpha-\sum_{i=1}^{s-1}j_i-1}{\beta+\alpha k'}
\end{aligned}$$

by Vandermonde Formula

$$= \sum_{k'} \sum_{j_1,,\ldots,j_{s-1}} \sum_{i_1,\ldots,i_{s-1}} \binom{\alpha}{i_1}\binom{\beta+\alpha k'}{j_1-i_1}\binom{i_1}{i_2}\binom{j_1-i_1}{j_2-i_2}\cdots\binom{i_{s-2}}{i_{s-1}}\binom{j_{s-2}-i_{s-2}}{j_{s-1}-i_{s-1}}\times$$
$$\times \binom{n-rk'-r-\alpha-\sum_{i=1}^{s-1} j_i - 1}{\beta+\alpha k'}$$

$(l_v \to j_v - i_v)$

$$= \sum_{k'} \sum_{l_1,l_2,\ldots,l_{s-1}} \sum_{i_1,i_2,\ldots,i_{s-1}} \binom{\alpha}{i_1}\binom{\beta+\alpha k'}{l_1}\binom{i_1}{i_2}\binom{l_1}{l_2}\cdots\binom{i_{s-2}}{i_{s-1}}\binom{l_{s-2}}{l_{s-1}}\times$$
$$\times \binom{n-rk'-r-\alpha-\sum_{j=1}^{s-1} i_j - \sum_{j=1}^{s-1} l_j - 1}{\beta+\alpha k'}$$

We take the summation as by block $i_1+i_2+\cdots+i_{s-1}=i$

$$= \sum_{k'}\sum_{i} \sum_{l_1,l_2,\ldots,l_{s-1}} \sum_{i_1+i_2+\cdots+i_{s-1}=i} \binom{\alpha}{i_1}\binom{\beta+\alpha k'}{l_1}\binom{i_1}{i_2}\binom{l_1}{l_2}\cdots\binom{i_{s-2}}{i_{s-1}}\binom{l_{s-2}}{l_{s-1}}\times$$
$$\times \binom{n-rk'-r-\alpha-i-\sum_{j=1}^{s-1} l_j - 1}{\beta+\alpha k'}$$
$$= \sum_{k'}\sum_{i} \sum_{l_1,\ldots,l_{s-1}} \binom{\beta+\alpha k'}{l_1}\binom{l_1}{l_2}\cdots\binom{l_{s-2}}{l_{s-1}}\binom{n-rk'-r-\alpha-i-\sum_{j=1}^{s-1} l_i - 1}{\beta+\alpha k'}$$
$$\times \sum_{i} \binom{\alpha}{i_1}\binom{i_1}{i_2}\binom{i_1}{i_2}\cdots\binom{i_{s-2}}{i_{s-1}}$$
$$= \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} \mathcal{T}_{n-\alpha-r-i,s}^{(\alpha,\beta,r)}.$$

□

For $\alpha = 1, r = 1, \beta = 0$, we obtain the terms of the s-bonacci sequence, for $s = 2$. We obtain Theorem 7 of [**2**].

Example 11. For $\alpha = 2, r = 1, \beta = 0$ and $n \geq 2s + 1$, we have the following recurrence relation

$$\begin{aligned} T_{n,s}^{(2,0,1)} &= \sum_{j=1}^{2} (-1)^{j+1}\binom{2}{j} T_{n-j,s}^{(2,0,1)} + \sum_{j=0}^{2(s-1)} \binom{2}{j}_{s-1} T_{n-j-3,s}^{(2,0,1)} \\ &= 2T_{n-1,s}^{(2,0,1)} - T_{n-2,s}^{(2,0,1)} + T_{n-3,s}^{(2,0,1)} + 2T_{n-4,s}^{(2,0,1)} + \cdots \\ &\quad \cdots + sT_{n-s-2,s}^{(2,0,1)} + \cdots + 2T_{n-2s,s}^{(2,0,1)} + T_{n-2s-1,s}^{(2,0,1)}, \end{aligned}$$

as $\binom{2}{s-1}_{s-1} = s$.

The following result establishes the generating function for the sequence $(\mathcal{T}_{n,s}^{(\alpha,\beta,r)})_n$ for quasi s-Pascal triangles.

Theorem 12. *The generating function of the sequence* $\{\mathcal{T}_{n,s}^{(\alpha,\beta,r)}\}_{n\geq 0}$ *is given by*

$$\sum_{n\geqslant 0}\mathcal{T}_{n+1,s}^{(\alpha,\beta,r)}x^n=\frac{(1-x)^{\alpha-\beta-1}(x+x^2+\cdots+x^s)^\beta}{(1-x)^\alpha-x^{r+\alpha}(1+x+\cdots+x^{s-1})^\alpha}.$$

Proof. We have

$$\begin{aligned}\sum_{n\geqslant 0}\mathcal{T}_{n+1,s}^{(\alpha,\beta,r)}x^n&=\sum_{n\geqslant 0}\sum_k\binom{n-rk}{\beta+\alpha k}_{[s]}x^n\\&=\sum_{n\geqslant qk}\sum_k\binom{n-rk}{\beta+\alpha k}_{[s]}x^{n-rk}x^{rk}\\&=\sum_k\frac{(x+x^2+\cdots+x^s)^{\beta+\alpha k}x^{rk}}{(1-x)^{\beta+\alpha k+1}}\\&=\frac{(x+x^2+\cdots+x^s)^\beta}{(1-x)^{\beta+1}}\sum_k\left(\frac{(x+x^2+\cdots+x^s)^\alpha x^r}{(1-x)^\alpha}\right)^k\\&=\frac{(x+x^2+\cdots+x^s)^\beta}{(1-x)^{\beta+1}}\frac{1}{1-(\frac{x^r(x+x^2+\cdots+x^s)^\alpha}{(1-x)^\alpha})}\\&=\frac{(1-x)^{\alpha-\beta-1}(x+x^2+\cdots+x^s)^\beta}{(1-x)^\alpha-x^{r+\alpha}(1+x+\cdots+x^{s-1})^\alpha}.\end{aligned}$$

□

4. THE DE MOIVRE SUMMATION AND OTHER NESTED SUMS

Butler and Karasik [**12**] showed how the binomial coefficient can be written as nested sums. In this section, we establish an identity for the quasi bi^{s}nomial coefficients $\binom{n}{k}_{[s]}$ equivalent to the de Moivre summation for bi^{s}nomial coefficient and we give some other nested sums for the coefficient $\binom{n}{k}_{[s]}$. The following identity is important in the sense that it gives a simple summation with a product of two binomials.

Theorem 13. *The following identity holds true*

$$\binom{n}{k}_{[s]}=\sum_j(-1)^j\binom{k}{j}\binom{n+k-sj}{2k}. \tag{12}$$

Proof. By Theorem 3, we have

$$\begin{aligned}\sum_{n\geq 0}\binom{n}{k}_{[s]}x^n &= \frac{x^k(1+x+x^2+\cdots+x^{s-1})^k}{(1-x)^{k+1}}\\ &= x^k(1-x^s)^k\frac{1}{(1-x)^{2k+1}}\\ &= x^k\sum_j(-1)^j\binom{k}{j}x^{js}\sum_i\binom{i+2k}{2k}x^i\\ &= \sum_n\sum_{i+sj=n}(-1)^j\binom{k}{j}\binom{i+2k}{2k}x^{n+k}\\ &= \sum_n\sum_j(-1)^j\binom{k}{j}\binom{n-sj+k}{2k}x^n.\end{aligned}$$

Identity (12) is a dual version of Relation (1). □

Now, we give an identity for $\binom{n}{k}_{[s]}$ dual to Relation (2).

Theorem 14. *For $w=\exp(2i\pi/s)$, we have*

$$\binom{n}{k}_{[s]} = \sum_j\binom{n-j}{k}(-1)^j\sum_{k_1+k_2+\cdots+k_{s-1}=j}\binom{k}{k_1}\binom{k}{k_2}\cdots\binom{k}{k_{s-1}}\times w^{-\sum_{r=1}^{s-1}rk_r}.$$

Proof. By Theorem 3, we have

$$\begin{aligned}&\sum_{n\geq 0}\binom{n}{k}_{[s]}x^n\\ &= \frac{x^k}{(1-x)^{k+1}}\prod_{j=1}^{s-1}(x-w^j)^k\\ &= \sum_{n\geq 0}\binom{n}{k}x^n\sum_{k_1}(-1)^{k-k_1}\binom{k}{k_1}w^{k-k_1}x^{k_1}\sum_{k_2}(-1)^{k-k_2}\binom{k}{k_2}w^{2(k-k_2)}x^{k_2}\cdots\\ &\cdots\sum_{k_{s-1}}(-1)^{k-k_{s-1}}\binom{k}{k_{s-1}}w^{(s-1)(k-k_{s-1})}x^{k_{s-1}}\\ &= \sum_{n\geq 0}\binom{n}{k}x^n\sum_{j\geq 0}\sum_{k_1+k_2+\cdots+k_{s-1}=j}\binom{k}{k_1}\binom{k}{k_2}\cdots\binom{k}{k_{s-1}}\times\\ &\times(-1)^{(s-1)k-j}w^{\sum_{r=1}^{s-1}r(k-k_r)}x^j\\ &= \sum_{n\geq 0}\sum_j\binom{n-j}{k}\sum_{k_1+k_2+\cdots+k_{s-1}=j}\binom{k}{k_1}\binom{k}{k_2}\cdots\binom{k}{k_{s-1}}\times\\ &\times(-1)^{(s-1)k-j}w^{\sum_{r=1}^{s-1}r(k-k_r)}x^n.\end{aligned}$$

$$= \sum_{n\geq 0}\sum_{j}\binom{n-j}{k}\sum_{k_1+k_2+\cdots+k_{s-1}=j}\binom{k}{k_1}\binom{k}{k_2}\cdots\binom{k}{k_{s-1}}(-1)^j w^{-\sum_{r=1}^{s-1} rk_r} x^n,$$

This yields the result.

□

Remark 15. We can also deduce Theorem 14 using Theorem 8 and Identity (2).

Using the generating function, we obtain the following beautiful nested relation.

Theorem 16. *The terms of the s-Pascal triangle satisfy the following identity*

$$\binom{n}{k}_{[s]} = \sum_{j_1,j_2,\ldots,j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-k-\sum_{i=1}^{s-2} j_i}{j_{s-1}}\times$$
$$\times(2)^{j_1}(3/2)^{j_2}\cdots(s/s-1)^{j_{s-1}}.$$

Proof. We have

$$\sum_{n\geq 0}\sum_{j_1,j_2,\ldots,j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-k-\sum_{i=1}^{s-2} j_i}{j_{s-1}}(2)^{j_1}(3/2)^{j_2}\cdots$$
$$\cdots(s/s-1)^{j_{s-1}}x^n$$
$$= x^k\sum_{j_1}\binom{k}{j_1}(2x)^{j_1}\sum_{j_2}\binom{j_1}{j_2}(3x/2)^{j_2}\cdots\sum_{j_{s-2}}\binom{j_{s-3}}{j_{s-2}}(x(s-1)/(s-2))^{j_{s-2}}\times$$
$$\times\sum_{j_{s-1}}\binom{j_{s-2}}{j_{s-1}}(s/s-1)^{j_{s-1}}\sum_{n\geq 0}\binom{n-k-\sum_{i=1}^{s-2} j_i}{j_{s-1}}x^{n-k-\sum_{i=1}^{s-2} j_i}$$
$$= \frac{x^k}{1-x}\sum_{j_1}\binom{k}{j_1}(2x)^{j_1}\sum_{j_2}\binom{j_1}{j_2}(3x/2)^{j_2}\cdots\sum_{j_{s-2}}\binom{j_{s-3}}{j_{s-2}}(x(s-1)/(s-2))^{j_{s-2}}\times$$
$$\times\sum_{j_{s-1}}\binom{j_{s-2}}{j_{s-1}}\left(\frac{(s/(s-1))x}{1-x}\right)^{j_{s-1}}$$
$$= \frac{x^k}{1-x}\sum_{j_1}\binom{k}{j_1}(2x)^{j_1}\sum_{j_2}\binom{j_1}{j_2}(3x/2)^{j_2}\cdots$$
$$\cdots\sum_{j_{s-2}}\binom{j_{s-3}}{j_{s-2}}\left(\frac{x(s-1)/(s-2)+x^2/(s-2)}{1-x}\right)^{j_{s-2}}$$

$$= \frac{x^k}{1-x}\sum_{j_1}\binom{k}{j_1}(2x)^{j_1}\sum_{j_2}\binom{j_1}{j_2}(3x/2)^{j_2}\cdots$$

$$\cdots\sum_{j_{s-3}}\binom{j_{s-4}}{j_{s-3}}\left(\frac{(x(s-2)+x^2+x^3)/(s-3)}{1-x}\right)^{j_{s-3}}$$

$$\vdots$$

$$= \frac{x^k}{1-x}\sum_{j_1}\binom{k}{j_1}\left(\frac{2x+x^2+\cdots+x^{s-1}}{1-x}\right)^{j_1}$$

$$= \frac{x^k(1+x+x^2+\cdots+x^{s-1})^k}{(1-x)^{k+1}}.$$

□

5. THE Q-ANALOGUE OF THE QUASI S-PASCAL TRIANGLE

In this section, we define the q-analogue of the quasi s-Pascal triangle. We denote by $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}$ these coefficients; for that, we give an explicit formula and generating function of $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}$, and finally, we propose a q-deformation for the s-bonacci sequence.

Definition 17. We define the q-quasi-bi^{s}nomial coefficient, according to Relation (8), as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{[s]} + \sum_{j=1}^{s} q^{n-j}\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{[s]}, \tag{13}$$

or equivalently

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_{[s]} + \sum_{j=1}^{s} q^{(k-1)j}\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{[s]}. \tag{14}$$

We use the convention $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{[s]} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = 0$ for $k \notin \{0, \ldots, n\}$.

Remark 18. For $s = 1$, we obtain Relations (3) and (4) respectively.

The generating function of $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}$ is given by

Theorem 19. *Let $\mathbb{F}_k(x) := \sum_{n\geq 0}\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} x^n$ be the generating function of the q-quasi bi^{s}nomial coefficient, then*

$$\mathbb{F}_k(x) = \frac{x^k q^{\binom{k}{2}}\prod_{j=0}^{k-1}(1+q^jx+(q^jx)^2+\cdots+(q^jx)^{s-1})}{\prod_{j=0}^{k}(1-q^jx)}.$$

Proof. We have $\mathbb{F}_k(x) = \sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_{[s]} x^n$, then using Relation (13)

$$\begin{aligned}\mathbb{F}_k(x) &= \sum_{n\geq 0} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{[s]} x^n + \sum_{n\geq 0}\sum_{j=1}^{s} q^{n-j} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{[s]} x^n \\ &= x \sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_{[s]} x^n + (x + x^2 + \cdots + x^s) \sum_{n\geq 0} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{[s]} (qx)^n,\end{aligned}$$

thus

$$\frac{(1-x)\mathbb{F}_k(x)}{(x + x^2 + \cdots + x^s)} = \sum_{n\geq 0} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{[s]} (qx)^n, \tag{15}$$

on the other side, using Relation (13) again

$$\begin{aligned}&\frac{(1-x)\mathbb{F}_k(x)}{(x + x^2 + \cdots + x^s)} \\ &= \sum_{n\geq 0} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{[s]} (qx)^n + \sum_{n\geq 0} q^{n-1} \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_{[s]} (qx)^n + \cdots + \sum_{n\geq 0} q^{n-s} \begin{bmatrix} n-s \\ k-2 \end{bmatrix}_{[s]} (qx)^n \\ &= qx \sum_{n\geq 0} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{[s]} (qx)^n + (qx + (qx)^2 + \cdots + (qx)^s) + \sum_{n\geq 0} \begin{bmatrix} n \\ k-2 \end{bmatrix}_{[s]} (q^2x)^n,\end{aligned}$$

and by Relation (15), we obtain

$$\frac{(1-x)(1-qx)\mathbb{F}_k(x)}{(x + x^2 + \cdots + x^s)(qx + (qx)^2 + \cdots + (qx)^s)} = \sum_{n\geq 0} \begin{bmatrix} n \\ k-2 \end{bmatrix}_{[s]} (q^2x)^n.$$

We repeat the process and get

$$\begin{aligned}\frac{(1-x)(1-qx)\cdots(1-q^{k-1}x)\mathbb{F}_k(x)}{\prod_{j=0}^{k-1}(q^jx + (q^jx)^2 + \cdots + (q^jx)^s} &= \sum_{n\geq 0} \begin{bmatrix} n \\ 0 \end{bmatrix}_{[s]} (q^kx)^n \\ &= \frac{1}{(1-q^kx)},\end{aligned}$$

and finally, we conclude to the result.

□

Remark 20. With the same method, we can also prove Theorem 19, using Relation (14).

The following result establishes the explicit formula of the q-quasi-bi^{s}nomial coefficient.

Theorem 21. *The coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}$ *satisfies*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_{j} \begin{bmatrix} n-j \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}^{(s-1)}.$$

Proof. We have

$$\begin{aligned}\sum_{n\geq 0} \sum_{j} \begin{bmatrix} n-j \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}^{(s-1)} x^n &= \sum_{j} \begin{bmatrix} k \\ j \end{bmatrix}^{(s-1)} x^j \sum_{n\geq 0} \begin{bmatrix} n-j \\ k \end{bmatrix} x^{n-j} \\ &= \frac{x^k q^{\binom{k}{2}} \prod_{j=0}^{k-1} (1 + q^j x + (q^j x)^2 + \cdots + (q^j x)^{s-1})}{\prod_{j=0}^{k} (1 - q^j x)}.\end{aligned}$$

This last equality comes from Relation (6) and Relation (7). □

Cigler and Carlitz propose the q-analogue of the Fibonacci sequence see [**13, 15**], the following result establishes the recurrence relation for the $q-$analogue of the $s-$bonacci sequence, this generalizes Theorem 6.

Theorem 22. *Let* $\mathbb{T}_{n+1}^{(s)}(x) := \sum_k \begin{bmatrix} n-k \\ k \end{bmatrix}_{[s]} x^k$ *for* $n \geq 0$ *and* $\mathbb{T}_0^{(s)}(x) = 0$*, then*

$$\mathbb{T}_{n+1}^{(s)}(x) = \mathbb{T}_n^{(s)}(x) + x \sum_{j=1}^{s} q^{n-j-1} \mathbb{T}_{n-j}^{(s)}(x/q), \tag{16}$$

and

$$\mathbb{T}_{n+1}^{(s)}(x) = \mathbb{T}_n^{(s)}(xq) + x \sum_{j=1}^{s} \mathbb{T}_{n-j}^{(s)}(xq^j). \tag{17}$$

Proof. For Relation (16), we have

$$\mathbb{T}_{n+1}^{(s)}(x) = \sum_{k} \begin{bmatrix} n-k \\ k \end{bmatrix}_{[s]} x^k,$$

then by Relation (13), we have

$$\begin{aligned}\mathbb{T}_{n+1}^{(s)}(x) &= \sum_k \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_{[s]} x^k + \sum_k \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_{[s]} x^k q^{n-k-1} + \cdots \\ &\qquad \cdots + \sum_k \begin{bmatrix} n-k-s \\ k-1 \end{bmatrix}_{[s]} x^k q^{n-k-s} \\ &= \sum_k \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_{[s]} x^k + x \sum_{k'} \begin{bmatrix} n-k'-2 \\ k' \end{bmatrix}_{[s]} x^{k'} q^{n-k'-2} + \cdots \\ &\qquad \cdots + x \sum_{k'} \begin{bmatrix} n-k'-s-1 \\ k' \end{bmatrix}_{[s]} x^{k'} q^{n-k'-s-1} \\ &= \mathbb{T}_n^{(s)}(x) + x \sum_{j=1}^{s} q^{n-j-1} \mathbb{T}_{n-j}^{(s)}(x/q).\end{aligned}$$

□

The proof is the same for the Relation (17), we use Relation (14).

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