Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **16** (2022), 350–364.
https://doi.org/10.2298/AADM200918012G

OSCILLATION OF ODD-ORDER DIFFERENTIAL EQUATIONS WITH A NONPOSITIVE SUBLINEAR NEUTRAL TERM AND DISTRIBUTED DEVIATING ARGUMENTS

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New sufficient conditions for the oscillation of all solutions to a class of odd-order differential equations with a nonpositive sublinear neutral term and distributed deviating arguments are established. Example are included to illustrate the results.

1. INTRODUCTION

The purpose of this paper is to study the oscillation of the odd-order differential equation with a nonpositive sublinear neutral term and distributed deviating arguments of the form

$$y^{(n)}(t) + \int_c^d q(t,\tau)x^{\beta}(\phi(t,\tau))d\tau = 0, \quad t \geq t_0 > 0, \tag{1}$$

where

$$y(t) = x(t) - p(t)x^{\alpha}(\sigma(t)),$$

$n \geq 3$ is an odd natural number, α and β are the ratios of positive odd integers with $0 < \alpha \leq 1$, and $0 < c < d$.

Throughout the paper, the following conditions are always assumed to hold:

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2020 Mathematics Subject Classification. 34K11, 34K40, 34C10.
Keywords and Phrases. Oscillation, Odd order, Neutral differential equation, Nonpositive neutral term, Sublinear neutral term.

(i) $p : [t_0, \infty) \to \mathbb{R}$ is a continuous function with $0 < p(t) \le p_0 < 1$;

(ii) $q : [t_0, \infty) \times [c, d] \to \mathbb{R}^+ = (0, \infty)$ is a continuous function;

(iii) $\sigma : [t_0, \infty) \to \mathbb{R}$ is a continuous function with $\sigma(t) \le t$, $\sigma'(t) > 0$, and $\lim_{t\to\infty} \sigma(t) = \infty$;

(iv) $\phi : [t_0, \infty) \times [c, d] \to \mathbb{R}$ is a continuous function such that $\phi(t, \tau)$ is nonincreasing in τ, $\phi(t, \tau) \le t$, and $\phi(t, \tau) \to \infty$ as $t \to \infty$, $\tau \in [c, d]$.

By a *solution* of equation (1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \ge t_0$ such that, $y \in C^n([t_x, \infty), \mathbb{R})$ and x satisfies equation (1) on $[t_x, \infty)$. We only consider those solutions of (1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \le t < \infty\} > 0 \quad \text{for any} \quad T_1 \ge t_x;$$

we tacitly assume that (1) possesses such solutions. Such a solution $x(t)$ of (1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, and it is called *nonoscillatory* otherwise. Equation (1) itself is termed oscillatory if all its solutions are oscillatory.

In what follows, we give some background details that motivate our analysis of equation (1). In recent years, the oscillation of various classes of odd-order ($n \ge 3$) neutral differential equations and/or neutral dynamic equations on time scales without distributed deviating arguments has been discussed by many authors and many interesting results have been obtained; see the papers [**2, 3, 5, 6, 7, 9, 10, 12, 13, 15, 16, 17, 20, 21, 23, 24, 27, 28, 31**] and the references cited therein. A commonly employed condition in these works is

$$-1 \le p(t) \le 0$$

as well as the condition

$$-\infty < p_1 \le p(t) \le 0.$$

There are only a few results dealing with the oscillation of differential and/or dynamic equations with a nonpositive neutral term. For an important initial contribution on such equations, we may refer to [**15**] where several oscillation results were obtained for equation

$$(x(t) - p_0 x(t - \sigma))^{(n)} + q(t)x(t - \tau) = 0.$$

Further contributions concerning the oscillatory and asymptotic properties of odd-order differential and/or dynamic equations with a nonpositive neutral term can be found in [**2, 5, 9, 10, 11, 14, 17, 21, 31**].

On the other hand, oscillation results for odd-order ($n \ge 3$) neutral differential equations and/or neutral dynamic equations on time scales with distributed deviating arguments are relatively scarce in the literature; for typical results, we refer the reader to the papers [**4, 8, 25, 26, 29, 30, 32**] and the references contained therein. It is worth noting that these results only apply to the equation with a linear neutral term, i.e., for $\alpha = 1$.

Based on the above observations, the aim of this paper is to establish oscillation criteria that can be applied to odd-order differential equations with a sublinear neutral term and distributed deviating arguments. We wish to point out that the results of this paper can easily be extended to more general odd-order ($n \geq 3$) differential equations with a sublinear neutral term and distributed deviating arguments (see Remark 2.2 below). It is therefore hoped that the present paper will contribute significantly to the study of oscillatory behavior of solutions of odd-order ($n \geq 3$) differential equations with a nonlinear neutral term and distributed deviating arguments.

2. MAIN RESULTS

We begin with some auxiliary results that will be needed in the proofs of our main theorems. For notational purposes, we set:

$$\phi_1(t) := \phi(t, d), \quad \phi_2(t) := \phi(t, c), \quad \text{and} \quad q_1(t) := \int_c^d q(t, \tau) d\tau.$$

In the sequel, we will assume that

$$h(t) := \sigma^{-1}(\phi_1(t)) \leq t \tag{2}$$

where σ^{-1} is the inverse function of σ.

Notice that since

$$\sigma(t) \leq t, \ \sigma'(t) > 0, \quad \text{and} \quad \lim_{t\to\infty} \sigma(t) = \infty,$$

we have $t \leq \sigma^{-1}(t)$, and so

$$\phi_1(t) \leq \sigma^{-1}(\phi_1(t)), \quad \text{i.e.,} \quad \phi_1(t) \leq h(t).$$

Moreover, since $\phi_1(t) \to \infty$ as $t \to \infty$, we also have $h(t) \to \infty$.

Lemma 2.1 ([**18**]). *Let $f \in C^n([t_0, \infty), (0, \infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$, with $n + l$ even for $f^{(n)}(t) \geq 0$, or $n + l$ odd for $f^{(n)}(t) \leq 0$ such that for $t \geq t_x$*

$$l > 0 \ \text{ implies that } f^{(k)}(t) > 0 \quad \text{for} \quad k = 0, 1, \ldots, l - 1, \tag{3}$$

and

$$l \leq n - 1 \ \text{ implies that } (-1)^{l+k} f^{(k)}(t) > 0 \quad \text{for} \quad k = l, l + 1, \ldots, n - 1. \tag{4}$$

Lemma 2.2 ([**1**, Lemma 2.2.3]). *Let $f \in C^n([t_0,\infty),(0,\infty))$, $f^{(n)}(t)f^{(n-1)}(t) \le 0$ for $t \ge t_x$ for some $t_x \ge t_0$, and assume that $\lim_{t\to\infty} f(t) \ne 0$. Then for every $\lambda \in (0,1)$, there exists a $t_\lambda \in [t_x,\infty)$ such that, for all $t \in [t_\lambda,\infty)$,*

$$f(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} \left| f^{(n-1)}(t) \right|.$$

Theorem 2.1. *Let conditions (i)–(iv) and (2) hold and assume that there exists a continuous function $\eta : [t_0,\infty) \to \mathbb{R}$ such that $\phi_2(t) \le \eta(t) \le t$ for $t \ge t_0$. If there exist constants $\lambda_1, \lambda_2 \in (0,1)$ such that the first-order delay differential equations*

$$Y'(t) + q_1(t) \left(\frac{\lambda_1}{(n-1)!} \right)^\beta \left(\phi_1^{n-1}(t) \right)^\beta Y^\beta(\phi_1(t)) = 0, \tag{5}$$

$$W'(t) + q_1(t) \left(\frac{(\eta(t)-\phi_2(t))^{n-1}}{(n-1)!} \right)^\beta W^\beta(\eta(t)) = 0, \tag{6}$$

and

$$Z'(t) + \left(\frac{\lambda_2}{(n-2)!} \right)^{\beta/\alpha} \left(h^{n-2}(t) \right)^{\beta/\alpha} \left(\int_t^\infty q_1(s) ds \right) Z^{\beta/\alpha}(h(t)) = 0,$$

are oscillatory, then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$, and $x(\phi(t,\tau)) > 0$ for $(t,\tau) \in [t_1,\infty) \times [c,d]$. It follows from (1) that

$$y^{(n)}(t) = -\int_c^d q(t,\tau) x^\beta(\phi(t,\tau)) d\tau < 0 \quad \text{for } t \ge t_1, \tag{7}$$

and hence $y^{(n-1)}(t)$ is decreasing and eventually of one sign on $[t_1,\infty)$. We claim that $y^{(n-1)}(t) > 0$ for $t \ge t_2$. To prove the claim, suppose to the contrary that $y^{(n-1)}(t) < 0$ for $t \ge t_2$. Then in view of (7), we have

$$y^{(n-1)}(t) \le y^{(n-1)}(t_2) =: -c < 0 \quad \text{for } t \ge t_2,$$

where $c > 0$ is a constant. Integrating the last inequality from t_2 to t consecutively $n-1$ times, we see that $\lim_{t\to\infty} y(t) = -\infty$.

If we assume that $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$, where $x(t_k) = \max\{x(s) : t_0 \le s \le t_k\}$. Since $\sigma(t) \to \infty$ as $t \to \infty$, we have $\sigma(t_k) > t_0$ for sufficiently large k. From the fact that $\sigma(t) \le t$, we have

$$x(\sigma(t_k)) = \max\{x(s) : t_0 \le s \le \sigma(t_k)\} \le \max\{x(s) : t_0 \le s \le t_k\} = x(t_k).$$

Therefore, for sufficiently large k, we obtain

$$y(t_k) = x(t_k) - p(t_k)x^{\alpha}(\sigma(t_k)) \geq \left(1 - \frac{p_0}{x^{1-\alpha}(t_k)}\right) x(t_k) > 0,$$

which contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$.

If $x(t)$ is bounded, then $y(t)$ is also bounded, which again contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$. This proves our claim that $y^{(n-1)}(t) > 0$ for $t \geq t_2$.

Next, we have two cases to consider: (I) $y(t) > 0$ or (II) $y(t) < 0$ for $t \geq t_2$.

Case (I): $y(t) > 0$ for $t \geq t_2$. From the definition of y, we have

$$x(t) \geq y(t) \quad \text{for } t \geq t_2. \tag{8}$$

Since $\lim_{t\to\infty} \phi(t, \tau) = \infty$ by (iv), we can choose $t_3 \geq t_2$ such that $\phi(t, \tau) \geq t_2$ for all $t \geq t_3$, and so inequality (8) yields

$$x(\phi(t, \tau)) \geq y(\phi(t, \tau)) \quad \text{for } t \geq t_3. \tag{9}$$

Using (9) in (7) gives

$$y^{(n)}(t) \leq -\int_c^d q(t, \tau) y^{\beta}(\phi(t, \tau)) d\tau < 0 \quad \text{for } t \geq t_3. \tag{10}$$

Since $y(t) > 0$ and $y^{(n)}(t) < 0$, it follows from Lemma 2.1 that there exist a $t_4 \geq t_3$ and an even integer $l \in \{0, 2, 4, \ldots, n-1\}$ such that (3) and (4) hold for all $t \geq t_4$.

If $l \geq 2$, then $y'(t) > 0$ for $t \geq t_4$, and so $\lim_{t\to\infty} y(t) \neq 0$. Thus, by Lemma 2.2, for every $\lambda \in (0, 1)$, there exists a $t_\lambda \geq t_4$ such that

$$y(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} y^{(n-1)}(t) \quad \text{for } t \geq t_\lambda. \tag{11}$$

In view of the fact that $y'(t) > 0$, it follows from (10) and (iv) that

$$y^{(n)}(t) + \left(\int_c^d q(t, \tau) d\tau\right) y^{\beta}(\phi(t, d)) < 0,$$

or

$$y^{(n)}(t) + q_1(t) y^{\beta}(\phi_1(t)) < 0 \quad \text{for } t \geq t_4. \tag{12}$$

Using (11) in (12) yields

$$y^{(n)}(t) + q_1(t) \left(\frac{\lambda}{(n-1)!}\right)^{\beta} \left(\phi_1^{n-1}(t)\right)^{\beta} \left(y^{(n-1)}(\phi_1(t))\right)^{\beta} < 0$$

for $t \geq t_6$ for some $t_6 \geq t_\lambda$. Letting $Y(t) = y^{(n-1)}(t)$, we see that $Y(t)$ is a positive solution of the inequality

$$Y'(t) + q_1(t) \left(\frac{\lambda}{(n-1)!}\right)^{\beta} \left(\phi_1^{n-1}(t)\right)^{\beta} Y^{\beta}(\phi_1(t)) < 0 \tag{13}$$

for every $0 < \lambda < 1$. Integrating inequality (13) from $t \geq t_6$ to u and letting $u \to \infty$, we obtain

$$Y(t) \geq \int_t^\infty q_1(s) \left(\frac{\lambda}{(n-1)!}\right)^\beta \left(\phi_1^{n-1}(s)\right)^\beta Y^\beta(\phi_1(s))ds.$$

The function $Y(t)$ is strictly decreasing on $[t_6, \infty)$ for every λ with $\lambda \in (0,1)$, and so by [**22**, Theorem 1], there exists a positive solution of equation (5) . This contradicts the fact that equation (5) is oscillatory.

Next, we let $l = 0$. Then, we again arrive at (10) for $t \geq t_3$, and $y(t)$ satisfies

$$(-1)^k y^{(k)}(t) > 0 \quad \text{for } t \geq t_4 \text{ and } k = 0, 1, 2, \ldots, n-1. \tag{14}$$

From (14) and the fact that $y^{(n)}(t) < 0$, we have, for $t_4 \leq u \leq v$,

$$-y^{(n-2)}(u) \geq y^{(n-2)}(v) - y^{(n-2)}(u) = \int_u^v y^{(n-1)}(s)ds \geq (v-u)y^{(n-1)}(v). \tag{15}$$

Integrating (15) from $u \geq t_4$ to v consecutively $n-2$ times and taking (14) into account, we see that

$$y(u) \geq \frac{(v-u)^{n-1}}{(n-1)!} y^{(n-1)}(v) \quad \text{for } v \geq u \geq t_4. \tag{16}$$

Setting $u = \phi_2(t)$ and $v = \eta(t)$ into (16), we obtain

$$y(\phi_2(t)) \geq \frac{(\eta(t) - \phi_2(t))^{n-1}}{(n-1)!} y^{(n-1)}(\eta(t)) \quad \text{for } t \geq t_4. \tag{17}$$

Since $y'(t) < 0$, inequality (10) yields

$$y^{(n)}(t) + \left(\int_c^d q(t,\tau)d\tau\right) y^\beta(\phi(t,c)) < 0,$$

or

$$y^{(n)}(t) + q_1(t)y^\beta(\phi_2(t)) < 0 \quad \text{for } t \geq t_4. \tag{18}$$

Using (17) in (18), we obtain

$$W'(t) + q_1(t)\left(\frac{(\eta(t) - \phi_2(t))^{n-1}}{(n-1)!}\right)^\beta W^\beta(\eta(t)) < 0 \quad \text{for } t \geq t_4, \tag{19}$$

where $W(t) = y^{(n-1)}(t) > 0$. As in the case $l \geq 2$, this contradicts the fact that equation (6) is oscillatory.

Case (II): $y(t) < 0$ for $t \geq t_2$. Letting $z(t) = -y(t) > 0$ for $t \geq t_2$, it follows from equation (1) that

$$z^{(n)}(t) = \int_c^d q(t,\tau)x^\beta(\phi(t,\tau)))d\tau \quad \text{for } t \geq t_2. \tag{20}$$

From the definition of $y(t)$, we have

$$z(t) = -y(t) = p(t)x^{\alpha}(\sigma(t)) - x(t) \leq p(t)x^{\alpha}(\sigma(t)) \leq x^{\alpha}(\sigma(t)),$$

and so

$$x(\sigma(t)) \geq z^{1/\alpha}(t) \quad \text{or} \quad x(t) \geq z^{1/\alpha}(\sigma^{-1}(t)).$$

From this we have

$$x(\phi(t,\tau)) \geq z^{1/\alpha}(\sigma^{-1}(\phi(t,\tau))) \quad \text{for } t \geq t_3,$$

where $\phi(t,\tau) \geq t_2$ for $t \geq t_3$, for some $t_3 \geq t_2$ by (iv). Using the last inequality in (20) gives

$$z^{(n)}(t) \geq \int_c^d q(t,\tau) z^{\beta/\alpha}(\sigma^{-1}(\phi(t,\tau)))d\tau \quad \text{for } t \geq t_3. \tag{21}$$

Thus, by Lemma 2.1, $z(t)$ satisfies

$$z(t) > 0,\ z'(t) > 0,\ z^{(n-2)}(t) > 0,\ z^{(n-1)}(t) < 0,\ z^{(n)}(t) > 0, \tag{22}$$

for $t \geq t_3$. Using the fact that $z'(t) > 0$, we obtain from (21) that

$$z^{(n)}(t) \geq \left(\int_c^d q(t,\tau)d\tau\right) z^{\beta/\alpha}(\sigma^{-1}(\phi_1(t)) = q_1(t) z^{\beta/\alpha}(h(t)) \quad \text{for } t \geq t_3. \tag{23}$$

Integrating inequality (23) from $t \geq t_3$ to u, letting $u \to \infty$, and noting (2), we have

$$-z^{(n-1)}(t) \geq \int_t^{\infty} q_1(s) z^{\beta/\alpha}(h(s))ds \geq \left(\int_t^{\infty} q_1(s)ds\right) z^{\beta/\alpha}(h(t)). \tag{24}$$

In view of (22), by Lemma 2.2, for every $\lambda \in (0,1)$, there exists a $t_\lambda \geq t_3$ such that

$$z(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} z^{(n-2)}(t) \tag{25}$$

for $t \geq t_\lambda$. Since $\lim_{t\to\infty} h(t) = \infty$, we can choose $t_5 \geq t_\lambda$ such that $h(t) \geq t_\lambda$ for all $t \geq t_5$, and so inequality (25) yields

$$z(h(t)) \geq \frac{\lambda}{(n-2)!} h^{n-2}(t) z^{(n-2)}(h(t)) \quad \text{for } t \geq t_5. \tag{26}$$

Using (26) in (24) gives

$$Z'(t) + \left(\frac{\lambda}{(n-2)!}\right)^{\beta/\alpha} \left(h^{n-2}(t)\right)^{\beta/\alpha} \left(\int_t^{\infty} q_1(s)ds\right) Z^{\beta/\alpha}(h(t)) \leq 0 \quad \text{for } t \geq t_5, \tag{27}$$

where $Z(t) = z^{(n-2)}(t) > 0$. The remainder of the proof in this case is similar to that of case (I), and hence is omitted. This completes the proof of the theorem. □

It is well known (see, e.g., [**19**]) that if

$$\liminf_{t\to\infty}\int_{\xi(t)}^{t}\psi(s)ds>\frac{1}{e}, \tag{28}$$

then the first-order delay differential equation

$$x'(t)+\psi(t)x(\xi(t))=0 \tag{29}$$

is oscillatory, where $\psi,\xi\in C([t_0,\infty),\mathbb{R})$ with $\psi(t)\geq 0$, $\xi(t)\leq t$, and $\lim_{t\to\infty}\xi(t)=\infty$. Thus, from Theorem 2.1, we have the following result.

Corollary 2.1. *Let conditions (i)–(iv) and* (2) *hold and let* $\beta=\alpha=1$. *Assume that there exists a continuous function* $\eta:[t_0,\infty)\to\mathbb{R}$ *such that* $\phi_2(t)\leq\eta(t)\leq t$ *for* $t\geq t_0$. *If*

$$\liminf_{t\to\infty}\int_{\phi_1(t)}^{t}q_1(s)\phi_1^{n-1}(s)ds>\frac{(n-1)!}{e}, \tag{30}$$

$$\liminf_{t\to\infty}\int_{\eta(t)}^{t}q_1(s)\left(\eta(s)-\phi_2(s)\right)^{n-1}ds>\frac{(n-1)!}{e},$$

and

$$\liminf_{t\to\infty}\int_{h(t)}^{t}h^{n-2}(s)\left(\int_{s}^{\infty}q_1(u)du\right)ds>\frac{(n-2)!}{e}, \tag{31}$$

then equation (1) *is oscillatory.*

Proof. From (30) and (31), we can choose positive constants $\lambda_1,\lambda_2\in(0,1)$ such that

$$\liminf_{t\to\infty}\lambda_1\int_{\phi_1(t)}^{t}q_1(s)\phi_1^{n-1}(s)ds>\frac{(n-1)!}{e},$$

and

$$\liminf_{t\to\infty}\lambda_2\int_{h(t)}^{t}h^{n-2}(s)\left(\int_{s}^{\infty}q_1(u)du\right)ds>\frac{(n-2)!}{e}.$$

By Theorem 2.1 and in view of (28) and (29), the conclusion of Corollary 2.1 follows immediately. □

Corollary 2.2. *Let conditions (i)–(iv) and* (2) *hold,* $\beta<\alpha$, *and assume that there exists a continuous function* $\eta:[t_0,\infty)\to\mathbb{R}$ *such that* $\phi_2(t)\leq\eta(t)\leq t$ *for* $t\geq t_0$. *If*

$$\int_{t_0}^{\infty}q_1(s)\left(\phi_1^{n-1}(s)\right)^{\beta}ds=\infty, \tag{32}$$

$$\int_{t_0}^{\infty}q_1(s)\left[\left(\eta(s)-\phi_2(s)\right)^{n-1}\right]^{\beta}ds=\infty, \tag{33}$$

and

$$\int_{t_0}^{\infty} \left(h^{n-2}(s)\right)^{\beta/\alpha} \left(\int_{s}^{\infty} q_1(u)du\right) ds = \infty, \tag{34}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$, and $x(\phi(t,\tau)) > 0$ for $(t,\tau) \in [t_1,\infty) \times [c,d]$. Proceeding exactly as in the proof of Theorem 2.1, we again arrive at (13) for $t \geq t_6$, (19) for $t \geq t_4$, and (27) for $t \geq t_5$. Since $\phi_1(t) \leq t$ and $Y(t)$ is positive and decreasing, inequality (13) takes the form

$$Y'(t) + q_1(t)\left(\frac{\lambda}{(n-1)!}\right)^{\beta} \left(\phi_1^{n-1}(t)\right)^{\beta} Y^{\beta}(t) < 0$$

or

$$\frac{Y'(t)}{Y^{\beta}(t)} + q_1(t)\left(\frac{\lambda}{(n-1)!}\right)^{\beta} \left(\phi_1^{n-1}(t)\right)^{\beta} < 0, \tag{35}$$

for $t \geq t_6$. Integrating (35) from t_6 to t and using then the fact that $Y(t) > 0$, we get

$$\int_{t_6}^{t} q_1(s)\left(\phi_1^{n-1}(s)\right)^{\beta} ds \leq \left(\frac{(n-1)!}{\lambda}\right)^{\beta} \frac{Y^{1-\beta}(t_6)}{1-\beta} < \infty \quad \text{as } t \to \infty,$$

which contradicts (32). The remainder of the proof follows from the facts that $\eta(t) \leq t$ and $h(t) \leq t$, and the inequalities (19) and (27). The proof is now complete. □

Next, we present the following interesting result in which we need to assume that $\phi(t,\tau)$ is nondecreasing in t. Notice that this implies that $h(t)$ in nondecreasing.

Theorem 2.2. *Let conditions (i)–(iv) and* (2) *hold,* $\beta \leq \alpha$, *and assume that* $\phi(t,\tau)$ *is nondecreasing in* t. *If*

$$\limsup_{t\to\infty} \left(\phi_1^{\beta(n-1)}(t) \int_t^{\infty} q_1(s)ds\right) = \infty, \tag{36}$$

$$\limsup_{t\to\infty} \int_{\phi_2(t)}^{t} q_1(s)\left[\phi_2(t) - \phi_2(s)\right]^{\beta(n-1)} ds = \infty, \tag{37}$$

and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q_1(s) h^{\beta(n-2)/\alpha}(s)(h(t)-h(s))^{\beta/\alpha} ds = \infty, \tag{38}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$, and $x(\phi(t,\tau)) > 0$ for $(t,\tau) \in [t_1,\infty) \times [c,d]$. As in the proof of Theorem 2.1, we again have two cases to consider: (I) $y(t) > 0$ or (II) $y(t) < 0$ for $t \geq t_2$.

Case (I). Proceeding as in the proof of Theorem 2.1, we see that (10) holds for $t \geq t_3$, and so there exist a $t_4 \geq t_3$ and an even integer $l \in \{0, 2, 4, \ldots, n-1\}$ such that (3) and (4) hold for all $t \geq t_4$.

If $l \geq 2$, we again see that (11) and (12) hold. Integrating (12) from t to u and letting $u \to \infty$, we get

$$y^{(n-1)}(t) \geq \left(\int_t^\infty q_1(s)ds\right) y^\beta(\phi_1(t)). \tag{39}$$

Using (11) in (39), we obtain

$$\begin{aligned} y^{(n-1)}(t) &\geq \left(\frac{\lambda}{(n-1)!}\phi_1^{n-1}(t)\right)^\beta \left(\int_t^\infty q_1(s)ds\right) \left[y^{(n-1)}(\phi_1(t))\right]^\beta \\ &\geq \left(\frac{\lambda}{(n-1)!}\phi_1^{n-1}(t)\right)^\beta \left(\int_t^\infty q_1(s)ds\right) \left[y^{(n-1)}(t)\right]^\beta. \end{aligned}$$

Thus, we have

$$\left[y^{(n-1)}(t)\right]^{1-\beta} \geq \left(\frac{\lambda}{(n-1)!}\phi_1^{n-1}(t)\right)^\beta \left(\int_t^\infty q_1(s)ds\right). \tag{40}$$

Taking $\limsup$ on both sides of (40) as $t \to \infty$, we get a contradiction to (36).

Next, we consider the case $l = 0$. Then, (16) and (18) hold for $t \geq t_4$. Integrating (18) from $\phi_2(t)$ to t gives

$$y^{(n-1)}(\phi_2(t)) \geq \int_{\phi_2(t)}^t q_1(s) y^\beta(\phi_2(s))ds. \tag{41}$$

For $t \geq s \geq t_4$, it follows from (16) that

$$y(\phi_2(s)) \geq \frac{[\phi_2(t) - \phi_2(s)]^{n-1}}{(n-1)!} y^{(n-1)}(\phi_2(t)) \quad \text{for } t \geq t_4.$$

Using this in (41) gives

$$\left[y^{(n-1)}(\phi_2(t))\right]^{1-\beta} \geq \int_{\phi_2(t)}^t q_1(s) \frac{[\phi_2(t) - \phi_2(s)]^{\beta(n-1)}}{((n-1)!)^\beta} ds.$$

Taking $\limsup$ on both sides of the last inequality as $t \to \infty$, we get a contradiction to (37).

Case (II). As in the proof of Theorem 2.1, we see that (22), (23) and (26) hold for $t \geq t_5$. Using (26) in (23), we obtain

$$z^{(n)}(t) \geq q_1(t) \left[\frac{\lambda}{(n-2)!} h^{n-2}(t)\right]^{\beta/\alpha} \left[z^{(n-2)}(h(t))\right]^{\beta/\alpha}. \tag{42}$$

Integrating (42) from $h(t)$ to t and taking (22) into account, we arrive at

$$\begin{aligned} -z^{(n-1)}(h(t)) &\geq z^{(n-1)}(t) - z^{(n-1)}(h(t)) \\ &\geq \int_{h(t)}^{t} q_1(s) \left[\frac{\lambda}{(n-2)!} h^{n-2}(s)\right]^{\beta/\alpha} \left[z^{(n-2)}(h(s))\right]^{\beta/\alpha} ds. \end{aligned} \tag{43}$$

Also for $t \geq s \geq t_3$, it follows from (22) that

$$\begin{aligned} z^{(n-2)}(h(s)) \geq z^{(n-2)}(h(s)) - z^{(n-2)}(h(t)) &= \int_{h(s)}^{h(t)} -z^{(n-1)}(u) du \\ &\geq (h(t) - h(s)) \left(-z^{(n-1)}(h(t))\right). \end{aligned} \tag{44}$$

Using (44) in (43) gives

$$\left[-z^{(n-1)}(h(t))\right]^{1-\frac{\beta}{\alpha}} \geq \int_{h(t)}^{t} q_1(s) \left[\frac{\lambda}{(n-2)!} h^{n-2}(s)\right]^{\beta/\alpha} (h(t) - h(s))^{\beta/\alpha} ds.$$

Taking lim sup on both sides of the last inequality as $t \to \infty$ gives a contradiction to (38). This proves he theorem. □

Example 2.1. Consider the differential equation with a nonpositive sublinear neutral term and distributed deviating argument

$$\left(x(t) - p(t)x^{\alpha}\left(\frac{t}{2}\right)\right)^{(n)} + \int_1^2 (t^2 + 2\tau) x^{\beta}\left(\frac{t}{2} - \tau\right) d\tau = 0, \quad t \geq 8, \tag{45}$$

where $n \geq 3$ is odd. Here we have $\sigma(t) = t/2$, $q(t,\tau) = t^2 + 2\tau$, and $\phi(t,\tau) = t/2 - \tau$, and we assume that $p(t)$ is a continuous function as in condition (i). Also, let α and β be ratios of odd positive integers with $\beta < \alpha$. Then,

$$\phi_1(t) = t/2 - 2, \ \phi_2(t) = t/2 - 1, \ \sigma^{-1}(t) = 2t, \ h(t) = t - 4,$$

and

$$q_1(t) := \int_c^d q(t,\tau) d\tau = \int_1^2 (t^2 + 2\tau) d\tau = t^2 + 3.$$

Since

$$\int_{t_0}^{\infty} q_1(s) \left(\phi_1^{n-1}(s)\right)^{\beta} ds = \int_8^{\infty} (s^2 + 3) \left(\frac{s}{2} - 2\right)^{\beta(n-1)} ds = \infty,$$

condition (32) holds.

With $\eta(t) = t/2$, condition (33) becomes

$$\int_{t_0}^{\infty} q_1(s)\left[(\eta(s)-\phi_2(s))^{n-1}\right]^{\beta} ds = \int_8^{\infty}(s^2+3)ds = \infty,$$

i.e, condition (33) holds.

Now

$$\int_{t_0}^{\infty}\left(h^{n-2}(s)\right)^{\beta/\alpha}\left(\int_s^{\infty} q_1(u)du\right)ds = \int_8^{\infty}\left((s-4)^{n-2}\right)^{\beta/\alpha}\int_s^{\infty}(u^2+3)duds = \infty$$

since $\int_s^{\infty}(u^2+3)du = \infty$ for $s \geq 8$, so condition (34) holds. By Corollary 2.2, equation (45) is oscillatory.

Example 2.2. Consider the differential equation

$$\left(x(t)-p(t)x\left(\frac{t}{2}\right)\right)^{(n)} + \int_1^2\left(\frac{1}{t^2}-3+2\tau\right)x\left(\frac{t}{2}-\tau\right)d\tau = 0, \quad t \geq 8, \tag{46}$$

where $n \geq 5$ is odd. Here we have $\sigma(t) = t/2$, $q(t,\tau) = \frac{1}{t^2} - 3 + 2\tau$, $\phi(t,\tau) = t/2-\tau$, $\alpha = \beta = 1$ and $p(t)$ is a continuous function with $0 < p(t) \leq p_0 < 1$. Then,

$$\phi_1(t) = t/2-2,\ \phi_2(t) = t/2-1,\ \sigma^{-1}(t) = 2t,\ h(t) = t-4,$$

and

$$q_1(t) := \int_c^d q(t,\tau)d\tau = \int_1^2\left(\frac{1}{t^2}-3+2\tau\right)d\tau = 1/t^2.$$

Now, (36)–(38) become, respectively,

$$\limsup_{t\to\infty}\left(\frac{t-4}{2}\right)^{n-1}\int_t^{\infty}\frac{1}{s^2}ds = \infty,$$

$$\limsup_{t\to\infty}\int_{t/2-1}^{t}\frac{1}{s^2}\left(\frac{t-s}{2}\right)^{n-1}ds \geq \limsup_{t\to\infty}\frac{1}{2^{n-1}t^2}\int_{t/2-1}^{t}(t-s)^{n-1}ds = \infty,$$

and

$$\limsup_{t\to\infty}\int_{t-4}^{t}\frac{1}{s^2}(s-4)^{n-2}(t-s)ds \geq \limsup_{t\to\infty}\frac{(t-8)^{n-2}}{t^2}\int_{t-4}^{t}(t-s)ds = \infty.$$

Therefore, all conditions of Theorem 2.2 are satisfied and so equation (46) is oscillatory.

Remark 2.1. The major difference between Theorems 2.1 and 2.2 is that the discovery of new results for the oscillation of solutions of first order delay equations expands the scope of equations covered by Theorem 2.1. On the other hand, the explicit integral conditions required in Theorem 2.1 make it easy to apply.

Remark 2.2. The results of this paper are presented in a form that can be extended to higher order equations of the type

$$\left[\left(a(t)\left(x(t)-p(t)x^{\alpha}(\sigma(t))\right)^{(n-1)}\right)^{\gamma}\right]' + \int_c^d q(t,\tau)f(t,x(\phi(t,\tau)))d\tau = 0,$$

where $n \geq 3$ is a positive integer, p, q, α, β, σ, and ϕ are as in this paper, γ is the ratio of positive odd integers, $a \in C\left([t_0,\infty),(0,\infty)\right)$, $f : [t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u) > 0$ for all $u \neq 0$, and there exists a positive constant μ such that

$$f(t,u)/u^{\beta} \geq \mu \quad \text{for } u \neq 0.$$

Remark 2.3. It will be of interest to study equation (1) with $\alpha \geq 1$.

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(Received 18. 09. 2020.)
(Revised 26. 03. 2022.)

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