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A MONOTONICITY THEOREM FOR THE GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND

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For $a \in (0, 1/2]$ and $r \in (0, 1)$, let $\mathscr{K}_a(r)$ ($\mathscr{K}(r)$) denote the generalized elliptic integral (complete elliptic integral, respectively) of the first kind. In this article, we mainly present a sufficient and necessary condition under which the function $a \mapsto [\mathscr{K}(r) - \mathscr{K}_a(r)]/(1-2a)^{\lambda}(\lambda \in \mathbb{R})$ is monotone on $(0, 1/2)$ for each fixed $r \in (0, 1)$. The obtained result leads to the conclusion that inequality

$$\mathscr{K}(r) - (1-2a)^{\alpha}\left[\mathscr{K}(r) - \frac{\pi}{2}\right] \leq \mathscr{K}_a(r) \leq \mathscr{K}(r) - (1-2a)^{\beta}\left[\mathscr{K}(r) - \frac{\pi}{2}\right]$$

holds for all $a \in (0, 1/2]$ and $r \in (0, 1)$ with the best possible constants $\alpha = \pi/2$ and $\beta = 2$.

1. INTRODUCTION

For $r \in (0, 1)$, Legendre's complete elliptic integral of the first kind [**5, 21**] is defined by

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 t}} dt.$$

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This integral is a special case of the Gaussian hypergeometric function [**3, 7, 9**]

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} x^n, \quad |x| < 1,$$

where (a,n) is the Pochhammer symbol or shifted factorial defined as $(a,0) = 1$ for $a \neq 0$, and $(a,n) = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a)$ for $n \in \mathbb{N} = \{1,2,3,\cdots\}$, and $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt (\mathrm{Re}x > 0)$ is the classical Euler Gamma function. Indeed, we have

$$\mathscr{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \tag{1}$$

For $a \in (0, 1/2]$ and $r \in (0,1)$, the generalized elliptic integral of the first kind is defined as [**3**]

$$\mathscr{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2), \tag{2}$$

when $a = 1/2$, the function $\mathscr{K}_a(r)$ reduces to the complete elliptic integral of the first kind $\mathscr{K}(r)$. It follows from integral representation of the Gaussian hypergeometric function that $\mathscr{K}_a(r)$ satisfies the following integral formula[**1, 21**]

$$\mathscr{K}_a(r) = [\sin(\pi a)] \int_0^{\pi/2} (\tan x)^{1-2a}(1 - r^2 \sin^2 x)^{-a} dx. \tag{3}$$

The complete elliptic integrals and generalized elliptic integrals have wide applications in many mathematical branches as well as in physics and engineering [**4, 8**]. In recent years, they have frequently occurred in geometric function theory, the theory of mean values and number theory. Especially, a lot of conformal invariants and distortion functions in conformal and quasiconformal maps, and modular functions in Ramanujan's generalized modular equation all depend on the two integrals $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$ [**3, 4, 14, 17, 18, 19, 20, 25, 26**]. In reviewing the research results of $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$ in the past few years, they can be broadly classified into two kinds. One is to prove monotonicity properties of certain combinations of $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$ with respect to $r \in (0,1)$, and thus extend some classical inequalities of $\mathscr{K}(r)$ to the case of $\mathscr{K}_a(r)$. The other one is to study the dependence of $\mathscr{K}_a(r)$ on the parameter a, and obtain some sharp upper and lower bounds for $\mathscr{K}_a(r)$ in terms of $\mathscr{K}(r)$. Relatively speaking, the study of the dependence on the parameter a of $\mathscr{K}_a(r)$ is more difficult and fewer achievements. For above, and the recent development of $\mathscr{K}_a(r)$ and $\mathscr{K}(r)$, the readers can refer to the literatures [**2, 10, 11, 12, 13, 22, 23, 24, 27, 29**].

In 2000, Anderson, Qiu, Vamanamurthy and Vuorinen [**3**, Corollary 7.3] proved that, for each $r \in (0,1)$, the function $a \mapsto \mathscr{K}_a(r)$ is strictly increasing from $[0, 1/2]$ onto $[\pi/2, \mathscr{K}(r)]$, and therefore derived that inequality

$$\mathscr{K}_a(r) \leq \mathscr{K}(r), \tag{4}$$

which holds for each $a \in [0,1/2]$ and $r \in (0,1)$.

Very recently, Qiu, Ma and Bao[**16**, Theorem 1.1] showed that, for each $r \in (0,1)$, the following functions

$$(5)\qquad f_{1,\lambda}(a) = \frac{\mathscr{K}_a(r)}{a^\lambda}\ (\lambda \in \mathbb{R}) \quad \text{and} \quad f_{2,\lambda}(a) = \frac{\mathscr{K}_a(r) - \pi/2}{a^\lambda}\ (\lambda \in \mathbb{R})$$

are both strictly increasing (decreasing) on $(0,1/2]$ if and only if $\lambda \le 0$ ($\lambda \ge 1$, respectively), and substantially obtained that

$$(6)\qquad \frac{\pi}{2}(1-2a) + 2a\mathscr{K}(r) \le \mathscr{K}_a(r) \le \min\left\{\mathscr{K}(r), \frac{\pi}{2} + \pi a \log\frac{1}{\sqrt{1-r^2}}\right\},$$

for each $a \in [0,1/2]$ and $r \in (0,1)$, with equality in each instance if and only if $a = 1/2$. Moreover, the authors[**16**, Theorem 3.6] also proved that the function $r \mapsto [\mathscr{K}_a(r)-\pi/2]/[\mathscr{K}(r)-\pi/2]$ is strictly decreasing from $(0,1)$ onto $(\sin(\pi a), 4a(1-a))$. Consequently, for each $a \in [0,1/2]$ and $r \in (0,1)$, the double inequality

$$(7)\qquad \frac{\pi}{2} + \left[\mathscr{K}(r) - \frac{\pi}{2}\right]\sin(\pi a) \le \mathscr{K}_a(r) \le \frac{\pi}{2} + 4a(1-a)\left[\mathscr{K}(r) - \frac{\pi}{2}\right]$$

holds true.

Motivated by the functions $f_{1,\lambda}(a)$ and $f_{2,\lambda}(a)$ defined in (5) and inequality (6), in this paper, we shall introduce another combinational function of $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$ with one free parameter $\lambda \in \mathbb{R}$,

$$g(a) = \frac{\mathscr{K}(r) - \mathscr{K}_a(r)}{(1-2a)^\lambda}, \quad a \in (0,1/2],$$

and establish a sufficient and necessary condition under which $g(a)$ is monotone on $(0,1/2]$ for each $r \in (0,1)$. Exactly speaking, we shall find all values of $\lambda \in \mathbb{R}$ such that $g(a)$ is strictly increasing or decreasing on $(0,1/2]$ for each $r \in (0,1)$, and thus establish several new inequalities between $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$. Besides, the refinement of inequality (7) will also be given in the following Theorem 2.

In the sequel, we always let $a \in (0,1/2]$, $\mathbb{N}$ ($\mathbb{R}$) denotes the set of positive integers (real numbers, respectively) as usual, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let

$$(8)\qquad a_k = \frac{(a,k)(1-a,k)}{(k!)^2}, \quad b_k = \left[\frac{(1/2,k)}{k!}\right]^2,$$

$$(9)\qquad c_k = (k!)^2 a_k = (a,k)(1-a,k), \quad d_k = (k!)^2 b_k = (1/2,k)^2,$$

$$(10)\qquad P_{1,n}(r) = \frac{\pi}{2}\sum_{k=0}^{n} a_k r^{2k}, \quad P_{2,n}(r) = \frac{\pi}{2}\sum_{k=0}^{n} b_k r^{2k},$$

$$(11)\qquad D_n = \frac{d_n}{4}\left[\frac{\pi^2}{2} - \psi'\left(n+\frac{1}{2}\right)\right], \quad D(r) = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{D_n}{(n!)^2}r^{2n}.$$

Recall that the Psi (Digamma) function is defined by [**1, 6**]

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \mathrm{Re}x > 0, \tag{12}$$

and the Gamma function $\Gamma(x)$ has the well-known Euler's reflection formula

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}. \tag{13}$$

Theorem 1. *Let $r \in (0,1)$, $\lambda \in \mathbb{R}$, and $D(r)$ be as in (11). Define the function g on $(0,1/2]$ by*

$$g(a) = \frac{\mathscr{K}(r) - \mathscr{K}_a(r)}{(1-2a)^\lambda}. \tag{14}$$

Then g is strictly increasing (decreasing) on $(0,1/2]$ if and only if $\lambda \geq 2$ ($\lambda \leq \pi/2$, respectively). Moreover, $g(0^+)\ |_{\lambda=2} = \mathscr{K}(r) - \pi/2$, $g(1/2)\ |_{\lambda=2} = D(r)$, and $g(0^+)\ |_{\lambda=\pi/2} = \mathscr{K}(r) - \pi/2$, $g(1/2)\ |_{\lambda=\pi/2} = 0$. In particular, for $a \in (0,1/2]$ and $r \in (0,1)$,

$$\begin{aligned} &\max\left\{\mathscr{K}(r) - (1-2a)^{\pi/2}\left[\mathscr{K}(r) - \frac{\pi}{2}\right], \mathscr{K}(r) - (1-2a)^2 D(r)\right\} \\ &\leq \mathscr{K}_a(r) \leq \mathscr{K}(r) - (1-2a)^2\left[\mathscr{K}(r) - \frac{\pi}{2}\right], \end{aligned} \tag{15}$$

with equality in each instance if and only if $a = 1/2$.

Theorem 2. *Let $a \in (0,1/2)$, $n \in \mathbb{N}_0$, and $P_{1,n}(r)$ be as in (10). Define the function f on $(0,1)$ by*

$$f(r) = \frac{\mathscr{K}_a(r) - P_{1,n}(r)}{\mathscr{K}(r) - P_{2,n}(r)},$$

then f is strictly decreasing on $(0,1)$ with

$$f(0^+) = \alpha = \frac{\Gamma(n+1+a)\Gamma(n+2-a)\sin(\pi a)}{\Gamma(3/2+n)^2}, \quad f(1^-) = \sin(\pi a).$$

In particular, for $a \in (0,1/2]$, $r \in (0,1)$ and $n \in \mathbb{N}_0$,

$$P_{1,n}(r) + [\mathscr{K}(r) - P_{2,n}(r)]\sin(\pi a) \leq \mathscr{K}_a(r) \leq P_{1,n}(r) + \alpha[\mathscr{K}(r) - P_{2,n}(r)], \tag{16}$$

with equality in each instance if and only if $a = 1/2$.

2. PRELIMINARIES

In this section, we shall prove several Lemmas, which will be used in the proof of our main results.

Lemma 3. ([**5**, Lemma 5.1]) *For $-\infty < a < b < \infty$, let f, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \textit{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 4. ([**15**, Lemma 2.1] and [**17**, Lemma 2.4]) *For $n \in \mathbb{N}_0$, let r_n and s_n be real numbers, and let the power series $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and $S(x) = \sum_{n=0}^{\infty} s_n x^n$ be convergent for $|x| < 1$. If $s_n \geq 0$ and not all vanish for $n \in \mathbb{N}_0$, and if r_n/s_n is strictly increasing (decreasing) in $n \in \mathbb{N}_0$, then the function $x \mapsto R(x)/S(x)$ is strictly increasing (decreasing, respectively) on $(0,1)$.*

Lemma 5. ([**28**, Theorem A]) *Let the functions f, g be defined on $(0,\infty)$ such that their Laplace transforms $\mathscr{L}(f) = \int_0^\infty f(t)e^{-xt}dt$ and $\mathscr{L}(g) = \int_0^\infty g(t)e^{-xt}dt$ exist with $g(t) \neq 0$ for all $t > 0$. Then the ratio $\mathscr{L}(f)/\mathscr{L}(g)$ is decreasing (increasing) on $(0,\infty)$ if f/g is increasing (decreasing) on $(0,\infty)$.*

Lemma 6. *For $a \in (0, 1/2]$, let $p(a) = (1-2a)\cos(\pi a)/[1 - \sin(\pi a)]$, then p is strictly increasing from $(0, 1/2]$ onto $(1, 4/\pi]$.*

Proof. Clearly $p(0) = 1$, and by l'Hôptial's rule we have $p(1/2) = 4/\pi$. Let $p_1(a) = (1-2a)\cos(\pi a)$, $p_2(a) = 1 - \sin(\pi a)$, $p_3(a) = 1 - 2a$ and $p_4(a) = \cot(\pi a)$. Then $p(a) = p_1(a)/p_2(a)$, $p_1(1/2) = p_2(1/2) = p_3(1/2) = p_4(1/2) = 0$ and

$$\frac{p_1'(a)}{p_2'(a)} = \frac{2}{\pi} + \frac{p_3(a)}{p_4(a)}, \quad \frac{p_3'(a)}{p_4'(a)} = \frac{2\sin^2(\pi a)}{\pi}.$$

Applying Lemma 3 twice, the monotonicity of $p(a)$ on $(0, 1/2]$ follows. □

Lemma 7. *Let $n \in \mathbb{N}_0$, c_n, d_n and D_n be as in (9) and (11). Then*

$$\lim_{a \to 1/2} \frac{d_n - c_n}{(1-2a)^2} = D_n.$$

Proof. By (12), (13) and l'Hôpital Rule twice,

$$\begin{aligned}\lim_{a\to 1/2}\frac{d_n-c_n}{(1-2a)^2}&=\frac{1}{4}\lim_{a\to 1/2}\frac{[\psi(a+n)-\psi(1-a+n)-\psi(a)+\psi(1-a)]c_n}{1-2a}\\&=-\frac{1}{8}\lim_{a\to 1/2}\Big\{[\psi'(a+n)+\psi'(1-a+n)-\psi'(a)-\psi'(1-a)]c_n\\&\quad+[\psi(a+n)-\psi(1-a+n)-\psi(a)+\psi(1-a)]^2c_n\Big\}\\&=\frac{d_n}{4}\left[\frac{\pi^2}{2}-\psi'\left(n+1/2\right)\right]=D_n.\end{aligned}$$

□

Lemma 8. *For each $a\in(0,1/2)$, define the function h on $(0,\infty)$ by*

$$h(x)=\frac{\psi'(x+a)-\psi'(x+1-a)}{2\psi(x+1/2)-\psi(1-a+x)-\psi(a+x)}.$$

Then h is positive and strictly decreasing on $(0,\infty)$.

Proof. Write $h(x)=h_1(x)/h_2(x)$, where

$$h_1(x)=\psi'(x+a)-\psi'(x+1-a),$$
$$h_2(x)=2\psi(x+1/2)-\psi(1-a+x)-\psi(a+x).$$

Since ψ' is strictly decreasing on $(0,\infty)$ and $a\in(0,1/2)$, then $h_1(x)>0$ for all $x\in(0,\infty)$. Also, due to

$$\frac{\partial h_2}{\partial a}=\psi'(1-a+x)-\psi'(a+x)<0,$$

one has $h_2(x)>h_2(x)\mid_{a=1/2}=0$. Thus $h(x)$ is a positive function defined on $(0,\infty)$.

According to (cf. [**1**])

$$\psi(x)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-xt}}{1-e^{-t}}\right)dt$$

and

$$\psi'(x)=\int_0^\infty\frac{te^{-xt}}{1-e^{-t}}dt=\mathscr{L}\left[\frac{t}{1-e^{-t}}\right],$$

where Laplace transform of a function $f(t)$ is denoted by $\mathscr{L}[f]$, we get

$$\begin{aligned}h_1(x)&=\int_0^\infty\frac{te^{-(x+a)t}}{1-e^{-t}}dt-\int_0^\infty\frac{te^{-(x+1-a)t}}{1-e^{-t}}dt\\&=\int_0^\infty\frac{t(e^{-at}-e^{-(1-a)t})}{1-e^{-t}}e^{-xt}dt\\&=\mathscr{L}\left[\frac{t(e^{-at}-e^{-(1-a)t})}{1-e^{-t}}\right],\end{aligned}$$

$$\begin{aligned} h_2(x) &= -2\int_0^\infty \frac{e^{-(x+1/2)t}}{1-e^{-t}}dt + \int_0^\infty \frac{e^{-(x+1-a)t}+e^{-(x+a)t}}{1-e^{-t}}dt \\ &= \int_0^\infty \frac{-2e^{-t/2}+e^{(a-1)t}+e^{-at}}{1-e^{-t}}e^{-xt}dt \\ &= \mathscr{L}\left[\frac{-2e^{-t/2}+e^{(a-1)t}+e^{-at}}{1-e^{-t}}\right], \end{aligned}$$

and thereby

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{\int_0^\infty h_3(t)e^{-xt}dt}{\int_0^\infty h_4(t)e^{-xt}dt} = \frac{\mathscr{L}[h_3(t)]}{\mathscr{L}[h_4(t)]}. \tag{17}$$

Here

$$h_3(t) = \frac{t(e^{-at}-e^{-(1-a)t})}{1-e^{-t}}, \quad h_4(t) = \frac{-2e^{-t/2}+e^{(a-1)t}+e^{-at}}{1-e^{-t}}. \tag{18}$$

are both real functions defined on $(0,\infty)$.

Noting that

$$\begin{aligned} \frac{h_3(t)}{h_4(t)} &= \frac{t[e^{(1-a)t}-e^{at}]}{e^{(1-a)t}+e^{at}-2e^{t/2}} = \frac{t[e^{(1/2-a)t}-e^{-(1/2-a)t}]}{e^{(1/2-a)t}+e^{-(1/2-a)t}-2} \\ &= \frac{2t\sinh[(1/2-a)t]}{4\sinh^2[(1/2-a)t/2]} = \frac{t\cosh[(1/2-a)t/2]}{\sinh[(1/2-a)t/2]} \\ &= \frac{2}{(1/2-a)}\frac{[(1/2-a)t/2]}{\tanh[(1/2-a)t/2]}, \end{aligned} \tag{19}$$

and $h_4(t) > 0$ for all $t \in (0,\infty)$ and $a \in (0,1/2)$.

Therefore, the monotonicity of h directly follows from (17)-(19) and Lemma 5 together with the fact that the function $x \mapsto \tanh(x)/x$ is strictly decreasing from $(0,\infty)$ onto $(0,1)$. □

Lemma 9. *For $a \in (0,1/2)$, let*

$$q(x) = \frac{\psi(x+a)-\psi(x+1-a)-\psi(a)+\psi(1-a)}{\Gamma(x+1/2)^2/[\Gamma(x+1-a)\Gamma(x+a)\sin(\pi a)]-1}, \quad x \in (0,\infty).$$

Then q is positive and strictly decreasing on $(0,\infty)$.

Proof. Let

$$q_1(x) = \psi(x+a)-\psi(x+1-a)-\psi(a)+\psi(1-a),$$

$$q_2(x) = \frac{\Gamma(x+1/2)^2}{\Gamma(x+1-a)\Gamma(x+a)\sin(\pi a)} - 1$$

and $q_3(x) = q_2(x)+1$, then $q_1(0^+) = q_2(0^+) = 0$ and

$$\frac{q_1'(x)}{q_2'(x)} = \frac{1}{q_3(x)} \cdot \frac{\psi'(x+a)-\psi'(x+1-a)}{2\psi(x+1/2)-\psi(1-a+x)-\psi(a+x)}.$$

It was proved in [**30**, Theorem 2.6] that the function q_3 is positive and strictly increasing on $(0,\infty)$. In combination with Lemma 8, we obtain that the ratio function q_1'/q_2' is a product of two positive and strictly decreasing functions, so that q is also strictly decreasing on $(0,\infty)$ by application of Lemma 3. Moreover, for all $x\in(0,\infty)$, $q_2(x)=q_3(x)-1>q_3(0^+)-1=0$, and $q_1(x)>q_1(0^+)=0$ since $q_1'(x)=\psi'(x+a)-\psi'(x+1-a)>0$ for each $a\in(0,1/2)$. □

Letting $x=n\in\mathbb{N}$ in $q(x)$ defined in Lemma 9, we get a positive decreasing sequence.

Corollary 10. *Let $n\in\mathbb{N}$, $a\in(0,1/2)$ and c_n, d_n be as in (9), the positive sequence*

$$Q_n=\frac{\psi(n+a)-\psi(n+1-a)-\psi(a)+\psi(1-a)}{d_n/c_n-1}$$

is strictly decreasing in n.

3. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. First of all, Corollary 7.3 in [**3**] gives

$$\lim_{a\to 0} g(a)\big|_{\lambda>0}=\lim_{a\to 0}\frac{\mathscr{K}(r)-\mathscr{K}_a(r)}{(1-2a)^{\lambda}}=\mathscr{K}(r)-\frac{\pi}{2}. \tag{20}$$

By (1), (2) and Lemma 7,

$$\begin{aligned}\lim_{a\to 1/2} g(a)\big|_{\lambda=2}&=\lim_{a\to 1/2}\frac{\mathscr{K}(r)-\mathscr{K}_a(r)}{(1-2a)^2}=\frac{\pi}{2}\lim_{a\to 1/2}\sum_{n=0}^{\infty}\frac{d_n-c_n}{(1-2a)^2}\frac{r^{2n}}{(n!)^2}\\&=\frac{\pi}{2}\sum_{n=0}^{\infty}\lim_{a\to 1/2}\frac{d_n-c_n}{(1-2a)^2}\frac{r^{2n}}{(n!)^2}\\&=\frac{\pi}{8}\sum_{n=0}^{\infty}d_n\left[\frac{\pi^2}{2}-\psi'\left(n+1/2\right)\right]\frac{r^{2n}}{(n!)^2}\\&=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{D_n}{(n!)^2}r^{2n}\\&=D(r),\end{aligned} \tag{21}$$

and thus

$$\lim_{a\to 1/2} g(a)\big|_{\lambda=\frac{\pi}{2}}=\lim_{a\to 1/2}\frac{\mathscr{K}(r)-\mathscr{K}_a(r)}{(1-2a)^{\pi/2}}=\lim_{a\to 1/2}\frac{\mathscr{K}(r)-\mathscr{K}_a(r)}{(1-2a)^2}(1-2a)^{2-\pi/2}=0. \tag{22}$$

Next, logarithmic differentiation of g yields

$$\frac{(1-2a)}{2}\frac{g'(a)}{g(a)} = \frac{1-2a}{2}F(a,r)+\lambda, \tag{23}$$

where

$$\begin{aligned} F(a,r) =& \frac{1}{\mathscr{K}(r)-\mathscr{K}_a(r)}\frac{\partial[\mathscr{K}(r)-\mathscr{K}_a(r)]}{\partial a} = -\frac{1}{\mathscr{K}(r)-\mathscr{K}_a(r)}\frac{\partial \mathscr{K}_a(r)}{\partial a} \\ =& -\left(\frac{\partial}{\partial a}\sum_{n=0}^{\infty}\frac{c_n}{(n!)^2}r^{2n}\right)\left(\sum_{n=0}^{\infty}\frac{d_n-c_n}{(n!)^2}r^{2n}\right)^{-1} \\ =& -\left(\sum_{n=1}^{\infty}\frac{\partial c_n}{\partial a}\frac{1}{(n!)^2}r^{2n}\right)\left(\sum_{n=1}^{\infty}\frac{d_n-c_n}{(n!)^2}r^{2n}\right)^{-1} \\ =& -\left(\sum_{n=0}^{\infty}A_n r^{2n}\right)\left(\sum_{n=0}^{\infty}B_n r^{2n}\right)^{-1}, \end{aligned} \tag{24}$$

$$A_n = \frac{\partial c_{n+1}}{\partial a}\frac{1}{[(n+1)!]^2}, \quad B_n = \frac{d_{n+1}-c_{n+1}}{[(n+1)!]^2}. \tag{25}$$

Lemma 9 and Corollary 10 imply that $B_n > 0$ for $n \in \mathbb{N}_0$, and

$$\begin{aligned} \frac{A_n}{B_n} &= \frac{\partial c_{n+1}}{\partial a}\left(\frac{1}{d_{n+1}-c_{n+1}}\right) \\ &= \frac{[\psi(n+1+a)-\psi(n+2-a)-\psi(a)+\psi(1-a)]c_{n+1}}{d_{n+1}-c_{n+1}} = Q_{n+1} \end{aligned}$$

is positive and strictly decreasing in $n \in \mathbb{N}_0$, so that $F(a,r)$ is strictly increasing in $r \in (0,1)$ by (24), (25) and Lemma 4.

Clearly, $F(a,0^+) = -Q_1 = 4/(2a-1)$. For $a \in (0,1/2)$ and $r \in (0,1)$, if we set

$$F_2(a,r) = \int_0^{\pi/2}(\tan x)^{1-2a}(1-r^2\sin^2 x)^{-a}dx,$$

$$F_3(a,x) = (\sin x)^{1-2a}(\cos x)^{-1}\log(\sin x),$$

$$F_4(a,x) = (\tan x)^{1-2a}(1-r^2\sin^2 x)^{-a}\left[2\log(\tan x)+\log\left(1-r^2\sin^2 x\right)\right].$$

Then $\mathscr{K}_a(r) = F_2(a,r)\sin(\pi a)$ by (3), so that

(26)
$$\begin{aligned}\lim_{r\to 1} F(a,r) &= \lim_{r\to 1}\frac{1}{\mathscr{K}_a(r)-\mathscr{K}(r)}\left[\pi\cos(\pi a)F_2(a,r)+\lim_{r\to 1}\frac{\partial F_2}{\partial a}\sin(\pi a)\right]\\ &= \lim_{r\to 1}\frac{\pi\cos(\pi a)\mathscr{K}_a(r)}{[\mathscr{K}_a(r)-\mathscr{K}(r)]\sin(\pi a)}+\lim_{r\to 1}\frac{\sin(\pi a)}{\mathscr{K}_a(r)-\mathscr{K}(r)}\frac{\partial F_2}{\partial a}\\ &= \lim_{r\to 1}\frac{\pi\cos(\pi a)}{[1-\mathscr{K}(r)/\mathscr{K}_a(r)]\sin(\pi a)}+\lim_{r\to 1}\frac{1}{\mathscr{K}(r)}\frac{\sin(\pi a)}{\mathscr{K}_a(r)/\mathscr{K}(r)-1}\frac{\partial F_2}{\partial a}\\ &= \frac{\pi\cos(\pi a)}{\sin(\pi a)-1}+\frac{\sin(\pi a)}{\sin(\pi a)-1}\lim_{r\to 1}\frac{1}{\mathscr{K}(r)}\frac{\partial F_2}{\partial a},\end{aligned}$$

(27) $$\frac{\partial F_2}{\partial a} = -\int_0^{\pi/2} F_4(a,x)dx, \quad \lim_{r\to 1}\frac{\partial F_2}{\partial a} = -2\int_0^{\pi/2} F_3(a,x)dx.$$

It is well known that for $n\in\mathbb{N}$ (cf. [**21**]),

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt = -\int_0^1 \frac{t^{x-1}}{1-t}(\log t)^n dt.$$

Hence by (27) and by using the substitution $t=\sin^2 x$, we have

$$\begin{aligned}\lim_{r\to 1}\frac{\partial F_2}{\partial a} &= -2\int_0^{\pi/2}\frac{(\sin x)^{1-2a}\log(\sin x)}{1-\sin^2 x}d(\sin x)\\ &= -\frac{1}{2}\int_0^{\pi/2}\frac{(\sin x)^{-2a}\log(\sin^2 x)}{1-\sin^2 x}d(\sin^2 x)\\ &= -\frac{1}{2}\int_0^1 \frac{t^{(1-a)-1}\log t}{1-t}dt = \frac{1}{2}\psi'(1-a).\end{aligned}$$

Combining with (26), $F(a,1) = \pi\cos(\pi a)/[\sin(\pi a)-1]$, so that $F(a,r)$ is strictly increasing in r from $(0,1)$ onto $(4/(2a-1), \pi\cos(\pi a)/[\sin(\pi a)-1])$ for each $a\in(0,1/2)$. Hence by (23) and Lemma 6, we conclude that

$$g'(a)\ge 0 \Longleftrightarrow \lambda \ge \sup_{a\in(0,1/2),r\in(0,1)}\frac{2a-1}{2}F(a,r) = \sup_{a\in(0,1/2)}\frac{2a-1}{2}F(a,0)=2,$$

$$g'(a)\le 0 \Longleftrightarrow \lambda \le \inf_{a\in(0,1/2),r\in(0,1)}\frac{2a-1}{2}F(a,r) = \inf_{a\in(0,1/2)}\frac{\pi(1-2a)\cos(\pi a)}{2[1-\sin(\pi a)]}=\frac{\pi}{2}$$

for all $a\in(0,1/2)$ and $r\in(0,1)$.

Finally, it follows from (20), (21) and (22) together with the monotonicity properties of the functions $g(a)$ with $\lambda=\pi/2$ and $\lambda=2$ that inequalities

$$\begin{aligned}&(1-2a)^2\left[\mathscr{K}(r)-\frac{\pi}{2}\right]\le \mathscr{K}(r)-\mathscr{K}_a(r)\\ &\le \min\left\{(1-2a)^{\pi/2}\left[\mathscr{K}(r)-\frac{\pi}{2}\right], (1-2a)^2 D(r)\right\}.\end{aligned}$$

hold for all $a \in (0,1/2)$ and $r \in (0,1)$, which is equivalent to inequality (15). The remaining conclusions are clear. $\square$

Proof of Theorem 2. If $n=0$, then Theorem 2 can be found in [**16**, Theorem 3.6]. Following we suppose $n \ge 1$. By (1) and (2),

$$\mathscr{K}(r) - P_{2,n}(r) = \frac{\pi}{2}\sum_{k=n+1}^{\infty} b_k r^{2k} > 0,$$

$$f(r) = \frac{\sum_{k=n+1}^{\infty} a_k r^{2k}}{\sum_{k=n+1}^{\infty} b_k r^{2k}} = \frac{\sum_{k=0}^{\infty} a_{k+n+1} r^{2k}}{\sum_{k=0}^{\infty} b_{k+n+1} r^{2k}}.$$

Let

$$E_k = \frac{a_{k+n+1}}{b_{k+n+1}} = \frac{\Gamma(k+n+1+a)\Gamma(k+n+2-a)\sin(\pi a)}{\Gamma(k+n+3/2)^2},$$

Then

$$E_{k+1} - E_k = -\frac{\Gamma(k+n+1+a)\Gamma(k+n+2-a)(1/2-a)^2\sin(\pi a)}{\Gamma(k+n+5/2)^2} < 0,$$

so that the sequence $\{E_k\}$ is strictly decreasing in $k \in \mathbb{N}_0$ for each fixed $n \in \mathbb{N}$. It shows that the function f is strictly decreasing on $(0,1)$ by Lemma 4. Moreover,

$$\lim_{r\to 0^+} f(r) = E_0 = \frac{\Gamma(n+1+a)\Gamma(n+2-a)\sin(\pi a)}{\Gamma(3/2+n)^2}, \tag{28}$$

$$\begin{aligned}\lim_{r\to 1^-} f(r) &= \lim_{r\to 1^-} \frac{\mathscr{K}_a(r)/\mathscr{K}(r) - \frac{\pi}{2}\left(\sum_{k=0}^{n} a_k r^{2k}\right)/\mathscr{K}(r)}{1 - \frac{\pi}{2}\left(\sum_{k=0}^{n} b_k r^{2k}\right)/\mathscr{K}(r)} \\ &= \lim_{r\to 1^-} \frac{\mathscr{K}_a(r)}{\mathscr{K}(r)} = \sin(\pi a).\end{aligned} \tag{29}$$

Therefore, Theorem 2 directly follows from (28) and (29) together with the monotonicity of f. $\square$

Remark 11. Rewrite inequality (15) as

$$\begin{aligned}\max\left\{\mathscr{K}(r) - (1-2a)^2 D(r),\ \frac{\pi}{2} + \left[\mathscr{K}(r) - \frac{\pi}{2}\right]\left(1-(1-2a)^{\pi/2}\right)\right\} \\ \le \mathscr{K}_a(r) \le \frac{\pi}{2} + 4a(1-a)\left[\mathscr{K}(r) - \frac{\pi}{2}\right],\end{aligned}$$

we clearly see that the upper bound of $\mathscr{K}_a(r)$ in inequality (15) is equal to that of inequality (7). But it is our view that Theorem 1 in this paper adds that the upper bound is optimal in some sense. On the other hand, computational and numerical experiments show that the lower bound of $\mathscr{K}_a(r)$ in (15) is not directly comparable to any one of (6) and (7) for $(a,r) \in (0,1/2)\times(0,1)$.

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