Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **16** (2022), 400–426.
https://doi.org/10.2298/AADM210518032Z

SOME GENERAL WILKER–HUYGENS INEQUALITIES

*Tie-Hong Zhao and Yu-Ming Chu**

In this paper, we provide a systematic way to study on some general Wilker–Huygens type inequalities for the trigonometric and hyperbolic functions, lemniscate and hyperbolic lemniscate functions, and their corresponding inverse functions. Our results are some extensions and refinements of the recently published results in [A. Mhanna, On a general Huygens-Wilker inequality, Appl. Math. E.-Notes, **20** (2020), 79-81; MR4076436], and improve many previous results involving Wilker–Huygens type inequalities.

1. INTRODUCTION

As is known, the following inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1}$$

($0 < |x| < \frac{\pi}{2}$) was first presented by Wilker [**32**]. It has attracted the attention of several researchers; see [**4, 12, 16, 33, 44**] and the references therein. Another inequality which is of interest to us is due to Huygens [**17**]:

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \tag{2}$$

($0 < |x| < \frac{\pi}{2}$). The hyperbolic counterpart of Wilker and Huygens inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2, \qquad 2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3 \tag{3}$$

*Corresponding author. Yu-Ming Chu.
2020 Mathematics Subject Classification. 26D07, 33E20.
Keywords and Phrases. Trigonometric functions, Arc lemniscate functions, Wilker–Huygens inequality, Monotonicity.

($x \neq 0$) had been respectively established by Zhu [**45**] and Neuman-Sándor [**22**]. Inequalities (1)-(3) had been studied extensively, we recommend [**5, 6, 7, 8, 9, 11, 13, 14, 20, 25, 26, 27, 28, 31, 34, 35, 36, 37, 42, 43, 46**] to see more generalizations and refinements.

Further, the Wilker–Huygens type inequalities had been extended to the (arc) lemniscate functions. Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine functions are defined, respectively, as follows:

$$\text{(4)} \quad \operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}} \quad (|x| \leq 1) \quad \text{and} \quad \operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1+t^4}} \quad (x \in \mathbb{R}),$$

see [**2**, p. 259] and [**3**, (2.5)-(2.6)]. An alternative pair of the arc lemniscate functions, following [**19**, Proposition 3.1], Gauss' arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions are respectively defined by

(5)

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right) \ (x \in \mathbb{R}) \text{ and } \operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right) \ (|x| < 1).$$

Recall that the *first lemniscate constant* (c.f. [**2**, Theorem 1.7] and [**23**, 19.20.2]) is given by

$$\omega = \operatorname{arcsl}(1) = \frac{1}{\sqrt{2}}\mathcal{K}(1/\sqrt{2}) = \frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{2\pi}} = 1.31103\cdots,$$

where

$$\mathcal{K}(r) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} \quad (0 < r < 1) \quad \text{and} \quad \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

are the complete elliptic integral of the first kind [**10, 29, 39, 40**] and the classical Euler gamma function [**24**], respectively. For later use, some other constants are presented as follows,

$$\kappa = \operatorname{arcslh}(+\infty) = \sqrt{2}\omega = 1.85407\cdots, \quad \sigma = \operatorname{arcslh}(1) = \omega/\sqrt{2} = 0.92703\cdots,$$
$$\tau = \operatorname{arctl}(1) = \operatorname{arcsl}(1/\sqrt[4]{2}) = 0.89558\cdots.$$

Lemniscate functions are the inverse functions of Gauss's arc lemniscate functions given in (4) and (5), which are denoted, respectively, by $\operatorname{sl}, \operatorname{slh}, \operatorname{tl}$ and tlh.

By (4) and (5), the derivative formula of an inverse function allows us to give the following formulas

$$\text{(6)} \qquad \frac{d\operatorname{sl}x}{dx} = \sqrt{1-\operatorname{sl}^4 x}, \qquad \frac{d\operatorname{tl}x}{dx} = (1+\operatorname{tl}^4 x)^{\frac{3}{4}}, \qquad |x| < \omega,$$

$$\text{(7)} \qquad \frac{d\operatorname{slh}x}{dx} = \sqrt{1+\operatorname{slh}^4 x}, \qquad \frac{d\operatorname{tlh}x}{dx} = (1-\operatorname{tlh}^4 x)^{\frac{3}{4}}, \qquad |x| < \kappa.$$

The lemniscate function counterparts of (5) (see [**21**, Proposition 2.1]) have the following relations

$$\operatorname{sl} x = \frac{\operatorname{tl} x}{\sqrt[4]{1+\operatorname{tl}^4 x}} \quad \text{and} \quad \operatorname{slh} x = \frac{\operatorname{tlh} x}{\sqrt[4]{1-\operatorname{tlh}^4 x}}. \tag{8}$$

Recently, Chen [**7**] proved that the inequalities

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^2 + \frac{\operatorname{arctl} x}{x} > 2, \quad 2\left(\frac{\operatorname{arcsl} x}{x}\right) + \frac{\operatorname{arctl} x}{x} > 3 \tag{9}$$

and

$$\frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x}\right)^2 > 2, \quad \frac{\operatorname{arcslh} x}{x} + 2\left(\frac{\operatorname{arctlh} x}{x}\right) > 3 \tag{10}$$

hold for $0 < |x| < 1$. A remarkable inequality [**19**, 2.16] for the lemniscate mean leads to more general Wilker–Huygens type inequalities of lemniscate (hyperbolic) functions (see [**21**, Theorems 3.1 and 3.3]), which are stated as follows:

Let $\lambda, \mu > 0$. Then the inequality

$$\lambda\left(\frac{\operatorname{sl} x}{x}\right)^{2p} + \mu\left(\frac{\operatorname{tl} x}{x}\right)^{2q} > \lambda + \mu \tag{11}$$

holds for $0 < |x| < \omega$ either if $q > 0$ and $2\lambda p \le 3\mu q$ or if $p \le q \le -1$ and $2\lambda \ge 3\mu$; the inequality

$$\lambda\left(\frac{\operatorname{slh} x}{x}\right)^{2p^*} + \mu\left(\frac{\operatorname{tlh} x}{x}\right)^{2q^*} > \lambda + \mu \tag{12}$$

holds for $0 < |x| < \kappa$ either if $p^* > 0$ and $2\lambda p^* \ge 3\mu q^*$ or if $q^* \le p^* \le -1$ and $2\lambda \le 3\mu$.

The eye-catching similarity between (1)-(3) and (9)-(12) makes sense to consider the following general question:

Question. *Under what conditions on a, b, m and n, the following general Wilker-Huygens type inequality*

$$a\left[\frac{f(x)}{x}\right]^m + b\left[\frac{g(x)}{x}\right]^n > a + b \tag{13}$$

holds for $x \in (0, r)$? Here two functions $f, g : (-r, r) \mapsto \mathbb{R}$ are differentiable, increasing and real odd functions satisfying $f(0) = g(0) = 0$ and

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{g(x)}{x} = 1 \tag{14}$$

with $r > 0$ (or $r = \infty$).

The inequality (13) will be called *a general Wilker–Huygens type inequality of* $f(x)$ *and* $g(x)$. Throughout this paper, we denote

$$\Phi(x) =: \Phi(x; a, b, m, n) = a\left[\frac{f(x)}{x}\right]^m + b\left[\frac{g(x)}{x}\right]^n$$

for $x \in (0, r)$, which simplify the inequality (13) as $\Phi(x) > a + b$ for $x \in (0, r)$. By the virtue of (14), our main focus is on the inequality $\Phi(x) > a + b$ in which $\Phi(x)$ is strictly increasing on $(0, r)$ under certain conditions on the parameters a, b, m and n. Further, the inverse inequality $\Phi(x) < a + b$ can be also derived. Our main results (Theorems 1 and 3) in this paper will improve the parameters' conditions under which the inequalities (11) and (12) are valid, and (13) holds for the case $(f, g) = (\sin, \tan)$ which had been treated in [**18**].

For the reader's convenience, in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

2. MAIN RESULTS

In this section, sometimes we will omit the variable x to represent a function if no risk for confusion, such as the symbols f and g will stand for $f(x)$ and $g(x)$ respectively. Moreover, the symbol $\nearrow$ ($\searrow$) represents that a function is increasing (decreasing).

For the sake of presentation, let us introduce the auxiliary function

$$\mathscr{L}_{f,g}(x) = \frac{f(xg' - g)}{g(f - xf')}. \tag{15}$$

In what follows, we denote $\mathcal{S}_1$ by the sets of pairs of (inverse) trigonometric, hyperbolic trigonometric functions and $\mathcal{S}_2$ by the sets of pairs of (arc) lemniscate, hyperbolic lemniscate functions, specifically,

$$\mathcal{S}_1 = \{(\sin, \tan), (\sinh, \tanh), (\arcsin, \arctan), (\operatorname{arcsinh}, \operatorname{arctanh})\},$$
$$\mathcal{S}_2 = \{(\mathrm{sl}, \mathrm{tl}), (\mathrm{slh}, \mathrm{tlh}), (\mathrm{arcsl}, \mathrm{arctl}), (\mathrm{arcslh}, \mathrm{arctlh})\}.$$

The monotonicity of f/x and g/x can be easily proved by Lemma 6 for $(f, g) \in \mathcal{S}_1$ and had been treated in [**30**, Lemma 3.1] for $(f, g) \in \mathcal{S}_2$. In order to present more clearly, we summarize all the monotonicity of f/x and g/x in Tabel 1 while the monotonicity of $\mathscr{L}_{f,g}(x)$ can be found in Propositions 10-17.

Tabel 1 clearly demonstrates that the monotonicity of f/x is opposite to g/x. This makes us to easily know that the inequality (13) or its inverse holds when $m \cdot n < 0$. In this paper, we mainly study the inequality (13) or its inverse inequality for the case of $m \cdot n > 0$.

We are now in a position to state our main theorems.

Table 1: A visual table for the monotonicity of $f/x, g/x$ and $\mathscr{L}_{f,g}$.

Functions \ (f,g)	$(\sin, \tan)$	$(\sinh, \tanh)$	$(\arcsin, \arctan)$	$(\text{arcsinh}, \text{arctanh})$
f/x	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
g/x	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
$\mathscr{L}_{f,g}$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
Functions \ (f,g)	(sl, tl)	(slh, tlh)	$(\text{arcsl}, \text{arctl})$	$(\text{arcslh}, \text{arctlh})$
f/x	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
g/x	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
$\mathscr{L}_{f,g}$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Theorem 1. *Let $a, b \geq 0$, $m \cdot n > 0$ and $(f,g) \in \mathcal{S}_1 \cup \mathcal{S}_2$. Then the general Wilker–Huygens type inequality of f and g*

$$\Phi(x) =: a\left[\frac{f(x)}{x}\right]^m + b\left[\frac{g(x)}{x}\right]^n > a+b$$

holds for $x \in (0, r)$, where $r > 0$ $(r = \infty)$ depends on the domains of f and g, if (m, n, a, b) satisfies one of the following conditions

(1) $\min\{m,n\} > 0$ *and*

$$\begin{cases} am \leq \mathscr{L}_{f,g}(0^+)bn, & if\ f/x\ \searrow, \\ am \geq \mathscr{L}_{f,g}(0^+)bn, & if\ f/x\ \nearrow. \end{cases}$$

(2) $\max\{m,n\} < 0$ *and*

(i) $am \leq 2bn$, $m \leq -(3 - \frac{\pi^2}{4})$, $n \leq -1$ *for* $(f,g) = (\sin, \tan)$;

(ii) $am \geq 2bn$, $\max\{m,n\} \leq -\frac{4}{5}$ *for* $(f,g) = (\sinh, \tanh)$;

(iii) $am > 2bn$, $a\left(\frac{\pi}{2}\right)^m + b\left(\frac{\pi}{4}\right)^n \geq a+b$, $\min\{m,n\} \geq -\frac{14}{5}$ *for* $(f,g) = (\arcsin, \arctan)$;

(iv) $am < 2bn$, $a(\text{arcsinh}\,1)^m \geq a+b$, $\min\{m,n\} \geq -\frac{9}{4}$ *for* $(f,g) = (\text{arcsinh}, \text{arctanh})$;

(v) $2am \leq 3bn$, $\max\{m,n\} \leq -2$ *for* $(f,g) = (\text{sl}, \text{tl})$;

(vi) $2am \geq 3bn$, $\max\{m,n\} \leq -2$ *for* $(f,g) = (\text{slh}, \text{tlh})$;

(vii) $2am > 3bn$, $a\omega^m + b\tau^n \geq a+b$, $\min\{m,n\} \geq -4$ *for* $(f,g) = (\text{arcsl}, \text{arctl})$;

(viii) $2am < 3bn$, $a\sigma^m + b\kappa^n \geq a+b$, $\min\{m,n\} \geq -4$ *for* $(f,g) = (\text{arcslh}, \text{arctlh})$;

Proof. Differentiating $\Phi(x)$ gives

$$\begin{aligned}\Phi'(x) &= am\left(\frac{f}{x}\right)^{m-1}\cdot\frac{xf'-f}{x^2}+bn\left(\frac{g}{x}\right)^{n-1}\cdot\frac{xg'-g}{x^2}\\ &=\left(\frac{f}{x}\right)^{m-1}\left(\frac{f}{x}\right)' bn\left[\frac{am}{bn}-\left(\frac{x}{f}\right)^m\left(\frac{g}{x}\right)^n\mathscr{L}_{f,g}(x)\right],\end{aligned} \tag{16}$$

where $\mathscr{L}_{f,g}(x)$ is defined as in (15).

We divide the proof into two cases.

Case 1: $\min\{m,n\} > 0$. If $f(x)/x$ is decreasing, then it follows from Table 1 that $g(x)/x$ and $\mathscr{L}_{f,g}(x)$ are increasing, so $(x/f)^m(g/x)^n\mathscr{L}_{f,g}(x)$ is increasing. Combining this with (14) and $am \le \mathscr{L}_{f,g}(0^+)bn$, we clearly see that

$$\frac{am}{bn}-\left(\frac{x}{f}\right)^m\left(\frac{g}{x}\right)^n\mathscr{L}_{f,g}(x)\le 0 \tag{17}$$

for $x\in(0,r)$. According to (16) and (17) with $\Phi(0^+)=a+b$, the desired result is completed. Similar for the case that $f(x)/x$ is increasing.

Observe that by the propositions in Section 3, $\mathscr{L}_{f,g}(0^+)=2$ for $(f,g)\in\mathcal{S}_1$ and $\mathscr{L}_{f,g}(0^+)=\frac{3}{2}$ for $(f,g)\in\mathcal{S}_2$.

Case 2: $\max\{m,n\}\le 0$.

- For $(f,g)=(\sin,\tan)$, we clearly see from Table 1 that $f(x)/x$ is decreasing and $g(x)/x$ is increasing. Combining this with Proposition 10, it follows that

$$\left(\frac{x}{f}\right)^m\left(\frac{g}{x}\right)^n\mathscr{L}_{f,g}(x)=\left(\frac{x}{f}\right)^{m+3-\frac{\pi^2}{4}}\left(\frac{g}{x}\right)^{n+1}\left(\frac{f}{x}\right)^{3-\frac{\pi^2}{4}}\left(\frac{x}{g}\right)\mathscr{L}_{f,g}(x)$$

is strictly decreasing if $m\le -(3-\frac{\pi^2}{4})$ and $n\le -1$. This in conjunction with $(f/x)'<0$, $n<0$ and (16) makes us to obtain $\Phi'(x)>0$ provided that

$$\frac{am}{bn}\ge\lim_{x\to 0^+}\left[\left(\frac{x}{f}\right)^m\left(\frac{g}{x}\right)^n\mathscr{L}_{f,g}(x)\right]=2,$$

namely, $am\le 2bn$. This completes the proof with $\Phi(0^+)=a+b$.

The similar analysis as above can be applied to the case for $(\sinh,\tanh)$, $(\mathrm{sl},\mathrm{tl})$ and $(\mathrm{slh},\mathrm{tlh})$. So the desired results of Theorem 1 are obtained from Propositions 11, 14 and 15.

- For $(f,g)=(\arcsin,\arctan)$, Table 1 enables us to know that $f(x)/x$ is increasing and $g(x)/x$ is decreasing. According to this with $\min\{m,n\}\ge -\frac{14}{5}$, it can be easily seen from Proposition 12 that

$$\frac{am}{bn}-\left(\frac{x}{f}\right)^m\left(\frac{g}{x}\right)^n\mathscr{L}_{f,g}(x)=\frac{am}{bn}-\left(\frac{x}{f}\right)^{m+\frac{14}{5}}\left(\frac{g}{x}\right)^{n+\frac{14}{5}}\left(\frac{f}{g}\right)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)$$

is strictly increasing. That is to say, there exists a number $\xi \in (0,1)$ such that $(am)/(bn) - (x/f)^m(g/x)^n\mathscr{L}_{f,g}(x) < 0$ for $x \in (0,\xi)$ and $(am)/(bn) - (x/f)^m(g/x)^n\mathscr{L}_{f,g}(x) > 0$ for $x \in (\xi, 1)$ provided that

$$\frac{am}{bn} - \mathscr{L}_{f,g}(0^+) = \frac{am}{bn} - 2 < 0.$$

This in conjunction with $(f/x)' > 0$, $n < 0$ and (16) implies that $\Phi(x)$ is strictly increasing on $(0,\xi)$ and strictly decreasing on $(\xi, 1)$. Therefore, $\Phi(x) > a + b$ for $x \in (0,1)$ follows from the piecewise monotonicity with its boundary limit values $\Phi(0^+) = a + b$ and $\Phi(1^-) = a(\frac{\pi}{2})^m + b(\frac{\pi}{4})^n$.

The inequality $\Phi(x) > a + b$ can be obtained from the similar analysis as above and Propositions 13, 16, 17 for the case $(f, g) = (\text{arcsinh}, \text{arctanh})$, $(\text{arcsl}, \text{arctl})$ and $(\text{arcslh}, \text{arctlh})$.

□

Remark 2. Theorem 1(2)(v) and (vi) improve the parameters' conditions satisfying (11) and (12). In our parameters, $a = \lambda$, $b = \mu$, $m = 2p$ and $n = 2q$. It can be easily seen that $p \le q \le -1$, namely $m \le n \le -2$ gives rise to $0 < n/m \le 1$. This together with $2\lambda \ge 3\mu$ yields $(2a)/(3b) \ge 1 \ge n/m$, equivalently, $2am \le 3bn$. In others words, our parameters' conditions are better than those given in (11). On the other hand, we only require the parameters m, n as $\max\{m, n\} \le -2$ containing the case $q < p \le -1$ which is not mentioned in (11). Similar analysis holds for (12).

Theorem 3. *Let $a, b \ge 0$, $m \cdot n > 0$, $(f, g) \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $\Phi(x)$ be defined as in Theorem 1. Then the inverse general Wilker–Huygens type inequality of f and g*

$$\Phi(x) < a + b$$

holds for $x \in (0, r)$, where $r > 0$ ($r = \infty$) depends on the domains of f and g, if (m, n, a, b) satisfies one of the following conditions

(i) $-\frac{\log(1+b/a)}{\log\frac{\pi}{2}} \le m \le -(3 - \frac{\pi^2}{4})$, $n \le -1$ *and* $am > 2bn$ *for* $(f, g) = (\sin, \tan)$*;*

(ii) $0 < m \le \frac{\log[1+\frac{b}{a}(1-(\frac{\pi}{4})^n)]}{\log\frac{\pi}{2}}$ *and* $am < 2bn$ *(or* $\min\{m, n\} \ge -\frac{15}{4}$ *and* $am \le 2bn < 0$*) for* $(f, g) = (\arcsin, \arctan)$*;*

(iii) $\min\{m, n\} \ge -\frac{9}{4}$ *and* $0 > am \ge 2bn$ *for* $(f, g) = (\text{arcsinh}, \text{arctanh})$*;*

(iv) $-\frac{\log(1+b/a)}{\log\omega} \le m \le -2$, $n \le -2$ *and* $2am > 3bn$ *for* $(f, g) = (\text{sl}, \text{tl})$*;*

(v) $-\frac{\log(1+a/b)}{\log\kappa} \le n \le -2$, $m \le -2$ *and* $2am < 3bn$ *for* $(f, g) = (\text{slh}, \text{tlh})$*;*

(vi) $0 < m \le \frac{\log[1+\frac{b}{a}(1-\tau^n)]}{\log\omega}$ *and* $2am < 3bn$ *(or* $\min\{m, n\} \ge -4$ *and* $2am \le 3bn < 0$*) for* $(f, g) = (\text{arcsl}, \text{arctl})$*;*

(vii) $0 < m \le \frac{\log[1+\frac{b}{a}(1-\kappa^n)]}{\log\sigma}$ *and* $2am > 3bn$ *(or* $\min\{m,n\} \ge -4$ *and* $0 > 2am \ge 3bn$*) for* $(f,g) = (\mathrm{arcslh}, \mathrm{arctlh})$.

Proof. For $(f,g) = (\sin, \tan)$, the inequality $\Phi(x) < a+b$ does not hold for $x \in (0, \frac{\pi}{2})$ when $\min\{m,n\} > 0$, since $\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{x} = \infty$. As in the proof of Case (2)(i) of Theorem 1, $(x/f)^m(g/x)^n\mathscr{L}_{f,g}(x)$ is strictly decreasing when $m \le -(3-\frac{\pi^2}{4})$ and $n \le -1$. This in conjunction with (16) implies that $\Phi(x)$ is strictly convex on $(0,\frac{\pi}{2})$ under stated condition $am > 2bn$. By the convexity and its boundary condition, $\Phi(x) < a+b$ holds for all $x \in (0,\frac{\pi}{2})$, where the boundary condition $\Phi(\frac{\pi}{2}^-) = a(\frac{2}{\pi})^m \le a+b$ is equivalent to $m \ge -\frac{\log(1+b/a)}{\log\frac{\pi}{2}}$. This completes the proof of Theorem 3(i).

The similar analysis as in the proof of Theorem 3(i) can be applied to the case for $(f,g) = (\mathrm{arcsinh}, \mathrm{arctanh})$, $(\mathrm{sl}, \mathrm{tl})$, $(\mathrm{slh}, \mathrm{tlh})$. Theorem 3$(iii)$, (iv) and (v) can be derived from Propositions 13, 14 and 15, respectively. A little difference between $(\sin, \tan)$ and $(\mathrm{arcsinh}, \mathrm{arctanh})$ is that $\Phi(x)$ is strictly decreasing under stated conditions for $(f,g) = (\mathrm{arcsinh}, \mathrm{arctanh})$. This gives $\Phi(x) < a+b$ provided only that $\Phi(0^+) = a+b$.

For $(f,g) = (\arcsin, \arctan)$, it follows from Table 1 that f/x is strictly increasing and $g/x, \mathscr{L}_{f,g}(x)$ are strictly decreasing. This in conjunction with (16) and $am < 2bn$ implies the convexity of $\Phi(x)$ when $\min\{m,n\} > 0$. In order to obtain the desired inequality, it requires to satisfy the boundary condition $\Phi(1^-) = a(\frac{\pi}{2})^m + b(\frac{\pi}{4})^n \le a+b$, which is equivalent to $m \le \frac{\log[1+\frac{b}{a}(1-(\frac{\pi}{4})^n)]}{\log\frac{\pi}{2}}$. When $\max\{m,n\} < 0$ and $\min\{m,n\} \ge -\frac{15}{4}$, by the proof of Case (2)(iii) of Theorem 1, $(am)/(bn) - (x/f)^m(g/x)^n\mathscr{L}_{f,g}(x)$ is strictly increasing. According to this with $am \le 2bn$, it can be easily seen that $(am)/(bn) - (x/f)^m(g/x)^n\mathscr{L}_{f,g}(x) > 0$ for $x \in (0,1)$. From (16), $(f/x)' > 0$ and $n < 0$, and hence $\Phi'(x) < 0$ for $x \in (0,1)$. This completes the proof of Theorem 3(ii) from $\Phi(0^+) = a+b$.

By the similar method as $(f,g) = (\arcsin, \arctan)$, Theorem 3$(vi)$ and (vii) can be obtained from Propositions 16 and 17. □

Remark 4. For $(f,g) = (\sinh, \tanh)$, it can be easily proved that

$$\lim_{x\to\infty} \frac{\sinh x}{x} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{\tanh x}{x} = 0.$$

This makes us to know that the inverse general Wilker–Huygens type inequality

$$a\left(\frac{\sinh x}{x}\right)^m + b\left(\frac{\tanh x}{x}\right)^n < a+b$$

cannot hold for $x \in (0,\infty)$ under the assumption $m \cdot n > 0$.

Remark 5. [**18**, Corollary 2] states that if $am \le 2bn$, $a\big((\frac{\pi}{2})^m - 1\big) \le b$, $m \cdot n > 0$ and $\min\{m,n\} < 0$ or $\min\{m,n\} \ge 1$, then the inverse general Wilker–Huygens type inequality of $\sin x$ and $\tan x$ holds for $x \in (0,\frac{\pi}{2})$.

It is worth noting that if $am = 2bn$ and $\min\{m,n\} \geq 1$, then we have $a\left[\left(\frac{\pi}{2}\right)^m - 1\right] > b$. More precisely,

$$\frac{m}{2\left[(\frac{\pi}{2})^m - 1\right]} \leq \frac{1}{2(\frac{\pi}{2} - 1)} = \frac{1}{\pi - 2} < 1 \leq n,$$

which gives

$$\left(\frac{\pi}{2}\right)^m - 1 > \frac{m}{2n} = \frac{b}{a}, \quad \text{or equivalently,} \quad a\left[\left(\frac{\pi}{2}\right)^m - 1\right] > b.$$

In other words, the parameters' condition given in [**18**, Corollary 2] is not correct. Clearly, their parameters m and n are opposite to ours. In our setting of $am = 2bn$ and $\max\{m,n\} \leq -1$, Theorem 1(2)(i) makes us to know $\Phi(x) > a + b$ for $x \in (0, \frac{\pi}{2})$.

In conclusion, Theorem 3(i) not only gives some improvements for the parameters' conditions satisfying $\Phi(x) < a + b$ but also makes some corrections.

3. TOOLS AND PROPOSITIONS

In this section we present some lemmas as tools and all the technical details which are used to prove our main results.

The following monotonicity rule comes from [**1**, Theorem 1.25].

Lemma 6. *Suppose $\varphi, \phi : (a,b) \to \mathbb{R}$ are differentiable with $\phi'(x) \neq 0$ such that $\varphi(a^+) = \phi(a^+) = 0$ or $\varphi(b^-) = \phi(b^-) = 0$. If φ'/ϕ' is (strictly) increasing (decreasing) on (a,b), then so is φ/ϕ .*

The following two lemmas offer a simple criterion to determine the sign of a class of special polynomial or series .

Lemma 7. ([**38**, Lemma 7]). *Let $n, m \in \mathbb{N} \cup \{0\}$ with $n > m$ and $P_n(x)$ be the polynomial of degree n defined by*

$$P_n(x) = \sum_{i=0}^{m} a_i x^i - \sum_{i=m+1}^{n} a_i x^i,$$

where $a_m, a_n > 0$ and $a_i \geq 0$ for $0 \leq i \leq n-1$ with $i \neq m$. Then there exist $x_0 \in (0,\infty)$ such that $P_n(x_0) = 0$ and $P_n(x) > 0$ for $x \in (0, x_0)$ and $P_n(x) < 0$ for $x \in (x_0, \infty)$.

Lemma 8. ([**41**, Lemma 2]). *Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$ and let*

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0, R)$ $(R > 0)$.

(i) If $S(R^-) \le 0$, then $S(t) < 0$ for all $t \in (0, R)$;

(ii) If $S(R^-) > 0$, then there is a unique $t_0 \in (0, R)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, R)$.

Lemma 9. *For $0 < x < 1$, we have the following inequalities*

$$p_1(x) < \arcsin x < p_2(x), \quad q_1(x) < \arctan x < q_2(x),$$

where

$$p_1(x) = \sum_{n=0}^{4} \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1}, \qquad p_2(x) = \sum_{n=0}^{3} \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1} + \frac{2x^9}{7},$$
$$q_1(x) = \sum_{n=0}^{5} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad q_2(x) = \sum_{n=0}^{4} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

Proof. For $0 < x < 1$, recall the series expansions of $\arcsin x$ and $\arctan x$ as

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1} \quad \text{and} \quad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

Elementary calculations lead to

$$p_1(x) - \arcsin x = -\sum_{n=5}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1} < 0,$$
$$\arcsin x - p_2(x) = -\frac{2059x^9}{8064} + \sum_{n=5}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1},$$
$$q_1(x) - \arctan x = -\sum_{k=3}^{\infty} \left(\frac{1}{4k+1} - \frac{x^2}{4k+3} \right) x^{4k+1} < -\sum_{k=3}^{\infty} \frac{2x^{4k+1}}{(4k+1)(4k+3)} < 0,$$
$$\arctan x - q_2(x) = -\sum_{k=3}^{\infty} \left(\frac{1}{4k-1} - \frac{x^2}{4k+1} \right) x^{4k-1} < -\sum_{k=3}^{\infty} \frac{2x^{4k-1}}{16k^2-1} < 0.$$

This completes the proof with Lemma 8 and $\arcsin 1 - p_2(1) < -0.001$. $\square$

For completeness, we still state the case for $(f, g) = (\sin, \tan)$ in the following proposition although it has been given in [**18**]. Further, Proposition 10(*ii*) also gives some refinements of Lemma 2 of [**18**].

Proposition 10. *Let $(f, g) = (\sin, \tan)$. Then the function*

(i) $\mathscr{L}_{f,g}(x)$ is strictly increasing from $(0, \frac{\pi}{2})$ onto $(2, \infty)$;

(ii) $(f/x)^s (x/g) \mathscr{L}_{f,g}(x)$ is strictly decreasing from $(0, \frac{\pi}{2})$ onto $((\frac{\pi}{2})^{2-s}, 2)$ if and only if $s \ge 3 - \frac{\pi^2}{4} = 0.53259 \cdots$.

Proof. (*i*) It has been proved in [**18**, Lemmas 2 and 3], by a matter of simple transformations, that $\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0, \frac{\pi}{2})$ and $(f/g)\mathscr{L}_{f,g}(x)$ is strictly decreasing from $(0, \frac{\pi}{2})$ onto $(2, \infty)$.

(*ii*) Differentiation gives

$$\left(\frac{x}{\sin x}\right)^s \frac{\sin^2 x}{x - \sin x \cos x}\Big[(f/x)^s (x/g)\mathscr{L}_{f,g}(x)\Big]' = 1 - s + G_1(x), \tag{18}$$

where

$$G_1(x) = \frac{x(2\sin^2 x - x^2 - x \sin x \cos x)\sin^2 x}{(x - \sin x \cos x)(\sin x - x\cos x)^2}.$$

We shall show the monotonicity of $G_1(x)$ on $(0, \frac{\pi}{2})$.

Differentiating $G_1(x)$ yields

$$G_1'(x) = \frac{\sin x}{16(\sin x - x\cos x)^3 (x - \sin x \cos x)^2}\widetilde{G}_1(x), \tag{19}$$

where

$$\begin{aligned}\widetilde{G}_1(x) = {} & -12x - 56x^3 + 32x^5 + 16x(1+2x^2)\cos(2x) - 4x(1-6x^2)\cos(4x) \\ & - (5 - 64x^2)\sin(2x) + 4(1 - 8x^2 + 2x^4)\sin(4x) - \sin(6x).\end{aligned}$$

By power series expansions, $\widetilde{G}_1(x)$ can be rewritten as

(20)

$$\begin{aligned}\widetilde{G}_1(x) = {} & -12x - 56x^3 + 32x^5 + 16x(1+2x^2)\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n} \\ & - 4x(1-6x^2)\sum_{n=0}^{\infty}\frac{(-1)^n 2^{4n}}{(2n)!}x^{2n} - (5-64x^2)\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n+1)!}x^{2n+1} \\ & + 4(1-8x^2+2x^4)\sum_{n=0}^{\infty}\frac{(-1)^n 2^{4n+2}}{(2n+1)!}x^{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+1}}{(2n+1)!}x^{2n+1} \\ = {} & \sum_{n=7}^{\infty}(-1)^{n-1}A_n x^{2n+1} = \sum_{k=4}^{\infty}(A_{2k-1} - A_{2k}x^2)x^{4k-1},\end{aligned}$$

where

$$\begin{aligned}A_n &= \frac{2^{2n+1}}{(2n+1)!}\Big[\widehat{A}_n + 32n^3 + 64n^2 + 8n - 3\Big], \\ \widehat{A}_n &= 3^{2n+1} - 2^{2n-2}(24 + 23n + 63n^2 - 28n^3 + 4n^4).\end{aligned}$$

Since

$$\widehat{A}_{n+1} - 9\widehat{A}_n = 2^{2n-2}\Big[4296 + 5105(n-7) + (n-7)^2(639 + 76n + 20n^2)\Big] > 0$$

for $n \geq 7$, it follows from $\widehat{A}_7 = 946795$ that $\widehat{A}_n > 0$ for $n \geq 7$ and so is A_n.

For $k \geq 4$, let us denote

(21)
$$\begin{aligned}&\frac{(4k+1)!}{2^{4k}(2k+5)(4k-9)}\left[A_{2k-1} - \frac{5}{2}A_{2k}\right]\\ &= a_k + \frac{1823735 + 2(k-4)(227965 + 56996k + 14400k^2 + 3840k^3 + 1024k^4)}{(2k+5)(4k-9)},\end{aligned}$$

where

$$a_k = 3^{4k-1} - \frac{128k^6 - 672k^5 + 872k^4 + 650k^3 - 1271k^2 - 182k - 120}{(2k+5)(4k-9)} \cdot 2^{4k-2}.$$

It is a matter of simple calculations to verify that $A_7 - \frac{5}{2}A_8 = 0$ $(k = 4)$ and

$$\begin{aligned}a_5 &= \frac{12483899195}{33},\\ a_{k+1} &- 81a_k\\ &= \frac{2^{4k-2}}{(2k+5)(2k+7)(4k-9)(4k-5)}\Big[4991200875 + 10871464440(k-5)\\ &\quad + (k-5)^2\big(1974641325 + 355015868k + 63185988k^2\\ &\qquad + 10918528k^3 + 1724032k^4 + 400384k^5 + 66560k^6\big)\Big] > 0,\end{aligned}$$

which gives $a_k > 0$ for $k \geq 5$, and hence $A_{2k-1} - \frac{5}{2}A_{2k} \geq 0$ for $k \geq 4$ by (21).

Due to $A_n > 0$ for $n \geq 7$, it follows from $(\frac{\pi}{2})^2 = 2.4674\cdots < \frac{5}{2}$ that

$$A_{2k-1} - A_{2k}x^2 > A_{2k-1} - \frac{5}{2}A_{2k} \geq 0 \quad (k \geq 4)$$

for $x \in (0, \frac{\pi}{2})$, which together with (19) and (20) implies that $G_1(x)$ is strictly increasing on $(0, \frac{\pi}{2})$. Therefore, by (18), we conclude that $(f/x)^s(x/g)\mathscr{L}_{f,g}(x)$ is strictly decreasing on $(0, \frac{\pi}{2})$ if and only if

$$s \geq 1 + \sup_{x\in(0,\frac{\pi}{2})}\{G_1(x)\} = 1 + G_1(\tfrac{\pi}{2}) = 3 - \frac{\pi^2}{4}.$$

Note that, by Taylor's formula,

$$\lim_{x\to 0^+}(f/x)^s(x/g)\mathscr{L}_{f,g}(x) = \lim_{x\to 0^+}\frac{2x - \sin(2x)}{2(\sin x - x\cos x)} = \lim_{x\to 0^+}\frac{\frac{4x^3}{3} + o(x^3)}{2\left[\frac{x^3}{3} + o(x^3)\right]} = 2$$

and

$$\lim_{x\to\frac{\pi}{2}^-}(f/x)^s(x/g)\mathscr{L}_{f,g}(x) = \lim_{x\to\frac{\pi}{2}^-}\left(\frac{x}{\sin x}\right)^{1-s}\frac{2x - \sin(2x)}{2(\sin x - x\cos x)} = \left(\frac{\pi}{2}\right)^{2-s}.$$

This completes the proof of (ii). □

Proposition 11. *Let* $(f,g)=(\sinh,\tanh)$. *Then the function*

(i) $\mathscr{L}_{f,g}(x)$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0,2)$;

(ii) $(f/g)^s\mathscr{L}_{f,g}(x)$ *is strictly increasing from* $(0,\infty)$ *onto* $(2,\infty)$ *if and only if* $s\geq\frac{4}{5}$.

Proof. (i) By (15), we clearly see that

$$\mathscr{L}_{f,g}(x)=\frac{\sinh x-\frac{x}{\cosh x}}{x\cosh x-\sinh x}.$$

Differentiation yields

$$\mathscr{L}'_{f,g}(x)=-\frac{\sinh x}{[\sinh(2x)-x\cosh(2x)-x]^2}F_2(x), \tag{22}$$

where $F_2(x)=\cosh(3x)+4x\sinh x-(1+8x^2)\cosh x$.

We can rewrite $F_2(x)$, by power series expansion, as

$$F_2(x)=\sum_{n=3}^{\infty}\frac{9^n-32n^2+24n-1}{(2n)!}x^{2n}>0,$$

since $(1+8)^n-32n^2+24n-1>1+8n+\frac{n(n-1)}{2}8^2-32n^2+24n-1=0$ for $n\geq 3$ by binomial expansion. This together with (22) shows that $\mathscr{L}_{f,g}(x)$ is strictly decreasing on $(0,\infty)$. Moreover, two limiting values can be obtained from

$$\lim_{x\to 0^+}\mathscr{L}_{f,g}(x)=\lim_{x\to 0^+}\frac{\sinh(2x)-2x}{2(x\cosh x-\sinh x)}=\lim_{x\to 0^+}\frac{\frac{4x^3}{3}+o(x^3)}{2\left[\frac{x^3}{3}+o(x^3)\right]}=2$$

and

$$\lim_{x\to\infty}\mathscr{L}_{f,g}(x)=\lim_{x\to\infty}\frac{1-\frac{2x}{\sinh(2x)}}{x-\tanh x}=0.$$

(ii) By (15), differentiating $(f/g)^s\mathscr{L}_{f,g}(x)$ yields

$$\frac{(\cosh x)^{2-s}(x\cosh x-\sinh x)}{\sinh x(\sinh x\cosh x-x)}\left[(f/g)^s\mathscr{L}_{f,g}(x)\right]'=s+G_2(x), \tag{23}$$

where

$$G_2(x)=\frac{(1+8x^2)\cosh x-\cosh(3x)-4x\sinh x}{4(x\cosh x-\sinh x)(\sinh x\cosh x-x)}.$$

We now show the the monotonicity of $G_2(x)$ on $(0,\infty)$.

By differentiations,

$$G'_2(x)=\frac{\widetilde{G}_2(x)}{16(x\cosh x-\sinh x)^2(\sinh x\cosh x-x)^2}, \tag{24}$$

where

$$\widetilde{G}_2(x) = 12x(1-4x^2) + 32x(1-x^2)\cosh(2x) - 4x(11+4x^2)\cosh(4x) \\ - (19-64x^2)\sinh(2x) + 8(1+5x^2)\sinh(4x) + \sinh(6x).$$

We can rewrite $\widetilde{G}_2(x)$, in terms of power series, as

$$\begin{aligned}\widetilde{G}_2(x) &= 12x(1-4x^2) + 32x(1-x^2)\sum_{n=0}^{\infty}\frac{2^{2n}x^{2n}}{(2n)!} \\ &\quad - 4x(11+4x^2)\sum_{n=0}^{\infty}\frac{2^{4n}x^{2n}}{(2n)!} - (19-64x^2)\sum_{n=0}^{\infty}\frac{2^{2n+1}x^{2n+1}}{(2n+1)!} \\ &\quad + 8(1+5x^2)\sum_{n=0}^{\infty}\frac{2^{4n+2}x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty}\frac{6^{2n+1}x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=6}^{\infty} B_n x^{2n+1},\end{aligned} \tag{25}$$

where $B_n = \widetilde{b}_n + 2^{2n}\widehat{b}_n$ and

$$\widetilde{b}_n = 3^{2n-2} - (3 - 72n - 64n^2 + 32n^3),$$

$$\widehat{b}_n = \frac{26}{9}\left(\frac{3}{2}\right)^{2n} - (6 + 33n - 20n^2 + 4n^3).$$

As a matter of fact, $\widetilde{b}_n > 0$ and $\widehat{b}_n > 0$ can be derived from

$$\widetilde{b}_6 = 54870, \quad \widehat{b}_6 = \frac{54933}{2048},$$

$$\widetilde{b}_{n+1} - 9\widetilde{b}_n = 32\Big[946 + (n-6)(157 + 29n + 8n^2)\Big] > 0,$$

$$\widehat{b}_{n+2} - \frac{9}{4}\widehat{b}_n = \frac{1}{4}\Big[616 + 661(n-6) + 4(n-6)^2(23+5n)\Big] > 0$$

for $n \geq 6$. This together with (24) and (25) implies that $G_2(x)$ is strictly increasing on $(0,\infty)$. Therefore, by (23), we conclude that $(f/g)^s\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0,\infty)$ if and only if

$$s \geq -\sup_{x\in(0,\infty)}\{G_2(x)\} = -G_2(0^+) = \frac{4}{5}.$$

This completes the proof of Proposition 11(ii) together with

$$\lim_{x\to 0^+}(f/g)^s\mathscr{L}_{f,g}(x) = \lim_{x\to 0^+}\mathscr{L}_{f,g}(x) = 2$$

and

$$\lim_{x\to\infty}(f/g)^s\mathscr{L}_{f,g}(x) = \lim_{x\to\infty}\frac{\cosh^s x}{x}\cdot\frac{1-\frac{2x}{\sinh(2x)}}{1-\frac{\tanh x}{x}} = s\lim_{x\to\infty}\cosh^s x\tanh x = \infty.$$

□

Proposition 12. *Let* $(f,g)=(\arcsin,\arctan)$. *Then the function*

(i) $(f/g)^s\mathscr{L}_{f,g}(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,2)$ *for every* $s\le\frac{14}{5}$*;*

(ii) In particular $\mathscr{L}_{f,g}(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,2)$.

Proof. In this case, it follows easily from Lemma 6 that f/g is strictly increasing on $(0,1)$. This together with $f(0^+)/g(0^+)=1$ allows us to know that Proposition 12(i) will be true for $s\le\frac{14}{5}$ if we can show $(f/g)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)$ is strictly decreasing from $(0,1)$ onto $(0,2)$.

By (15), we can split $(f/g)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)$ into the product of two functions as

$$(f/g)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)=F_3(x)\cdot G_3(x), \tag{26}$$

where

$$F_3(x)=\left(\frac{\arcsin x}{\arctan x}\right)^{\frac{4}{5}}\frac{\arcsin^3 x}{x/\sqrt{1-x^2}-\arcsin x},\quad G_3(x)=\frac{\arctan x-x/(1+x^2)}{\arctan^3 x}.$$

It is apparent that $F_3(x)>0$ and $G_3(x)>0$ for $x\in(0,1)$.

Differentiation yields

$$F_3'(x)=\left(\frac{\arcsin x}{\arctan x}\right)^{\frac{9}{5}}\frac{\widetilde{F}_3(x)\arcsin x}{5(1+x^2)\sqrt{1-x^2}\left(x-\sqrt{1-x^2}\arcsin x\right)^2}, \tag{27}$$

$$G_3'(x)=\frac{(3+x^2)\widetilde{G}_3(x)}{(1+x^2)^2\arctan^4 x}, \tag{28}$$

where

$$\begin{aligned}\widetilde{F}_3(x)=&\left[4(1-x^2)\arcsin^2 x+19x(1+x^2)\arctan x\right]\sqrt{1-x^2}\\&-\left[4x(1-x^2)+(1+x^2)(19-14x^2)\arctan x\right]\arcsin x,\\ \widetilde{G}_3(x)=&\frac{3x}{3+x^2}-\arctan x.\end{aligned}$$

By applying Lemma 9 and simplifying, we have

$$\begin{aligned}\widetilde{F}_3(x)<&\left[4(1-x^2)p_2(x)+19x(1+x^2)q_2(x)\right]\sqrt{1-x^2}\\&-\left[4x(1-x^2)+(1+x^2)(19-14x^2)q_1(x)\right]p_1(x),\\ =&-\frac{S_1(x^2)-S_2(x^2)\sqrt{1-x^2}}{x^2},\end{aligned} \tag{29}$$

where

$$S_1(x) = 23 - \frac{3x}{2} - \frac{3971x^2}{360} + \frac{1153x^3}{720} - \frac{270229x^4}{201600} + \frac{2579x^5}{20790} - \frac{51366773x^6}{23284800} + \frac{4782181x^7}{4656960} + \frac{111187x^8}{1995840} + \frac{61637x^9}{1995840} - \frac{485x^{10}}{114048} + \frac{245x^{11}}{6336},$$

$$S_2(x) = 23 + 10x - \frac{142x^2}{45} + \frac{262x^3}{315} + \frac{32947x^4}{25200} + \frac{34x^5}{225} - \frac{10741x^6}{47040} - \frac{1213x^7}{15680} + \frac{11x^8}{49} - \frac{16x^9}{49}.$$

Moreover, it is a matter of simple transformations to verify that

$$S_1(x) = (1-x)\widetilde{S}_1(x) + \frac{68431721}{6985440} + \frac{485x^{10}}{57024}$$

and

$$\widetilde{S}_1(x) = \frac{92233399}{6985440} + \frac{81755239x}{6985440} + \frac{940391x^2}{1397088} + \frac{15888361x^3}{6985440} + \frac{130498523x^4}{139708800} + \frac{147829403x^5}{139708800} - \frac{32074247x^6}{27941760} - \frac{483023x^7}{3991680} - \frac{28961x^8}{443520} - \frac{3925x^9}{114048} - \frac{245x^{10}}{6336} > 0$$

for $x \in (0,1)$ by Lemma 7 and $\widetilde{S}_1(1) = \frac{331108073}{11642400}$, which implies $S_1(x) > 0$ for $x \in (0,1)$.

We shall determine the sign of $S_1(x) - S_2(x)\sqrt{1-x}$.

- If $S_2(x) \le 0$, then it follows easily from $S_1(x) > 0$ that $S_1(x) - S_2(x)\sqrt{1-x} > 0$ for $x \in (0,1)$;
- If $S_2(x) > 0$, then by scaling the inequality and turning it into a polynomial with integer coefficients, it can be obtained that

$$\begin{aligned} S_1^2(x) - (1-x)S_2^2(x) &> 2 + (1-x)(28+9x+9x^2) + 5x^2 + 38x^4 + 31x^5 \\ &+ (1-x)\Big[(1+x)(28+28x^2+21x^4+8x^6+3x^8) \\ &+ x^{10}(1+x+x^2)(2+x^3)\Big] > 0. \end{aligned}$$

In other words, $S_1(x) - S_2(x)\sqrt{1-x} > 0$ for $x \in (0,1)$.

Combining this with (27) and (29), it follows that $F_3(x)$ is strictly decreasing on $(0,1)$. On the other hand, the monotonicity of $G_3(x)$ can be easily derived from (28), $\widetilde{G}_3(0) = 0$ and

$$\widetilde{G}_3'(x) = -\frac{4x^4}{(1+x^2)(3+x^2)^2} < 0.$$

Therefore, by (26), two positive decreasing functions $F_3(x)$ and $G_3(x)$ lead to the conclusion that $(f/g)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)$ is strictly decreasing on $(0,1)$.

To this end, by l'Hopital rule, we have

$$\lim_{x\to 0^+}(f/g)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)=\lim_{x\to 0^+}\frac{\arctan x-x/(1+x^2)}{x/\sqrt{1-x^2}-\arcsin x}=\lim_{x\to 0^+}\frac{2(1-x^2)^{\frac{3}{2}}}{(1+x^2)^2}=2,$$

$$\lim_{x\to 1^-}(f/g)^{\frac{14}{5}}\mathscr{L}_{f,g}(x)=2^{\frac{9}{5}}(\pi-2)\lim_{x\to 1^-}\frac{\sqrt{1-x^2}}{x-\sqrt{1-x^2}\arcsin x}=0.$$

□

Proposition 13. *Let* $(f,g)=(\mathrm{arcsinh},\mathrm{arctanh})$. *Then the function*

(i) $(f/g)^s\mathscr{L}_{f,g}(x)$ *is strictly increasing from* $(0,1)$ *onto* $(2,\infty)$ *for every* $s\le\frac{9}{4}$;

(ii) In particular $\mathscr{L}_{f,g}(x)$ *is strictly increasing from* $(0,1)$ *onto* $(2,\infty)$.

Proof. As in the proof of Proposition 12, it suffices to show that $(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0,1)$ since f/g is strictly decreasing from $(0,1)$ onto $(2,\infty)$.

We divide the proof into two cases:

Case 4.1: $0<x<\frac{7}{10}$. We first prove that

$$\mathrm{arctanh}\,x>\sqrt{1+x^2}\,\mathrm{arcsinh}\,x\quad\text{for } x\in(0,1). \tag{30}$$

Let $\zeta_0(x)=\mathrm{arctanh}\,x-\sqrt{1+x^2}\,\mathrm{arcsinh}\,x$ and $\zeta_1(x)=\frac{x\sqrt{1+x^2}}{1-x^2}-\mathrm{arcsinh}\,x$. Then simple calculations yield

$$\zeta_0'(x)=\frac{x\zeta_1(x)}{\sqrt{1+x^2}}\quad\text{and}\quad\zeta_1'(x)=\frac{x^2(5-x^2)}{(1-x^2)^2\sqrt{1+x^2}}>0$$

for $x\in(0,1)$. This together with $\zeta_0(0)=\zeta_1(0)=0$ gives (30).

By (15), we can rewrite $(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x)$ as

$$(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x)=F_4(x)\cdot G_4(x), \tag{31}$$

where

$$F_4(x)=\frac{\mathrm{arcsinh}^{\frac{13}{4}}x}{\mathrm{arctanh}^{\frac{1}{4}}x\left[\mathrm{arcsinh}\,x-x/\sqrt{1+x^2}\right]},\quad G_4(x)=\frac{x/(1-x^2)-\mathrm{arctanh}\,x}{\mathrm{arctanh}^3x}.$$

Differentiating $F_4(x)$ yields

$$F_4'(x)=\frac{\mathrm{arcsinh}^{\frac{9}{4}}x}{4(1-x^2)\sqrt{1+x^2}(x-\sqrt{1+x^2}\,\mathrm{arcsinh}\,x)^2\,\mathrm{arctanh}^{\frac{5}{4}}x}\widetilde{F}_4(x), \tag{32}$$

where

$$\widetilde{F}_4(x) = (1+x^2)(x - \sqrt{1+x^2}\operatorname{arcsinh} x)\operatorname{arcsinh} x \\ + (1-x^2)(13+9x^2)\left(\operatorname{arcsinh} x - \frac{13x\sqrt{1+x^2}}{13+9x^2}\right)\operatorname{arctanh} x.$$

It can be easily proved that

$$\eta(x) = \operatorname{arcsinh} x - \frac{13x\sqrt{1+x^2}}{13+9x^2} > 0 \tag{33}$$

for $0 < x < 1$ by $\eta(0) = 0$ and $\eta'(x) = \left[x^2(13+81x^2)\right] / \left[(13+9x^2)^2\sqrt{1+x^2}\right] > 0$. From (30) and (33) we clearly see that

$$\widetilde{F}_4(x) > (12 - 5x^2 - 9x^4)\sqrt{1+x^2}\,\zeta_2(x)\operatorname{arcsinh} x, \tag{34}$$

where

$$\zeta_2(x) = \operatorname{arcsinh} x - \frac{x(12-13x^2)\sqrt{1+x^2}}{12-5x^2-9x^4}.$$

Elementary computations lead to

$$\zeta_2(0) = 0, \quad \zeta_2(\tfrac{7}{10}) = 0.00162\cdots, \tag{35}$$

$$\zeta_2'(x) = \frac{81x^4(x^2-x_1^2)(x^2-x_2^2)}{(12-5x^2-9x^4)^2\sqrt{1+x^2})}, \tag{36}$$

where $x_1 = \frac{1}{9}\sqrt{\frac{139-\sqrt{5065}}{2}} = 0.64707\cdots$ and $x_2 = \frac{1}{9}\sqrt{\frac{139+\sqrt{5065}}{2}} = 1.13901\cdots$. It follows from (36) that $\zeta_2(x)$ is strictly increasing on $(0, x_1)$ and strictly decreasing on $(x_1, 0.7)$. This in conjunction with (35) gives $\zeta_2(x) > 0$ for $0 < x < \frac{7}{10}$. According to this with $12-5x^2-9x^4 > 7.3891$, it can be easily seen from (32) and (34) that $F_4(x)$ is strictly increasing on $(0, \frac{7}{10})$ and so $F_4(x) > 0$ from $F_4(0^+) = \frac{6}{5}$. Further, differentiation of $G_4(x)$ yields

$$G_4'(x) = \frac{(3-x^2)}{(1-x^2)^2\operatorname{arctanh}^4 x}\widetilde{G}_4(x), \quad \widetilde{G}_4(x) = \operatorname{arctanh} x - \frac{3x}{3-x^2},$$

$$\widetilde{G}_4(0) = 0, \qquad \widetilde{G}_4'(x) = \frac{4x^4}{(1-x^2)(3-x^2)^2} > 0$$

for $0 < x < 1$. This gives the monotonicity of $G_4(x)$ on $(0,1)$ and so $G_4(x) > 0$ due to $G_4(0^+) = \frac{2}{3}$.

Therefore, the monotonicity of $F_4(x)$ and $G_4(x)$ and (31) together with $F_4(x) > 0$ and $G_4(x) > 0$ lead to the conclusion that $(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0, \frac{7}{10})$.

Case 4.2: $\frac{7}{10} \le x < 1$. In this case, we can split $(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x)$ into the product of alternative two functions as

$$(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x) = \widetilde{F}_4(x) \cdot \widetilde{G}_4(x), \tag{37}$$

where

$$\widetilde{F}_4(x) = \frac{\operatorname{arcsinh}^{\frac{13}{4}} x}{\operatorname{arcsinh} x - x/\sqrt{1+x^2}}, \qquad \widetilde{G}_4(x) = \frac{x/(1-x^2) - \operatorname{arctanh} x}{\operatorname{arctanh}^{\frac{13}{4}} x}.$$

Simple computations with (33) lead to

$$\widetilde{F}_4'(x) = \frac{(13+9x^2)\eta(x)\operatorname{arcsinh}^{\frac{9}{4}} x}{4\sqrt{1+x^2}(x-\sqrt{1+x^2}\operatorname{arcsinh}^2 x)^2} > 0, \tag{38}$$

$$\widetilde{G}_4'(x) = \frac{(13-5x^2)\zeta_3(x)}{4(1-x^2)^2 \operatorname{arctanh}^{1\frac{7}{4}} x}, \tag{39}$$

$$\zeta_3(\tfrac{7}{10}) = 0.00474\cdots, \quad \zeta_3'(x) = \frac{2x^2\left[\frac{181}{20} + 45(x^2 - (\frac{7}{10})^2)\right]}{(1-x^2)(13-5x^2)^2} > 0. \tag{40}$$

Equations (37)-(40) lead to the conclusion that $(f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x)$ is strictly increasing on $[\frac{7}{10}, 1)$.

Therefore, the proof of Proposition 13 is completed from Cases 4.1-4.2 and

$$\lim_{x\to 0^+} (f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x) = \lim_{x\to 0^+} \frac{\operatorname{arctanh} x - \frac{x}{1-x^2}}{\frac{x}{\sqrt{1+x^2}} - \operatorname{arcsinh} x} = 2\lim_{x\to 0^+} \frac{(1+x^2)^{\frac{3}{2}}}{(1-x^2)^2} = 2,$$

$$\lim_{x\to 1^-} (f/g)^{\frac{9}{4}}\mathscr{L}_{f,g}(x) = \frac{\sqrt{2}\operatorname{arcsinh}^{\frac{13}{4}} 1}{\sqrt{2}\operatorname{arcsinh} 1 - 1} \lim_{x\to 1^-} \frac{x - (1-x^2)\operatorname{arctanh} x}{(1-x^2)\operatorname{arctanh}^{\frac{13}{4}} x} = \infty.$$

□

Proposition 14. *Let* $(f,g) = (\mathrm{sl}, \mathrm{tl})$. *Then the function*

(i) $\mathscr{L}_{f,g}(x)$ *is strictly increasing from* $(0,\omega)$ *onto* $(\frac{3}{2}, \infty)$*;*

(ii) $(f/g)^s\mathscr{L}_{f,g}(x)$ *is strictly decreasing on* $(0,\omega)$ *onto* $(\omega, \frac{3}{2})$ *if* $s \ge 2$.

Proof. (i) By (15) and formulas (6), (8), we obtain

$$\mathscr{L}_{f,g}(x) = \frac{\mathrm{sl}\, x\,[x(\mathrm{tl}\, x)' - \mathrm{tl}\, x]}{\mathrm{tl}\, x\,[\mathrm{sl}\, x - x(\mathrm{sl}\, x)']} = \frac{x\,\mathrm{tl}^4 x}{\mathrm{tl}\, x(1+\mathrm{tl}^4 x)^{\frac{1}{4}} - x} - 1 = F_5(\mathrm{tl}\, x) - 1, \tag{41}$$

where

$$F_5(x) = \frac{x^4 \operatorname{arctl} x}{x(1+x^4)^{\frac{1}{4}} - \operatorname{arctl} x}.$$

Due to the monotonicity of $\operatorname{tl} x$, it suffices to show that $F_5(x)$ is strictly increasing on $(0,\infty)$.

Let $F_{51}(x) = x^4 \operatorname{arctl} x$ and $F_{52}(x) = x(1+x^4)^{\frac14} - \operatorname{arctl} x$. Then it can be easily seen that $F_5(x) = F_{51}(x)/F_{52}(x)$, $F_{51}(0) = F_{52}(0) = 0$ and

$$\frac{F'_{51}(x)}{F'_{52}(x)} = \frac{2(1+x^4)^{\frac34}\operatorname{arctl} x}{x} + \frac12, \tag{42}$$

$$\left[\frac{(1+x^4)^{\frac34}\operatorname{arctl} x}{x}\right]' = \frac{F_{52}(x) + 2x^4\operatorname{arctl} x}{x^2(1+x^4)^{\frac14}} > 0, \tag{43}$$

where the inequality (43) follows from $F_{52}(0) = 0$ and $F'_{52}(x) = 2x^4/(1+x^4)^{\frac34} > 0$.

Therefore, Lemma 6 and (42), (43) lead to the conclusion that $F_5(x)$ is strictly increasing on $(0,\infty)$. Moreover, by l'Hopital rule and (42), we obtain

$$\mathscr{L}_{f,g}(0^+) = \lim_{x\to 0^+} F_5(x) - 1 = \lim_{x\to 0^+}\frac{F'_{51}(x)}{F'_{52}(x)} - 1 = \frac32,$$
$$\mathscr{L}_{f,g}(\omega^+) = \lim_{x\to\infty} F_5(x) - 1 = \infty.$$

(*ii*) Since $f/g = \operatorname{sl} x/\operatorname{tl} x = \sqrt[4]{1-\operatorname{sl}^4 x}$ is strictly decreasing on $(0,\omega)$, Proposition 14 will be proved if we can show that $(f/g)^2\mathscr{L}_{f,g}(x)$ is strictly decreasing on $(0,\omega)$.

According to (7) and (41), we can express $(f/g)^2\mathscr{L}_{f,g}(x)$ as

$$(f/g)^2\mathscr{L}_{f,g}(x) = \frac{1}{\sqrt{1+\operatorname{tl}^4 x}}\left[\frac{x\operatorname{tl}^4 x}{\operatorname{tl} x(1+\operatorname{tl}^4 x)^{\frac14} - x} - 1\right] = G_5(\operatorname{tl} x), \tag{44}$$

where

$$G_5(x) = \frac{1}{\sqrt{1+x^4}}\left[\frac{x^4\operatorname{arctl} x}{x(1+x^4)^{\frac14} - \operatorname{arctl} x} - 1\right].$$

Since $\operatorname{tl} x$ is strictly increasing on $(0,\omega)$, it suffices to show that $G_5(x)$ is strictly decreasing on $(0,\infty)$.

Several differentiations yield

$$G'_5(x) = -\frac{x^3}{\sqrt{1+x^4}\left[x(1+x^4)^{\frac14} - \operatorname{arctl} x\right]^2} G_{51}(x), \tag{45}$$

$$G_{51}(x) = 2\operatorname{arctl}^2 x + \frac{x\operatorname{arctl} x}{(1+x^4)^{\frac34}} - \frac{3x^2}{\sqrt{1+x^4}},$$

$$G'_{51}(x) = \frac{2x^4+5}{(1+x^4)^{\frac74}} G_{52}(x), \tag{46}$$

$$G_{52}(x) = \operatorname{arctl} x - \frac{5x(1+x^4)^{\frac14}}{2x^4+5},$$

$$G'_{52}(x) = \frac{24x^8}{(2x^4+5)^2(1+x^4)^{\frac34}} > 0. \tag{47}$$

Therefore, the monotonicity of $G_5(x)$ follows from (45)-(47) and $G_{51}(0) = G_{52}(0) = 0$ and so is $(f/g)^2\mathscr{L}_{f,g}(x)$. Further, two limiting values can be obtained from (44) together with $G_5(0^+) = \frac{3}{2}$ and

$$\lim_{x\to\infty} G_5(x) = \lim_{x\to\infty}\left[\frac{\operatorname{arctl} x}{\sqrt{1+1/x^4}\left((1+1/x^4)^{\frac{1}{4}} - \operatorname{arctl} x/x^2\right)} - \frac{1}{\sqrt{1+x^4}}\right] = \omega.$$

□

Proposition 15. *Let* $(f,g) = (\mathrm{slh}, \mathrm{tlh})$*. Then the function*

(i) $\mathscr{L}_{f,g}(x)$ *is strictly decreasing from* $(0,\kappa)$ *onto* $(0, \frac{3}{2})$*;*

(ii) $(f/g)^s\mathscr{L}_{f,g}(x)$ *is strictly increasing on* $(0,\kappa)$ *onto* $(\frac{3}{2}, \infty)$ *if* $s \geq 2$*.*

Proof. (i) As in (41), it follows from (7) and (8) that

$$\mathscr{L}_{f,g}(x) = \frac{x\,\mathrm{tlh}^4\,x}{x - \mathrm{tlh}\,x(1-\mathrm{tlh}^4\,x)^{\frac{1}{4}}} - 1 = F_6(\mathrm{tlh}\,x) - 1, \tag{48}$$

where

$$F_6(x) = \frac{x^4 \operatorname{arctlh} x}{\operatorname{arctlh} x - x(1-x^4)^{\frac{1}{4}}}.$$

It suffices to show that $F_6(x)$ is strictly decreasing on $(0,1)$ since $\mathrm{tlh}\,x$ is strictly increasing on $(0,\kappa)$.

Let $F_{61}(x) = x^4 \operatorname{arctlh} x$ and $F_{62}(x) = \operatorname{arctlh} x - x(1-x^4)^{\frac{1}{4}}$. Then we clearly see that $F_6(x) = F_{61}(x)/F_{62}(x)$, $F_{61}(0) = F_{62}(0) = 0$ and

$$F'_{62}(x) = \frac{2x^4}{(1-x^4)^{\frac{3}{4}}} > 0, \tag{49}$$

$$\frac{F'_{61}(x)}{F'_{62}(x)} = \frac{2(1-x^4)^{\frac{3}{4}} \operatorname{arctl} x}{x} + \frac{1}{2}. \tag{50}$$

According to (49), it follows that $F_{62}(x) > 0$, which gives rise to

$$\left[\frac{(1-x^4)^{\frac{3}{4}} \operatorname{arctl} x}{x}\right]' = -\frac{F_{62}(x) + 2x^4 \operatorname{arctlh} x}{x^2(1-x^4)^{\frac{1}{4}}} < 0. \tag{51}$$

Therefore, the monotonicity of $F_6(x)$ follows easily from Lemma 6 and (50), (51). Moreover, by l'Hopital rule and (50), we obtain

$$\mathscr{L}_{f,g}(0^+) = \lim_{x\to 0^+} F_6(x) - 1 = \lim_{x\to 0^+} \frac{F'_{61}(x)}{F'_{62}(x)} - 1 = \frac{3}{2},$$
$$\mathscr{L}_{f,g}(\kappa^+) = \lim_{x\to 1^-} F_6(x) - 1 = 0.$$

(*ii*) We only need to prove that $(f/g)^2\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0,\kappa)$, since $f/g = \operatorname{slh} x/\operatorname{tlh} x = \sqrt[4]{1+\operatorname{slh}^4 x}$ is strictly increasing on $(0,\kappa)$.

From (8) and (48), $(f/g)^2\mathscr{L}_{f,g}(x)$ can be rewritten as

$$(f/g)^2\mathscr{L}_{f,g}(x) = \frac{1}{\sqrt{1-\operatorname{tlh}^4 x}}\left[\frac{x\operatorname{tlh}^4 x}{x-\operatorname{tlh} x(1-\operatorname{tlh}^4 x)^{\frac14}}-1\right] = G_6(\operatorname{tlh} x), \tag{52}$$

where

$$G_6(x) = \frac{1}{\sqrt{1-x^4}}\left[\frac{x^4\operatorname{arctlh} x}{\operatorname{arctlh} x - x(1-x^4)^{\frac14}}-1\right].$$

Elementary computations lead to

$$G_6'(x) = \frac{x^3}{\sqrt{1-x^4}\left[\operatorname{arctlh} x - x(1-x^4)^{\frac14}\right]^2}G_{61}(x), \tag{53}$$

$$G_{61}(x) = 2\operatorname{arctlh}^2 x + \frac{x\operatorname{arctlh} x}{(1-x^4)^{\frac34}} - \frac{3x^2}{\sqrt{1-x^4}},$$

$$G_{61}'(x) = \frac{5-2x^4}{(1-x^4)^{\frac74}}G_{62}(x), \tag{54}$$

$$G_{62}(x) = \operatorname{arctlh} x - \frac{5x(1-x^4)^{\frac14}}{5-2x^4},$$

$$G_{62}'(x) = \frac{24x^8}{(5-2x^4)^2(1-x^4)^{\frac34}} > 0. \tag{55}$$

Therefore, (52)-(55) and $G_{61}(0) = G_{62}(0) = 0$ together with the monotonicity of $\operatorname{tlh} x$ lead to the conclusion that $(f/g)^2\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0,1)$. Further, two limiting values can be obtained from (52) together with $G_6(0^+) = \frac32$ and

$$G_6(1^-) = \lim_{x\to1^-}\frac{x-(1-x^4)^{\frac34}\operatorname{arctlh} x}{(1-x^4)^{\frac14}\left[\operatorname{arctlh} x - x(1-x^4)^{\frac14}\right]} = \infty.$$

□

Proposition 16. *Let* $(f,g) = (\operatorname{arcsl}, \operatorname{arctl})$*. Then the function*

(i) $(f/g)^s\mathscr{L}_{f,g}(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\frac32)$ *for every* $s\le 4$*;*

(ii) In particular $\mathscr{L}_{f,g}(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\frac32)$*.*

Proof. Proposition 16 is the arc lemniscate counterpart of Proposition 12. As in the proof of Proposition 12, it suffices to show that $(f/g)^4\mathscr{L}_{f,g}(x)$ is strictly decreasing from $(0,1)$ onto $(0,\frac32)$ by Tab. 1 and $f(0^+)/g(0^+) = 1$.

By (15), we can rewrite $(f/g)^4\mathscr{L}_{f,g}(x)$ as

$$(f/g)^4\mathscr{L}_{f,g}(x) = F_7(x)\cdot G_7(x), \tag{56}$$

where

$$F_7(x) = \frac{\operatorname{arcsl}^5 x}{x/\sqrt{1-x^4} - \operatorname{arcsl} x}, \quad G_7(x) = \frac{\operatorname{arctl} x - x/(1+x^4)^{\frac{3}{4}}}{\operatorname{arctl}^5 x}.$$

We now show that $F_7(x)$ and $G_7(x)$ are strictly decreasing on $(0,1)$.

Let $F_{71}(x) = \operatorname{arcsl}^5 x$ and $F_{72}(x) = x/\sqrt{1-x^4} - \operatorname{arcsl} x$. Then it can be easily seen that $F_7(x) = F_{71}(x)/F_{72}(x)$, $F_{71}(0) = F_{72}(0) = 0$ and

$$\frac{F_{71}'(x)}{F_{72}'(x)} = \frac{5}{2}\left[\frac{(1-x^4)^{\frac{1}{4}}\operatorname{arcsl} x}{x}\right]^4 \triangleq \frac{5}{2}\zeta_4^4(x). \tag{57}$$

Simple calculations give

$$x^2(1-x^4)^{\frac{3}{4}}\zeta_4'(x) = x\sqrt{1-x^4} - \operatorname{arcsl} x \triangleq \zeta_5(x), \tag{58}$$

$$\zeta_5(0) = 0, \quad \zeta_5'(x) = -\frac{3x^4}{\sqrt{1-x^4}} < 0. \tag{59}$$

Lemma 6 and (57)-(59) lead to the conclusion that $F_7(x)$ is strictly decreasing on $(0,1)$.

Similarly, let $G_{71}(x) = \operatorname{arctl} x - x/(1+x^4)^{\frac{3}{4}}$, $G_{72}(x) = \operatorname{arctl}^5 x$, $G_{73}(x) = x/(1+x^4)^{\frac{1}{4}}$ and $G_{74}(x) = \operatorname{arctl} x$. Then we clearly see that $G_7(x) = G_{71}(x)/G_{72}(x)$ and

$$G_{71}(0) = G_{72}(0) = 0, \quad G_{73}(0) = G_{74}(0) = 0, \tag{60}$$

$$\frac{G_{71}'(x)}{G_{72}'(x)} = \frac{3}{5}\left[\frac{G_{73}(x)}{G_{74}(x)}\right]^4, \quad \frac{G_{73}'(x)}{G_{74}'(x)} = \frac{1}{\sqrt{1+x^4}}. \tag{61}$$

According to Lemma 6 and (60), (61), it follows that $G_7(x)$ is strictly decreasing on $(0,1)$.

Therefore, Proposition 16 follows from (56) and the monotonicity of $F_7(x)$, $G_7(x)$ together with

$$\lim_{x\to 0^+}(f/g)^4\mathscr{L}_{f,g}(x) = \lim_{x\to 0^+}\frac{\operatorname{arctl} x - x/(1+x^4)^{\frac{3}{4}}}{x/\sqrt{1-x^4} - \operatorname{arcsl} x} = \lim_{x\to 0^+}\frac{3(1-x^4)^{\frac{3}{2}}}{2(1+x^4)^{\frac{7}{4}}} = \frac{3}{2},$$

$$\lim_{x\to 1^-}(f/g)^4\mathscr{L}_{f,g}(x) = \left(\frac{\omega}{\tau}\right)^5\left(\tau - \frac{1}{2^{\frac{3}{4}}}\right)\lim_{x\to 1^-}\frac{\sqrt{1-x^4}}{x - \sqrt{1-x^4}\operatorname{arcsl} x} = 0.$$

□

Proposition 17. *Let* $(f,g) = (\operatorname{arcslh}, \operatorname{arctlh})$. *Then the function* *(i)* $(f/g)^s \mathscr{L}_{f,g}(x)$ *is strictly increasing from* $(0,1)$ *onto* $(\frac{3}{2},\infty)$ *for every* $s \le 4$*;*

(ii) In particular $\mathscr{L}_{f,g}(x)$ *is strictly increasing from* $(0,1)$ *onto* $(\frac{3}{2},\infty)$*.*

Proof. As the hyperbolic counterpart of Proposition 16, by the similar method, we only need to show that $(f/g)^4\mathscr{L}_{f,g}(x)$ is strictly increasing on $(0,1)$.

By (15), in this case $(f/g)^4\mathscr{L}_{f,g}(x)$ can be split into

$$(f/g)^4\mathscr{L}_{f,g}(x) = F_8(x)\cdot G_8(x), \tag{62}$$

where

$$F_8(x) = \frac{\operatorname{arcslh}^5 x}{\operatorname{arcslh} x - x/\sqrt{1+x^4}}, \quad G_8(x) = \frac{x/(1-x^4)^{\frac{3}{4}} - \operatorname{arctlh} x}{\operatorname{arctlh}^5 x}.$$

We now prove that $F_8(x)$ and $G_8(x)$ are strictly increasing on $(0,1)$.

Let $F_{81}(x) = \operatorname{arcslh}^5 x$ and $F_{82}(x) = \operatorname{arcslh} x - x/\sqrt{1+x^4}$. Then we clearly see that $F_8(x) = F_{81}(x)/F_{82}(x)$, $F_{81}(0) = F_{82}(0) = 0$ and

$$\frac{F'_{81}(x)}{F'_{82}(x)} = \frac{5}{2}\left[\frac{(1+x^4)^{\frac{1}{4}}\operatorname{arcslh} x}{x}\right]^4 \triangleq \frac{5}{2}\zeta_6^4(x). \tag{63}$$

Simple calculations yield

$$x^2(1+x^4)^{\frac{3}{4}}\zeta_6'(x) = x\sqrt{1+x^4} - \operatorname{arcslh} x \triangleq \zeta_7(x), \tag{64}$$

$$\zeta_7(0) = 0, \quad \zeta_7'(x) = \frac{3x^4}{\sqrt{1+x^4}} > 0. \tag{65}$$

Lemma 6 and (63)-(65) lead to the conclusion that $F_8(x)$ is strictly increasing on $(0,1)$.

Similarly, let $G_{81}(x) = x/(1-x^4)^{\frac{3}{4}} - \operatorname{arctlh} x$, $G_{82}(x) = \operatorname{arctlh}^5 x$, $G_{83}(x) = x/(1-x^4)^{\frac{1}{4}}$ and $G_{84}(x) = \operatorname{arctlh} x$. Then it can be easily seen that $G_8(x) = G_{81}(x)/G_{82}(x)$ and

$$G_{81}(0) = G_{82}(0) = 0, \quad G_{83}(0) = G_{84}(0) = 0, \tag{66}$$

$$\frac{G'_{81}(x)}{G'_{82}(x)} = \frac{3}{5}\left[\frac{G_{83}(x)}{G_{84}(x)}\right]^4, \quad \frac{G'_{83}(x)}{G'_{84}(x)} = \frac{1}{\sqrt{1-x^4}}. \tag{67}$$

According to Lemma 6 and (60), (61), it follows that $G_8(x)$ is strictly decreasing on $(0,1)$.

Therefore, Proposition 17 follows from (62) and the monotonicity of $F_8(x)$, $G_8(x)$ together with

$$\lim_{x\to 0^+}(f/g)^4\mathscr{L}_{f,g}(x) = \lim_{x\to 0^+}\frac{x/(1-x^4)^{\frac{3}{4}} - \operatorname{arctlh} x}{\operatorname{arcslh} x - x/\sqrt{1+x^4}} = \lim_{x\to 0^+}\frac{3(1+x^4)^{\frac{3}{2}}}{2(1-x^4)^{\frac{7}{4}}} = \frac{3}{2},$$

$$\lim_{x\to 1^-}(f/g)^4\mathscr{L}_{f,g}(x) = \left(\frac{\sigma}{\kappa}\right)^5\frac{\sqrt{2}}{\sqrt{2}\sigma - 1}\lim_{x\to 1^-}\frac{x-(1-x^4)^{\frac{3}{4}}\operatorname{arctlh} x}{(1-x^4)^{\frac{3}{4}}} = \infty.$$

□

Acknowledgements. The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions. This research was supported by the National Natural Science Foundation of China (11971142) and the Natural Science Foundation of Zhejiang Province (LY19A010012).

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Tie-Hong Zhao
School of Mathematics,
Hangzhou Normal University, Hangzhou 311121,
China
Email: *tiehong.zhao@hznu.edu.cn*

(Received 18. 05. 2021.)
(Revised 26. 09. 2022.)

Yu-Ming Chu
Department of Mathematics,
Huzhou University, Huzhou 313000,
China
Email: *chuyuming@zjhu.edu.cn*