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AN IDENTITY INVOLVING DERANGEMENT NUMBERS AND BELL NUMBERS

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The aim of this paper is to establish a new identity between derangement numbers and Bell numbers. Two known identities can be recovered. We provide a combinatorial interpretation for the new identity and a representation of the derangement numbers in terms of the determinants of Hessenberg matrices.

1. PRELIMINARIES

A permutation of the numbers of $\{1, 2, \ldots, n\}$ is called an nth *derangement*, if none of the elements is left at its original position, i.e., a permutation that has no fixed points. The number of nth derangements is denoted by $!n$ throughout this paper, called nth *derangement number* or *subfactorial* of n. The first few derangement numbers are

$$0\,,1\,,2\,,9\,,44\,,265\,,1854\,,14833\,,133496\,,1334961\,,14684570$$

and the sequence is known in The On-Line Encyclopedia of Integer Sequences [14] as A000166.

Derangement numbers were combinatorially studied by Pierre Rémond de Montmort in the *Essay d'analyse sur les jeux de hazard*, published in 1708. The recurrence relations

(1) $$!n = (n-1)(!(n-1)+!(n-2))\,, \quad \text{for } n \geqslant 2,$$

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and

$$!n = n\,(!(n-1)) + (-1)^n\,, \quad \text{for } n \geqslant 1,$$

with $!0 = 1$ and $!1 = 0$, were established and proved by Euler in *Calcul de la probabilité dans le jeu de rencontre*, in 1753.

By the inclusion-exclusion principle, one can get the following explicit form of the derangement numbers (see [15]):

$$(2) \qquad !n = n!\sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!\,.$$

Moreover, taking into account the binomial inversion formula, we get

$$(3) \qquad n! = \sum_{k=0}^{n} \binom{n}{k} (!k)\,.$$

In terms of matrices, we can identify the nth derangement number with the permanent of the all ones matrix minus the identity matrix, all of order n [10] (for more details on this matter, the reader is referred to [15, Chapter 2]). Interestingly, it is also the determinant of a tridiagonal matrix as a direct consequence of (1) (cf.[1, 7, 8]).

Recently, Qi, Wang, and Guo in [13] established the following identity

$$(4) \qquad !n = \sum_{k=0}^{n-2} \binom{n}{k} (n-k-1)(!k)\,, \quad \text{for } n \geqslant 2\ .$$

A short proof for (4) can be found in [4].

A more elaborated identity was established in [2]:

$$(5) \qquad \sum_{k=0}^{n} k^s \binom{n}{k} (!k) = n! \sum_{j=0}^{s} (-1)^j \binom{s}{j} n^{s-j} B_j, \quad \text{if } n \geqslant s\,,$$

where B_n is the nth Bell number. Formally, the Bell number is defined to be the sum of Stirling numbers of the second kind:

$$(6) \qquad B_n = \sum_{k=0}^{n} S_2(n,k)\,,$$

where $S_2(n,k)$ represents the Stirling number of the second kind. The proof of (5) is of a combinatorial nature by counting permutations of $\{1, 2, \ldots, n\}$, where s labels are attached to the "displaced" elements, in two different ways. Indeed, we can find in the literature several identities involving both numbers and their generalizations as, for example, in [5, 6, 12, 16].

Our goal is to provide a new identity and the corresponding proof, where (3), (4), and (5) are particular cases. We also provide a combinatorial interpretation and establish a general determinantal representation for it in terms of a Hessenberg matrix.

2. THE IDENTITIES

Let

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

for $n \geqslant 1$. In particular, $(x)_0 = 1$.

The following lemma is straightforward.

Lemma 1. *For* $n, j \geqslant 0$,

$$(n)_j(n-j)! = \begin{cases} n! & \text{if } n \geqslant j \\ 0 & \text{if } n \leqslant j-1 \end{cases} .$$

The equation in the following lemma is an immediate consequence of the definition of the Stirling number of second kind.

Lemma 2. *For* $n \geqslant 0$,

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k .$$

The next identity will be useful in the proof of our main result.

Lemma 3. *For* $n \geqslant k$, $k \geqslant 0$, $j \geqslant 0$,

$$\binom{n}{k}(n-k)_j = (n)_j\binom{n-j}{k} .$$

Proof. We prove it by induction on j. The identity is clearly true when $j = 0$. Assume that it is true for $j-1 \geqslant 0$. In particular,

$$\binom{n-1}{k}(n-1-k)_{j-1} = (n-1)_{j-1}\binom{n-j}{k}.$$

So

$$\begin{aligned} \binom{n}{k}(n-k)_j &= \binom{n}{k}(n-k)(n-k-1)_{j-1} \\ &= n\binom{n-1}{k}(n-k-1)_{j-1} \\ &= n(n-1)_{j-1}\binom{n-j}{k} \\ &= (n)_j\binom{n-j}{k}. \end{aligned}$$

The desired result follows. □

Our main result is stated in the next theorem.

Theorem 4. *Let* $f(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ *be a polynomial on* x *of degree* r*. For* $n \geqslant 0$,

$$\sum_{k=0}^{n} \binom{n}{k} f(n-k)(!k) = n! \sum_{i=0}^{r} a_i B_i \tag{7}$$

when $n \geqslant r$*, and*

$$\sum_{k=0}^{n} \binom{n}{k} f(n-k)(!k) = n! \left(\sum_{i=0}^{r} a_i B_i - \sum_{n+1 \leqslant j \leqslant i \leqslant r} a_i S_2(i,j) \right)$$

when $n \leqslant r-1$.

Proof. The proof of the first part follows successively:

$$\begin{aligned}
& \sum_{k=0}^{n} \binom{n}{k} f(n-k)(!k) \\
= \; & \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=0}^{r} a_i (n-k)^i \right) (!k) \\
= \; & \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=0}^{r} a_i \sum_{j=0}^{i} S_2(i,j)(n-k)_j \right) (!k) \qquad \text{(from Lemma 2)} \\
= \; & \sum_{i=0}^{r} a_i \sum_{j=0}^{i} S_2(i,j) \sum_{k=0}^{n} \binom{n}{k} (n-k)_j (!k) \\
= \; & \sum_{i=0}^{r} a_i \sum_{j=0}^{i} S_2(i,j)(n)_j \sum_{k=0}^{n-j} \binom{n-j}{k} (!k) \qquad \text{(from Lemma 3)} \\
= \; & \sum_{i=0}^{r} a_i \sum_{j=0}^{i} S_2(i,j)(n)_j (n-j)! \qquad \text{(from (3))} \\
= \; & n! \sum_{i=0}^{r} a_i \sum_{j=0}^{\min\{n,i\}} S_2(i,j) \, . \qquad \text{(from Lemma 1)}
\end{aligned}$$

If $n \geqslant r$, then $\min\{n,i\} = i$, and furthermore

$$\sum_{k=0}^{n} \binom{n}{k} f(n-k)(!k) = n! \sum_{i=0}^{r} a_i \sum_{j=0}^{i} S_2(i,j) = n! \sum_{i=0}^{r} a_i B_i \, ,$$

where the relation between the Stirling numbers of the second kind and Bell numbers (6) is involved.

When $n \leqslant r-1$, it is clearly that $\min\{n,i\} = i$, for $0 \leqslant i \leqslant n$, and $\min\{n,i\} = n$, for $n+1 \leqslant i \leqslant r$, so

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} f(n-k)(!k) &= n! \left(\sum_{i=0}^{n} a_i \sum_{j=0}^{i} S_2(i,j) + \sum_{i=n+1}^{r} a_i \sum_{j=0}^{n} S_2(i,j) \right) \\ &= n! \left(\sum_{i=0}^{r} a_i B_i - \sum_{n+1 \leqslant j \leqslant i \leqslant r} a_i S_2(i,j) \right). \end{aligned}$$

The proof is now completed. □

Two direct consequences of (7) are established next.

Corollary 5. *Let $f(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0$ be a polynomial on x of degree r. For $n \geqslant 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} f(k)(!k) = n! \sum_{0 \leqslant j \leqslant i \leqslant r} a_i \binom{i}{j} (-1)^j n^{i-j} B_j \tag{8}$$

when $n \geqslant r$, and

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} f(k)(!k) &= n! \left(\sum_{0 \leqslant j \leqslant i \leqslant r} a_i \binom{i}{j} (-1)^j n^{i-j} B_j \right. \\ &\qquad \left. - \sum_{n+1 \leqslant t \leqslant j \leqslant i \leqslant r} a_i \binom{i}{j} (-1)^j n^{i-j} S_2(j,t) \right) \end{aligned} \tag{9}$$

when $n \leqslant r-1$.

Proof. We just prove (8). The proof of (9) is similar but somewhat lengthy. We can obtain (8) step by step as follows:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}f(k)(!k) &= \sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{r}a_i k^i\right)(!k)\\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{r}a_i (n-(n-k))^i\right)(!k)\\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{r}a_i \sum_{j=0}^{i}\binom{i}{j}(-1)^j n^{i-j}(n-k)^j\right)(!k)\\ &= \sum_{i=0}^{r}a_i\sum_{j=0}^{i}\binom{i}{j}(-1)^j n^{i-j}\sum_{k=0}^{n}\binom{n}{k}(n-k)^j(!k)\\ &= n!\sum_{i=0}^{r}a_i\sum_{j=0}^{i}\binom{i}{j}(-1)^j n^{i-j}B_j \qquad \text{(from (7))}\\ &= n!\sum_{0\leqslant j\leqslant i\leqslant r}a_i\binom{i}{j}(-1)^j n^{i-j}B_j\,,\end{aligned}$$

as desired. □

Applying the binomial inversion formula to Corollary 5, another identity connecting factorials and subfactorials is obtained.

Corollary 6. *Let $f(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ be a polynomial on x of degree r. For $n \geqslant 0$,*

$$f(n)(!n) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k!\sum_{0\leqslant j\leqslant i\leqslant r}a_i\binom{i}{j}(-1)^j k^{i-j}B_j \tag{10}$$

when $n \geqslant r$, and

$$\begin{aligned}f(n)(!n) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k!\left(\sum_{0\leqslant j\leqslant i\leqslant r}a_i\binom{i}{j}(-1)^j k^{i-j}B_j\right.\\ &\qquad \left.-\sum_{k+1\leqslant t\leqslant j\leqslant i\leqslant r}a_i\binom{i}{j}(-1)^j k^{i-j}S_2(j,t)\right)\end{aligned}$$

when $n \leqslant r-1$.

Remark 1. In particular, when $f(x) = 1$, (10) is reduced to (2). If we set $f(x) = x-1$, then (7) can be reduced into

$$\sum_{k=0}^{n}\binom{n}{k}(n-k-1)(!k) = n!(-B_0 + B_1) = 0\,,$$

which is an equivalent form of (4).

3. A COMBINATORIAL INTERPRETATION

When one sets $f(x) = x^r$, Theorem 4 would be reduced into

$$\sum_{k=0}^{n} \binom{n}{k} k^r (!(n-k)) = n! B_r \tag{11}$$

for $n \geqslant r$, and

$$\sum_{k=0}^{n} \binom{n}{k} k^r (!(n-k)) = n! \left(B_r - \sum_{t=n+1}^{r} S_2(r,t) \right) \tag{12}$$

for $n \leqslant r-1$. In this section, we present a unified combinatorial interpretation for both (11) and (12).

Assume that a company is holding a lottery draw. There are n staffs in this company, and r different presents are prepared (it is possibly that one person gets more than one present). Each staff possesses a lottery ticket with some number among $1, 2, \ldots, n$ (different tickets have different numbers).

The lottery draw has two rounds. In the first round, there is a box with n balls, each of which has a label among $1, 2, \ldots, n$. All the staffs draw a ball one by one from the box. If one draws a ball with the same label as his ticket, then he could go into the second round. In the second round, the winner for each present is determined by a lottery draw. Each time we place the same number of balls into the box as the number of participants in the second round, in which exactly one ball is in white, others are all in black. The present would be given to the one who draws the white ball. Let us enumerate the possible results in two ways.

Assume that there are k persons passing the first round, where $0 \leqslant k \leqslant n$, in which there are $\binom{n}{k}$ ways to determine the k persons. Observe that the remaining $n-k$ persons get a "wrong" ball, clearly there are $!(n-k)$ such ways. In the second round, each of the r presents is won by one of k persons, thus there are k^r possibilities. After k takes over from 0 to n, the left-hand side of (11) comes, as well as the left-hand side of (12).

Next, let us see how to obtain the right-hand sides of (11) and (12) uniformly. Assume that there are t winners (there are $\binom{n}{t}$ ways to choose such t winners), $0 \leqslant t \leqslant n$, that is to say, the r presents are won by the t persons, there are $t! S_2(r,t)$ ways to realize it. For the remaining $n-t$ persons, there are $(n-t)!$ possibilities in the first round (it does not matter whether a "right" ball is drawn in the first round or not, since they cannot be winners in the second round). In summary, it leads to

$$\sum_{t=0}^{n} \binom{n}{t} t! S_2(r,t)(n-t)! = n! \sum_{t=0}^{n} S_2(r,t) \,.$$

In particular,

$$\sum_{t=0}^{n} S_2(r,t) = \sum_{t=0}^{r} S_2(r,t) = B_r$$

when $n \geqslant r$, and

$$\sum_{t=0}^{n} S_2(r,t) = \sum_{t=0}^{r} S_2(r,t) - \sum_{t=n+1}^{r} S_2(r,t) = B_r - \sum_{t=n+1}^{r} S_2(r,t)$$

for $n \leqslant r-1$. At this time, the right-hand sides of (11) and (12) come.

4. HESSENBERG MATRICES WHOSE DETERMINANTS ARE DERANGEMENT NUMBERS

In this last section, we discuss a determinantal interpretation for the derangement numbers in terms of Hessenberg matrices.

Theorem 7. *Let* $f(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ *be a polynomial on* x *of degree* $r \geqslant 1$, *satisfying* $a_0 = -1$ *and* $\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} a_i B_i = 1$. *Let*

$$p_{n,k} = \binom{n}{k-1} f(n-k+1)$$

for $n \geqslant k \geqslant 1$. *Set* $A = (a_{ij})$ *be a Hessenberg matrix of order* n *such that*

$$a_{ij} = \begin{cases} -1 & \text{if } j = i-1 \\ p_{i,j} & \text{if } i \leqslant j \\ 0 & \textit{otherwise} \end{cases} .$$

Then

$$\det A = !n$$

when $n \geqslant \max\{r, 2\}$. *In particular, when* $r \geqslant 3$, *there are infinitely many constructions for such Hessenberg matrix* A.

Proof. Involving in the hypotheses $a_0 = -1$ and $\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} a_i B_i = 1$, we can change (7) into

$$\sum_{k=0}^{n-2} \binom{n}{k} f(n-k)(!k) = !n\,. \tag{13}$$

It is known that (cf. [7, 11]) if $a_1, a_2, \ldots$ is a sequence such that

$$a_{n+1} = p_{n,1}a_1 + \cdots + p_{n,n}a_n\,,$$

then

$$a_{n+1} = a_1 \det \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & p_{n,n} \end{pmatrix} .$$

Setting

$$a_n = !(n-1) \quad \text{and} \quad p_{n,k} = \binom{n}{k-1} f(n-k+1)\,,$$

then $\det A = !n$ follows immediately from (13).

Finally, it is worth mentioning that there are infinitely many $a_1, a_2, \ldots, a_r$ satisfying $\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} a_i B_i = 1$, when $r \geqslant 3$. We may regard $a_1, a_2, \ldots, a_r$ as unknown variables, $\sum_{i=1}^{r} a_i = 1$ and $\sum_{i=1}^{r} a_i B_i = 1$ as two equations, since the number of variables is more than the number of equations, there are infinitely many solutions for such system of two equations. □

Remark 2. Clearly, we may interpret this theorem in terms of the permament of an analogous Hessenberg matrix, with the subdiagonal of -1's being replaced by 1's (for more details, the reader is referred to [3]).

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