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A BERNSTEIN-SCHNABL TYPE OPERATOR: APPLICATIONS TO DIFFERENCE EQUATIONS

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We consider a sequence of positive linear operators L_n of Bernstein-Schnabl type. It was studied in the literature from various points of view; we provide new properties of it. The eigenstructure of these operators is described. We investigate the kernel of L_n which is related with the set of solutions of a difference equation. Several algorithms are proposed in order to solve the involved problems.

1. INTRODUCTION

Let $-1 \le x_{n,0} < x_{n,1} < \cdots < x_{n,n} \le 1$. For $a > 0$ and $n \ge 1$ we consider the following operators $L_n : C[-a-1, a+1] \to C[-a, a]$,

$$L_n f(x) = n![x + x_{n,0}, \ldots, x + x_{n,n}; f_n], \tag{1}$$

where $f \in C[-a-1, a+1]$, $x \in [-a, a]$, $f_n \in C^n[-a-1, a+1]$, $f_n^{(n)} = f$ and $[t_0, \ldots, t_n; g]$ is the n-th order divided difference of the function g at the points $t_0, \ldots, t_n$. These operators are of Bernstein-Schnabl type; see [**12**] and [**5**]. Others representations of L_n can be found in [**3**] and [**12**].

Their approximation properties and shape preserving properties were intensively investigated: see [**1**], [**2**], [**6**], [**10**], [**12**]. The preservation of higher order convexity was established in [**12**]. Extending results from [**6**], the authors of [**1**]

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described the rate of pointwise convergence for L_n. The complete asymptotic expansion for the operators L_n as n tends to infinity was presented in [**1**] and [**2**]. The central factorial numbers of first and second kind played an important role in the asymptotic expansion. Another essential tool was a representation for divided differences of monomials in terms of exponential complete Bell polynomials.

In [**3**] we showed that the operators L_n commute with the ordinary differential operator; consequently, they are invariant under the Kantorovich type modification. We gave explicit expressions for the images of exponentials and trigonometric functions under L_n. Moreover, we showed that for each fixed n the images of the monomials under L_n form a sequence of Appell polynomials.

In this paper we consider operators given by (1), acting on $C(\mathbb{R})$. Section 2 is devoted to the operators L_n restricted to polynomials. We describe the matrices of $L_n : \Pi_{2k} \to \Pi_{2k}$, respectively $L_n : \Pi_{2k+1} \to \Pi_{2k+1}$, with respect to the monomial bases. Here Π_m is the space of polynomial functions of degree at most m and Π denotes the space of all polynomial functions. The eigenstructure of these operators is described; in particular, the Jordan canonical form is remarkably simple. The kernel of the operator L_n acting on $C(\mathbb{R})$ is subject of study in Section 3. The kernel is important from at least two points of view. First, as we will see, it is related to the set of solutions of a difference equation. Second, a good control on the kernel is useful in order to investigate the Ulam stability of L_n; this will be the subject of a forthcoming paper. Given $T > 0$ and $p \in \Pi_m$, we provide three algorithms in order to find $q \in \Pi_{m+1}$ such that $q(x+T) - q(x) = p(x), x \in \mathbb{R}$. Together with the eigenstructure studied in Section 2, the algorithms are useful in inverting the operator $L_n : \Pi \to \Pi$ and, ultimately, in solving some difference equations.

Throughout the paper we use the notation $e_i(x) = x^i, i \in \mathbb{N}_0 = \{0, 1, \dots\}$.

2. EIGENSTRUCTURE OF THE OPERATOR L_N ACTING ON POLYNOMIALS

In this section we consider the operator $L_n : \Pi \to \Pi$,

$$L_n f(x) = n!\,[x + x_{n,0}, \dots, x + x_{n,n}; f_n]\,, \tag{2}$$

where $x \in \mathbb{R}$, $f \in \Pi$, $f_n^{(n)} = f$.

Denote $p_{n,j}(x) := (L_n e_j)(x)$.

Since $p'_{n,j} = j p_{n,j-1}$, $j \geq 1$, $p_{n,0}(x) = 1$, $p_{n,2i+1}(0) = 0$, $i \geq 0$, we get (see [**3**, (4.9)])

$$\begin{aligned}
p_{n,1}(x) &= x\\
p_{n,2}(x) &= x^2 + p_{n,2}(0)\\
p_{n,3}(x) &= x^3 + 3p_{n,2}(0)x\\
p_{n,4}(x) &= x^4 + 6p_{n,2}(0)x^2 + p_{n,4}(0)\\
&\dots
\end{aligned}$$

$$p_{n,2k}(x) = \sum_{l=0}^{k} \binom{2k}{2l} p_{n,2l}(0) x^{2k-2l}$$

$$p_{n,2k+1}(x) = \sum_{l=0}^{k} \binom{2k+1}{2l} p_{n,2l}(0) x^{2k-2l+1}.$$

With a slight abuse of notation we may consider the operators $L_n : \Pi_{2k} \to \Pi_{2k}$ and $L_n : \Pi_{2k+1} \to \Pi_{2k+1}$, $k \geq 0$. Using the above relations it is easy to determine the matrices $M_{n,2k}$ and $M_{n,2k+1}$ of these operators with respect to the monomial bases.

In both cases $\lambda = 1$ is the unique eigenvalue, with algebraic multiplicity $2k+1$, respectively $2k+2$. The eigenvectors are 1 and x. Then by straightforward calculation one determines the corresponding Jordan bases and Jordan matrices. For the sake of brevity we consider only some particular examples.

Denote $b_j := p_{n,2j}(0)$. Let $L_n : \Pi_{2k} \to \Pi_{2k}$. The associated matrix of the operator L_n is

$$M_{n,2k} = \begin{pmatrix} 1 & 0 & \binom{2}{0}b_1 & 0 & \binom{4}{0}b_2 & 0 & \dots & 0 & \binom{2k}{0}b_k \\ 0 & 1 & 0 & \binom{3}{1}b_1 & 0 & \binom{5}{1}b_2 & \dots & \binom{2k-1}{1}b_{k-1} & 0 \\ 0 & 0 & 1 & 0 & \binom{4}{2}b_1 & 0 & \dots & 0 & \binom{2k}{2}b_{k-1} \\ 0 & 0 & 0 & 1 & 0 & \binom{5}{3}b_1 & \dots & \binom{2k-1}{3}b_{k-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \binom{2k}{2k-4}b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \binom{2k-1}{2k-3}b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \binom{2k}{2k-2}b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The system which gives the coordinates of the eigenpolynomials corresponding to the eigenvalue $\lambda = 1$ is:

$$(M_{n,2k} - I_{2k+1}) \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_{2k-1} \\ t_{2k} \end{pmatrix} = O_{2k+1},$$

where I_n and O_n denote the identity matrix of size n and null vector of size n, respectively.

From the above system we get $t_i = 0,\ i = 2, \dots, 2k$. So we can choose $t_0 = 1$, $t_1 = 0$ and get the eigenpolynomial 1; then $t_0 = 0$, $t_1 = 1$ with eigenpolynomial x. Then it is easy to see that the eigenpolynomial 1 brings in the Jordan basis k even polynomials of exact degrees $2, 4, \dots, 2k$, while x brings $k-1$ odd polynomials of exact degrees $3, 5, \ldots, 2k-1$. Consequently, the Jordan matrix will consist of two Jordan cells of dimensions $k+1$, respectively k.

Consider now the case of $L_n : \Pi_{2k+1} \to \Pi_{2k+1}$. Its associated matrix is

$$M_{n,2k+1} = \begin{pmatrix} 1 & 0 & \binom{2}{0}b_1 & 0 & \binom{4}{0}b_2 & \ldots & \binom{2k}{0}b_k & 0 \\ 0 & 1 & 0 & \binom{3}{1}b_1 & 0 & \ldots & 0 & \binom{2k+1}{1}b_k \\ 0 & 0 & 1 & 0 & \binom{4}{2}b_1 & \ldots & \binom{2k}{2}b_{k-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 & \binom{2k+1}{3}b_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \binom{2k+1}{2k-3}b_2 \\ 0 & 0 & 0 & 0 & 0 & \ldots & \binom{2k}{2k-2}b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \binom{2k+1}{2k-1}b_1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Now the unique eigenvalue $\lambda = 1$ has algebraic multiplicity $2k+2$, with eigenvectors 1 and x. The eigenpolynomial 1 brings in the Jordan basis k even polynomials of exact degrees $2, 4, \ldots, 2k$, and x brings k odd polynomials of exact degrees $3, 5, \ldots, 2k+1$. Therefore, the Jordan matrix has two Jordan cells of the same dimensions $k+1$.

Example 1. *Let $L_n : \Pi_2 \to \Pi_2$. Then the associated matrix is*

$$M_{n,2} = \begin{pmatrix} 1 & 0 & p_{n,2}(0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain the Jordan basis $\left\{1, \frac{1}{p_{n,2}(0)}x^2, x\right\}$ *and the Jordan matrix* $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *with* $J^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Example 2. *Let $L_n : \Pi_3 \to \Pi_3$. Then the associated matrix is*

$$M_{n,3} = \begin{pmatrix} 1 & 0 & p_{n,2}(0) & 0 \\ 0 & 1 & 0 & 3p_{n,2}(0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain the Jordan basis $\left\{1, \frac{1}{p_{n,2}(0)}x^2, x, \frac{1}{3p_{n,2}(0)}x^3\right\}$ *and the Jordan matrix* $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *with* $J^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Example 3. *Let $L_n : \Pi_4 \to \Pi_4$. Then the associated matrix is*

$$M_{n,4} = \begin{pmatrix} 1 & 0 & p_{n,2}(0) & 0 & p_{n,4}(0) \\ 0 & 1 & 0 & 3p_{n,2}(0) & 0 \\ 0 & 0 & 1 & 0 & 6p_{n,2}(0) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain the Jordan basis

$$\left\{1, \frac{1}{p_{n,2}(0)}x^2, \frac{1}{6p_{n,2}(0)^2}x^4 - \frac{p_{n,4}(0)}{6p_{n,2}(0)^2}x^2, x, \frac{1}{3p_{n,2}(0)}x^3\right\}$$

and the Jordan matrix

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } J^{-1} = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 1. *In order to invert L_n it is useful to have also the inverses of the matrices M. For example,*

$$M_{n,2}^{-1} = \begin{pmatrix} 1 & 0 & -p_{n,2}(0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{n,3}^{-1} = \begin{pmatrix} 1 & 0 & -p_{n,2}(0) & 0 \\ 0 & 1 & 0 & -3p_{n,2}(0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{n,4}^{-1} = \begin{pmatrix} 1 & 0 & -p_{n,2}(0) & 0 & -p_{n,4}(0) + 6p_{n,2}(0)^2 \\ 0 & 1 & 0 & -3p_{n,2}(0) & 0 \\ 0 & 0 & 1 & 0 & -6p_{n,2}(0) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. THE KERNEL OF THE OPERATOR L_N ACTING ON $C(\mathbb{R})$

As we will see the kernel of the operator L_n is related to the set of solutions of some difference equations. In particular, we are interested in equations of the form

$$\Delta_T^n f(x) = g(x), \; x \in \mathbb{R}.$$

Such equations are investigated in the literature from several points of view: see, e.g., [**4**], [**8**], [**9**], [**11**].

Let $L_n : C(\mathbb{R}) \to C(\mathbb{R})$ and let $N(L_n)$ be its null space. Theorem 1 puts $N(L_n)$ in relation with the set

$$\left\{f \in C(\mathbb{R}) \,|\, \Delta_{\frac{2}{n}}^n f(x-1) = 0, \, x \in \mathbb{R}\right\}.$$

Theorem 2 describes a subset of $N(L_n)$ formed with periodic functions. Proposition 1 and Algorithms 1-4 are useful tools in this study.

By straightforward calculation one can prove

Proposition 1. *(i) Let $T>0$, $\varphi:\mathbb{R}\to\mathbb{R}$, $a:[0,T)\to\mathbb{R}$. Then $f:\mathbb{R}\to\mathbb{R}$,*

$$f(t)=\begin{cases} a(t),\ t\in[0,T),\\ a(t-kT)+\varphi(t-T)+\cdots+\varphi(t-kT),\ t\in[kT,(k+1)T),\ k\geq 1,\\ a(t+kT)-\varphi(t)-\varphi(t+T)-\cdots-\varphi(t+(k-1)T),\ t\in[-kT,-(k-1)T),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k\geq 1,\end{cases}$$

satisfies

$$f(t+T)-f(t)=\varphi(t),\ t\in\mathbb{R}. \tag{3}$$

(ii) Suppose that $n\geq 0$, $\varphi\in C^n(\mathbb{R})$, $a\in C^n[0,T)$, and there exists

$$a^{(j)}(T):=\lim_{\substack{t\to T\\ t<T}} a^{(j)}(t),\ j=0,1,\ldots,n.$$

If

$$a^{(j)}(T)-a^{(j)}(0)=\varphi^{(j)}(0), \tag{4}$$

for $j=0,1,\ldots,n$, then $f\in C^n(\mathbb{R})$.

Remark 2. *Let $m\geq 0$, $1\leq n\leq m+1$, $\varphi\in\Pi_m$, $a\in\Pi_n$. If (4) is satisfied for $j=0,1,\ldots,n-1$, then $f\in C^{n-1}(\mathbb{R})$ is a spline function, i.e., the restriction of f to each interval $[kT,(k+1)T)$, $k\in\mathbb{Z}$, is a polynomial of degree less than or equal to* $\max\{m,n\}$.

In particular, let $n=m+1$ and suppose that (4) is satisfied for $j=0,1,\ldots,m+1$. Then $f\in\Pi_{m+1}$, $\varphi\in\Pi_m$ and they satisfy (3). It is useful to remark that in fact (4) is trivially satisfied for $j=m+1$, and $a(0)$ can be chosen arbitrarily.

To resume the above discussion, given a function φ, Proposition 1 provides a function f satisfying (3).

The case when φ and f are polynomials is particularly useful for applications. In what follows we give three algorithms in order to solve the following problem: Given $p\in\Pi_m$, $T>0$, find $q\in\Pi_{m+1}$ such that

$$q(x+T)-q(x)=p(x),\ x\in\mathbb{R}. \tag{5}$$

Algorithm 1. *Let $p(x)=\sum_{i=0}^{m}\gamma_i x^i$ and $q(x)=\sum_{j=0}^{m+1}c_j x^j$. Then (5) leads to*

$$\sum_{j=i}^{m}\binom{j+1}{i}T^{j-i}c_{j+1}=\gamma_i,\ i=0,1,\ldots,m. \tag{6}$$

Therefore, $c_0\in\mathbb{R}$ is arbitrary and $c_1,\ldots,c_{m+1}$ are determined from the system (6) which has an upper triangular matrix.

Algorithm 2. *We have $q(x+T)-q(x)=p(x)$, $x\in\mathbb{R}$, i.e., $q(x+T)=q(x)+p(x)$, $x\in\mathbb{R}$. We may take $q(0)=0$. Then,*

$$\begin{aligned}&q(T)=p(0),\\&q(2T)=p(0)+p(T),\\&q(3T)=p(0)+p(T)+p(2T),\\&\dots\\&q\left((m+1)T\right)=p(0)+p(T)+\cdots+p(mT).\end{aligned}$$

We know $q(0),q(T),q(2T),\dots,q((m+1)T)$; using these $m+2$ values of q we can determine $q\in\Pi_{m+1}$ by using Lagrange interpolation, namely

$$q(x)=\frac{1}{T^{m+1}}\sum_{i=1}^{m+1}\sum_{j=0}^{i-1}p(jT)\frac{(-1)^{m+1-i}}{i!(m+1-i)!}\prod_{\substack{l=0\\l\neq i}}^{m+1}(x-lT).$$

Moreover, considering Newton interpolation we get:

$$q(x)=\sum_{i=1}^{m+1}\frac{1}{i!T^i}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{i-1-j}p(jT)\prod_{l=0}^{i-1}(x-lT).$$

Algorithm 3. *Let $p(x)=\sum_{j=0}^{m}a_jx^j\in\Pi_m$, $T>0$. The problem is to find $q\in\Pi_{m+1}$ such that $\Delta_T^1q(x)=p(x)$. Let us start with the Bernoulli polynomials $\varphi_{j+1}(t)$, $j\geq 0$, satisfying*

$$\varphi_{j+1}(t+1)-\varphi_{j+1}(t)=(j+1)t^j,\ j\geq 0,\ t\in\mathbb{R}. \tag{7}$$

For the fixed $T>0$ let $\alpha_{j+1}(x):=\varphi_{j+1}\left(\frac{x}{T}\right)$, $x\in\mathbb{R}$. Setting $t=\frac{x}{T}$ we get $\alpha_{j+1}(Tt)=\varphi_{j+1}(t)$, and $\alpha_{j+1}(Tt+T)=\varphi_{j+1}(t+1)$. It follows from (7) that

$$\alpha_{j+1}(Tt+T)-\alpha_{j+1}(Tt)=(j+1)t^j,\ \text{i.e.,}$$

$$\alpha_{j+1}(x+T)-\alpha_{j+1}(x)=\frac{j+1}{T^j}x^j,\ j\geq 0, x\in\mathbb{R}.$$

Briefly,

$$\Delta_T^1\alpha_{j+1}(x)=\frac{j+1}{T^j}x^j,\ j\geq 0,\ x\in\mathbb{R},$$

namely $x^j=\frac{T^j}{j+1}\Delta_T^1\alpha_{j+1}(x)$ and

$$p(x)=\sum_{j=0}^{m}a_jx^j=\sum_{j=0}^{m}a_j\frac{T^j}{j+1}\Delta_T^1\alpha_{j+1}(x)=\Delta_T^1\sum_{j=0}^{m}\frac{a_jT^j}{j+1}\alpha_{j+1}(x).$$

Therefore,

$$q = \sum_{j=0}^{m} \frac{a_j T^j}{j+1} \alpha_{j+1},$$

and this solves our problem.

*This is a slight extension of a known result (see [**7**], where the case $T = 1$ is presented).*

The previous three algorithms are concerned with polynomials. Now we consider a more general problem involving arbitrary functions from $C(\mathbb{R})$.

Let $n \geq 1$, $T > 0$ and $g \in C(\mathbb{R})$ be given. The problem is to find all the functions $f \in C(\mathbb{R})$ satisfying

$$\Delta_T^n f(x) = g(x),\ x \in \mathbb{R}. \tag{8}$$

We describe a procedure to solve this problem.

Algorithm 4. Step 1. *Set $\Delta_T^{n-1} f = h_1$. Then (8) becomes $\Delta_T^1 h_1 = g$. The homogeneous equation $\Delta_T^1 h_0 = 0$ has the space of solutions $C_T(\mathbb{R}) := \{h \in C(\mathbb{R}) \,|\, h \text{ is } T-\text{periodic}\}$. The space of solutions of $\Delta_T^1 h_1 = g$ is $C_T(\mathbb{R}) + u_1$, where u_1 is a particular solution (i.e., $\Delta_T^1 u_1 = g$) constructed as in Proposition 1.*

Step 2. *For each $h_1 \in C_T(\mathbb{R}) + u_1$, we have to solve the equation $\Delta_T^{n-1} f = h_1$. Set $\Delta_T^{n-2} f = h_2$; then $\Delta_T^1 h_2 = h_1$, and its space of solutions is $C_T(\mathbb{R}) + u_2$, with $\Delta_T^1 u_2 = h_1$, u_2 constructed with the help of Proposition 1.*

Going on in the same way, at Step n we will find all the solutions f to (8).

In the sequel we consider the operators $L_n : C(\mathbb{R}) \to C(\mathbb{R})$,

$$L_n f(x) := n! \left[x-1, x-1+\frac{2}{n}, \ldots, x+1 \right] f_n,\ x \in \mathbb{R}, \tag{9}$$

and $D_n : C(\mathbb{R}) \to C(\mathbb{R})$,

$$D_n f(x) := \Delta_{\frac{2}{n}}^n f(x-1) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f\left(x - 1 + \frac{2i}{n}\right),\ x \in \mathbb{R}. \tag{10}$$

Therefore we have

$$L_n f(x) = \left(\frac{n}{2}\right)^n \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_n\left(x - 1 + \frac{2i}{n}\right) = \left(\frac{n}{2}\right)^n D_n f_n(x). \tag{11}$$

Let $N(L_n) := \{f \in C(\mathbb{R}) \,|\, L_n f = 0\}$ and similarly can be defined $N(D_n)$.

We are interested in the relationship between the null spaces of L_n and D_n. First of all, let us remark that $N(D_n)$ can be described by using Algorithm 4.

We can use also the following

Remark 3. *Algorithms 1-3 are useful when we want to invert $L_n : \Pi \to \Pi$. Indeed, given $u \in \Pi_m$ we need to find the unique $v \in \Pi_m$ with $L_n v = u$, i.e. $\left(\frac{n}{2}\right)^n \Delta^n_{\frac{2}{n}} v_n(x-1) = u(x)$, $x \in \mathbb{R}$, where $v_n^{(n)} = v$. With an obvious change of notation this reduces to $\Delta^n_T v_n(t) = r(t) \in \Pi_m$, i.e. $\underbrace{\Delta^1_T \Delta^1_T \dots \Delta^1_T}_{n \text{ times}} v_n = r \in \Pi_m$. Applying n times one of the Algorithms 1-3, we get $v_n \in \Pi_{m+n}$, and finally $v = v_n^{(n)} \in \Pi_m$.*

Now we are in the position to prove

Theorem 1. *For $n \geq 1$ one has*

$$N(D_n) = N(L_n) + \Pi_{n-1}. \tag{12}$$

Proof. Let $f \in N(L_n)$. Then

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_n \left(x - 1 + \frac{2i}{n}\right) = 0, \; x \in \mathbb{R}.$$

Taking the n-th derivative we see that $D_n f = 0$, and so $N(L_n) \subset N(D_n)$. Since $\Pi_{n-1} \subset N(D_n)$, it follows that $N(L_n) + \Pi_{n-1} \subset N(D_n)$.

Now let $f \in N(D_n)$. We start by describing a constructive procedure in order to find $L_n f$. First,

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f \left(x - 1 + \frac{2i}{n}\right) = 0, \; x \in \mathbb{R},$$

implies

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_1 \left(x - 1 + \frac{2i}{n}\right) = C_0, \quad (C_0 \text{ is a constant}),$$

where

$$C_0 = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_1 \left(-1 + \frac{2i}{n}\right) = \left(\frac{2}{n}\right)^n n! \left[-1, -1 + \frac{2}{n}, \cdots, 1; f_1\right].$$

Next,

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_2 \left(x - 1 + \frac{2i}{n}\right) = C_0 x + C_1,$$

where

$$C_1 = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_2 \left(-1 + \frac{2i}{n}\right) = \left(\frac{2}{n}\right)^n n! \left[-1, -1 + \frac{2}{n}, \cdots, 1; f_2\right].$$

Step by step we find that

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f_n\left(x-1+\frac{2i}{n}\right)=C_0x^{n-1}+\cdots+C_{n-2}x+C_{n-1}, \tag{13}$$

where

$$C_j=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f_{j+1}\left(-1+\frac{2i}{n}\right)=\left(\frac{2}{n}\right)^n n!\left[-1,-1+\frac{2}{n},\cdots,1;f_{j+1}\right].$$

Now (11) and (13) show that

$$\begin{aligned} L_nf(x)&=\left(\frac{n}{2}\right)^n\left(C_0x^{n-1}+\cdots+C_{n-2}x+C_{n-1}\right)\\ &=n!\sum_{j=0}^{n-1}x^{n-1-j}\left[-1,-1+\frac{2}{n},\ldots,1;f_{j+1}\right]. \end{aligned} \tag{14}$$

Briefly, $L_nf\in\Pi_{n-1}$. Then we can find (see, e.g., Remark 3) $p\in\Pi_{n-1}$ such that $L_np=L_nf$. This entails $f-p\in N(L_n)$, and so $f\in N(L_n)+\Pi_{n-1}$. Therefore $N(D_n)\subset N(L_n)+\Pi_{n-1}$, and the proof is finished. □

Corollary 1. *Let $n\geq 1$. Then $f\in N(D_n)$ if and only if $L_nf\in\Pi_{n-1}$.*

Proof. The relation (14) shows that if $f\in N(D_n)$, then $L_nf\in\Pi_{n-1}$. So let $L_nf=q\in\Pi_{n-1}$. Then

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f_n\left(x-1+\frac{2}{n}\right)=\left(\frac{2}{n}\right)^n q(x),\ x\in\mathbb{R}.$$

Taking the n-th derivative we conclude that $\Delta^n_{\frac{2}{n}}f(x-1)=0$, $x\in\mathbb{R}$, and so $f\in N(D_n)$. □

Theorem 2. *Let $\varphi\in C(\mathbb{R})$ be periodic, with period $\frac{2}{n}$, and satisfying*

$$\int_{-1}^{1}\cdots\int_{-1}^{1}\varphi\left(\frac{t_1+\cdots+t_n}{n}\right)dt_1\ldots dt_n=0. \tag{15}$$

Then $\varphi\in N(L_n)$.

Proof. Since $g_1(t_1):=\varphi\left(\frac{t_1+t_2+\cdots+t_n}{n}\right)$ is 2-periodic as a function of t_1, it follows

$$\int_{x-1}^{x+1}f\left(\frac{t_1+\cdots+t_n}{n}\right)dt_1=\int_{x-1}^{x+1}\varphi\left(\frac{t_1+\cdots+t_n}{n}\right)dt_1=\int_{-1}^{1}\varphi\left(\frac{t_1+\cdots+t_n}{n}\right)dt_1.$$

Moreover, $g_2(t_2) := \int_{-1}^{1} \varphi\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1$ is 2-periodic as a function of t_2. Then,

$$\int_{x-1}^{x+1}\int_{x-1}^{x+1} f\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1dt_2 = \int_{x-1}^{x+1}\left(\int_{-1}^{1} \varphi\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1\right) dt_2$$
$$= \int_{-1}^{1}\int_{-1}^{1} \varphi\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1dt_2.$$

Step by step we get finally

$$L_nf(x) = 2^{-n}\int_{-1}^{1}\cdots\int_{-1}^{1} \varphi\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1\dots dt_n = 0,\ x\in\mathbb{R}.$$

□

Remark 4. *Let $\Phi \in C(\mathbb{R})$, $\frac{2}{n}$-periodic, $I := \int_{-1}^{1}\cdots\int_{-1}^{1} \Phi\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1\dots dt_n$. Then*
$\varphi := \Phi - \frac{I}{2^n}$ is $\frac{2}{n}$-periodic and $\int_{-1}^{1}\cdots\int_{-1}^{1} \varphi\left(\frac{t_1+\cdots+t_n}{n}\right) dt_1\dots dt_n = 0$.

Remark 5. *The converse of Theorem 2 is true for $n = 1$, but not for $n = 2$. Indeed, if $\varphi \in N(L_1)$ then $\frac{1}{2}\int_{x-1}^{x+1}\varphi(t)dt = 0$, $x \in \mathbb{R}$. Taking the derivative we get $\varphi(x+1) - \varphi(x-1) = 0$, $x \in \mathbb{R}$, so that φ is periodic with period 2. Setting $x = 0$ yields $\int_{-1}^{1}\varphi(t)dt = 0$. Now let $n = 2$, and $\varphi(t) := (k+1)\sin^2\pi t$, $t \in [k, k+1)$, $k \in \mathbb{Z}$.*

Then $\varphi(t+1) - \varphi(t) = \sin^2\pi t, t \in \mathbb{R}$, and consequently $\varphi(t+1) - 2\varphi(t) + \varphi(t-1) = 0$, $t \in \mathbb{R}$. Integrating twice we get

$$\varphi_2(t+1) - 2\varphi_2(t) + \varphi_2(t-1) = p(t),\ p \in \Pi_1.$$

According to (11), $L_2\varphi = p$. But $L_2p = p$, so that $\varphi - p \in N(L_2)$. Using (14) we obtain $p(t) = \frac{1}{2}t + \frac{1}{4}$. It follows that $\varphi - p$ is not 1-periodic.

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