Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **16** (2022), 508–523.
https://doi.org/10.2298/AADM210930019B

THE EXTENDED EULERIAN NUMBERS OVER FUNCTION FIELDS

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In this article, we introduce the extended Eulerian numbers for a large class of zeta functions, which includes the zeta functions associated to function fields, and to schemes over finite fields. This construction generalizes the extended Eulerian numbers defined by Carlitz. We give an asymptotic expansion for the summatory function associated to these numbers. Our main result generalizes the well known result on the asymptotic behavior of the extended Eulerian numbers associated to the Riemann zeta function.

1. INTRODUCTION

Carlitz in [**5**] introduced and investigated the following extended Eulerian numbers $(H(n,\lambda))_{n\in\mathbb{N}_{\geq 1}}$ which are defined by means of the generating function

$$\frac{1-\lambda}{\zeta(s)-\lambda}=\sum_{n=1}^{\infty}\frac{H(n,\lambda)}{n^s}$$

where $\zeta(s)$ is the Riemann zeta function.

Let $H(p_1\cdots p_r,\lambda) := H_r(\lambda)$, where $n = p_1\cdots p_r$ is a product of distinct primes. Then $H_r(\lambda)$ are generated by the following function

$$\frac{1-\lambda}{e^t-\lambda}=\sum_{r=0}^{\infty}H_r(\lambda)\frac{t^r}{r!}.$$

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2020 Mathematics Subject Classification: Primary 11A25, 11N37 Secondary: 11G40; 11R42.
Keywords and Phrases: Eulerian numbers, Function fields, Hasse-Weil zeta function.

which, upon setting $\lambda = -1$, generates the usual Eulerian numbers [**4**].

We put,

$$(\log \zeta(s))^k = \sum_{n=1}^{\infty} \frac{\alpha_k(n)}{n^s} \quad \text{for } k = 1, 2, \dots, \text{ and } \mathrm{Re}(s) \gg 1.$$

We have $\alpha_k(n) = \sum_{d_1 \cdots d_k = n} \alpha_1(d_1) \cdots \alpha_1(d_k)$. So,

$$H(n, \lambda) = \sum_{k=0}^{\Omega(n)} H_k(\lambda) \frac{\alpha_k(n)}{k!},$$

where $\Omega(n) = \sum a$, if $n = \prod p^a$ (see [**5**, (3.12)]). Note that, by Euler's product of ζ, we know that $\alpha_1(n) = \frac{1}{r}$ if $n = p^r$ with $r \geq 1$ and $\alpha_1(n) = 0$ otherwise. Further algebraic properties of the extended Eulerian numbers are given in [**5**], most of which are generalizations of the corresponding properties of Eulerian numbers.

Let us consider the case of $\lambda = 2$. For any positive integer n, let $f(n)$ be the number of representations of n as a product of factors greater than one. We have,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{2 - \zeta(s)},$$

The summatory function $F(n) = \sum_{m=0}^{n} f(m)$ was studied by Kalmàr in [**13**]. He proved that

$$F(n) = -\frac{n^\rho}{\rho \zeta'(\rho)}(1 + o(1)), \quad \forall n \gg 1$$

where ρ is the positive root of the equation $\zeta(s) = 2$. In [**10**], Hille extended this result to multiplicative systems of primes.

For $\lambda \in \mathbb{R} \setminus \{1\}$, Evans [**7**] proved an asymptotic expansion for the extended Eulerian numbers $H(n, \lambda)$ in terms of a_i where $n = \prod_i p_i^{a_i}$ is the prime factorisation of the integer n. Moreover, Grosswald [**9**] established the following asymptotic expansion

$$\text{(1)} \qquad \sum_{n \leq x}^{*} H(n, \lambda) = (1 - \lambda) \frac{x^\rho}{\rho \zeta'(\rho)}(1 + o(1)) \quad \text{for } \lambda > 1,$$

where ρ is the unique real root of $\zeta(s) = \lambda$, where the $*$ indicates that when x is an integer the final summand is $\frac{1}{2}H(x, \lambda)$. The behavior of the functions $\lambda \mapsto C(\lambda) = \frac{1-\lambda}{\rho(\lambda)\zeta'(\rho(\lambda))}$ and $\lambda \mapsto \rho(\lambda)$ is studied in [**3**].

It is well known that many results which hold for $\mathbb{Z}$ have their analogue of $\mathbb{F}[T]$, the ring of polynomials in one variable with coefficients in a finite field $\mathbb{F}$. In [**15**], Weil described the analogy between the function fields and the number fields.

In this article, we generalize the construction of Carlitz to a large class of zeta functions. Our construction gives a partial answer to a question of Hille (see [**10**, §6.]).

Let G be a $\mathbb{Z}$-module, and $\mathcal{B}$ a subset of G which we shall call the set of prime elements of G, and we assume that G has a norm-map N satisfying a finiteness condition. For example, we can take G to be the Picard group of a Dedekind domain, and N is the usual norm-map. To G, we associate a zeta function

$$\zeta_G(s) = \sum_{\mathfrak{a}\in G^+} \frac{1}{N(\mathfrak{a})^s} \quad \forall\ \mathrm{Re}(s) \gg 1,$$

where G^+ is the set of non-zero elements of G which can be written as linear combinations of elements of $\mathcal{B}$ with coefficients in $\mathbb{N}$. For instance when $G = \mathbb{Z}$ and $\mathcal{B}$ is the set of primes numbers, we obtain that

$$\zeta_G(s) = \sum_{n\in\mathbb{N}_{\geq 1}} \frac{1}{n^s},$$

is the Riemann zeta function. If $G = \mathbb{F}[T]$ and $\mathcal{B}$ is the set of irreducible monic polynomials, and N is the map defined by $N(f) = q^{\deg f}$ for any $f \in \mathbb{F}[T]$ (q is the cardinal of $\mathbb{F}$), we have

$$\zeta_G(s) = \sum_{f \text{ monic } \in\mathbb{F}[T]} \frac{1}{N(f)} = \sum_{n\in\mathbb{N}} \frac{b_n}{q^{ns}}$$

where b_n is the number of monic polynomials of degree n. This zeta function is an analogue of the classical zeta function which was first introduced by Euler and then Riemann. In this case, this zeta function is a much simpler object. Using the Riemann-Roch theorem for curves over finite fields, we can show that the function $\zeta_G(s)$ is a rational function. As a consequence one easily obtains a sharp version of the prime number theorem for polynomials without the need for any complicated analytic investigations.

Let $\lambda \in \mathbb{C} \setminus \{1\}$. We define the Extended Eulerian numbers of ζ_G, denoted by $H_G(n, \lambda)$ as follows

$$\frac{\lambda - 1}{\lambda - \zeta_G(s)} = \sum_{n\in\mathbb{N}_{\geq 1}} \frac{H_G(n, \lambda)}{n^s} \quad \text{for any } \ \mathrm{Re}(s) \gg 1.$$

It is worth noting that when $\zeta_G = \zeta_K$ where K is an arbitrary number field of degree ≥ 2, nothing is known on the behavior of $H_K(n, \lambda)$. This is due to the fact that it is difficult to find a closed formula for the $d_{G,k}(\mathfrak{a})$ in this case (see the definition in Section **2** and equation (7)).

In the sequel, we suppose that there exists a $\mathbb{Z}$-linear map $\deg : G \to \mathbb{Z}$ which is positive on $\mathcal{B}$, such that $N(\mathfrak{a}) = q^{\deg \mathfrak{a}}$ for all $\mathfrak{a} \in G$ where $q > 1$. For instance, one can consider G to be the ring of integers of a function field. We put $t = q^{-s}$, and we set

$$Z_G(t) := \zeta_G(s) = \sum_{n=0}^{\infty} b_n t^n, \tag{2}$$

where $b_n = \#\{\mathfrak{a} \mid \deg \mathfrak{a} = n\}$. We define the extended Eulerian numbers of Z_G, denoted by $\mathcal{H}_G(n, \lambda)$, by the following equation,

$$\frac{\lambda - 1}{\lambda - Z_G(t)} = \sum_{n=1}^{\infty} \mathcal{H}_G(n, \lambda) t^n.$$

By construction, it follows that

$$H_G(q^n, \lambda) = \mathcal{H}_G(n, \lambda) \quad \forall n \in \mathbb{N}.$$

We define the summatory function of G as follows,

$$\sum_{n \leq x} \mathcal{H}_G(n, \lambda), \quad \forall x \in \mathbb{N}. \tag{3}$$

The goal of this article is to study asymptotic behavior of this summatory function when x tends to infinity. When $\lambda = 0$, then $\mathcal{H}_G(n, 0)$ can be seen as an analogue of the Möbius function $\mu(n)$, and (3) as an analogue of the Mertens function $M(x)$ associated to the Riemann zeta function. It is known that the Riemann Hypothesis is equivalent to the following assertion:

$$M(x) = O(x^{\frac{1}{2}+\epsilon}), \quad \forall \epsilon > 0.$$

Theorem 1 (Theorem 13)**.** *We keep the same notations as above. Assume that* $\{n \in \mathbb{N} \mid b_n \neq 0\}$ *contains an arithmetic progression of the form* $\{ck + d | k \in \mathbb{N}\}$ *with* $\gcd(c, d) = 1$. *Then, for any* $x \in \mathbb{N}$ *large enough,*

$$\sum_{1 \leq n \leq x} \mathcal{H}_G(n, \lambda) = (1 - \lambda) \frac{(\log q)^{m_{s_\lambda}} m_{s_\lambda} (-1)^{m_{s_\lambda} - 1}}{(1 - q^{-s_\lambda}) s_\lambda \zeta^{(m_{s_\lambda})}(s_\lambda)} q^{x s_\lambda} x^{m_{s_\lambda} - 1} (1 + o(x)). \tag{4}$$

where s_λ *is the positive root of the equation* $\zeta_G(s) = \lambda$ *where* m_{s_λ} *is its multiplicity.*

Note that if we put $X = q^x$, then (4) becomes,

$$\sum_{q \leq N \leq X} H_G(N, \lambda) = C(\lambda) X^{s_\lambda} (\log_q X)^{m_{s_\lambda} - 1} (1 + o(X)) \quad X \gg 1,$$

with $C(\lambda) = (1 - \lambda) \frac{(\log q)^{m_{s_\lambda}} m_{s_\lambda} (-1)^{m_{s_\lambda} - 1}}{(1 - q^{-s_\lambda}) s_\lambda \zeta^{(m_\lambda)}(s_\lambda)}$. Thus, we have obtained a generalization of (1).

Remark 2. *Note that this theorem holds for any $\lambda \in \mathbb{R}\setminus\{1\}$, which is not the case when one considers the classical extended Eulerian numbers.*

In this article, $\mathbb{N}$ denotes $\{0,1,2,\ldots\}$. For $x \in \mathbb{R}$, we denote by $[x]$ the largest integer $\leq x$.

2. GENERAL CONSTRUCTION

In the sequel G will be a $\mathbb{Z}$-module which satisfies the following conditions:

1. There exists $\mathcal{B}$, a subset of G such that any element in G can be written in a unique way as a $\mathbb{Z}$-linear combination of elements in $\mathcal{B}$. We define G^+ to be the set of elements in G with non-negative coefficients.
2. There exists a map $N : G \to \mathbb{Q}$ which we call the norm-map satisfying $N(a+b) = N(a)N(b)$ for every $a, b \in G$, and $N(\mathcal{B}) \subset \mathbb{N}$.
3. The set $\{\mathfrak{a} \in G^+ \mid N(\mathfrak{a}) \leq n\}$ is finite for any $n \in \mathbb{N}$.

To such G we associate,

$$\zeta_G(s) = \sum_{\mathfrak{a}\in G^+} \frac{1}{N(\mathfrak{a})^s} \qquad \mathrm{Re}(s) \gg 1.$$

We assume that there exists $c > 0$ such that $\zeta_G(s)$ converges for $\mathrm{Re}(s) > c$ and it admits a meromorphic continuation to $\mathbb{C}$ with finitely many poles. We call ζ_G the zeta function of G. We denote also this continuation by ζ_G. We assume that c is a pole of ζ_G. Note that

$$\lim_{s>0, s\to\infty} \zeta_G(s) = b_0.$$

(see (2) for the definition of b_0).

Let $\lambda > b_0$. There exists a unique real number $s_\lambda > c$ such that $\zeta_G(s_\lambda) = \lambda$. We define the Möbius function μ_G of ζ_G as follows: Let $\mathfrak{a} = \sum'_{\mathfrak{p}\in\mathcal{B}} a_{\mathfrak{p}}\mathfrak{p}$ be an element of G^+ (where $'$ means that the sum if finite) then we set

$$\mu_G(\mathfrak{a}) = \begin{cases} 0 & \text{if } \exists\, \mathfrak{p} \in \mathcal{B} \text{ with } a_{\mathfrak{p}} \geq 2 \\ (-1)^r & \text{if } a_{\mathfrak{p}} = 1 \text{ for any } \mathfrak{p} \text{ such that } a_{\mathfrak{p}} \neq 0. \end{cases} \tag{5}$$

Let

$$\mu_{G,k}(\mathfrak{a}) := \sum_{\mathfrak{a}_1+\ldots+\mathfrak{a}_k=\mathfrak{a}, \mathfrak{a}_i\geq 0} \mu_G(\mathfrak{a}_1)\cdots\mu_G(\mathfrak{a}_k),$$

and

$$d_{G,k}(\mathfrak{a}) := \#\{(\mathfrak{a}_1,\ldots,\mathfrak{a}_k) \in (G^+)^k \mid \mathfrak{a}_1+\ldots+\mathfrak{a}_k = \mathfrak{a}\}.$$

Then, for any $\operatorname{Re}(s) \gg 1$ and $k \in \mathbb{N}$

$$\zeta_G(s)^k = \sum_{\mathfrak{a} \in G^+} \frac{d_{G,k}(\mathfrak{a})}{N(\mathfrak{a})^s} \quad \text{and} \quad \frac{1}{\zeta_G(s)^k} = \sum_{\mathfrak{a} \in G^+} \frac{\mu_{G,k}(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

Let $\lambda \neq 1$, for $\mathfrak{a} \in G^+$ we define $H_G(\mathfrak{a}, \lambda)$ as the coefficients of the following series

$$\frac{\lambda - 1}{\lambda - \zeta_G(s)} = \sum_{\mathfrak{a} \in G^+} \frac{H(\mathfrak{a}, \lambda)}{N(\mathfrak{a})^s} \quad \forall \operatorname{Re}(s) \gg 1.$$

Lemma 3. *Set, for $|\lambda| < 1$,*

$$H_G(n, \lambda) := \sum_{k=0}^{\infty} \left(\sum_{N(\mathfrak{a})=n} \mu_{G,k}(\mathfrak{a}) \right) \lambda^k (1 - \lambda). \tag{6}$$

and for $|\lambda| > 1$,

$$H_G(n, \lambda) := \sum_{k=0}^{\infty} \left(\sum_{N(\mathfrak{a})=n} d_{G,k}(\mathfrak{a}) \right) \lambda^{-k} (1 - \lambda). \tag{7}$$

Then these series converge absolutely. We have

$$\frac{\lambda - 1}{\lambda - \zeta_G(s)} = \sum_{n \in \mathbb{N}_{\geq 1}} \frac{H_G(n, \lambda)}{n^s} \quad \forall \lambda \in \mathbb{C} \setminus \{1\}.$$

and

$$\sum_{\mathfrak{a} \in G^+, N(\mathfrak{a}) \leq y} H_G(\mathfrak{a}, \lambda) = \sum_{n \leq y} H_G(n, \lambda) \quad \forall y \geq 0 \tag{8}$$

Proof. Let $n \geq 1$, we have

$$\begin{aligned} \left| \sum_{N(\mathfrak{a})=n} \mu_{G,k}(\mathfrak{a}) \right| &= \left| \sum_{N(\mathfrak{a})=n} \sum_{\mathfrak{a}_1 + \ldots + \mathfrak{a}_k = \mathfrak{a}} \mu_G(\mathfrak{a}_1) \cdots \mu_G(\mathfrak{a}_k) \right| \\ &\leq \sum_{N(\mathfrak{a})=n} d_{G,k}(\mathfrak{a}), \end{aligned}$$

By assumption, the set $\{\mathfrak{a} \in G^+ \mid N(\mathfrak{a}) \leq n\}$ is finite, then the term in the preceding inequality is bounded from above by a constant which depends only on n. It follows that (6) converges normally. The rest of the lemma is a straightforward exercise. □

When $G = \mathbb{Z}$, we get the usual Eulerian numbers introduced by Carlitz in [**5**]. More generally, if $G = \mathrm{Pic}(\mathcal{O}_K)$ where K is a Dedekind domain, then G possesses a norm-map and it satisfies the finiteness condition. We can then associate to G a zeta function ζ_G as above. This function is known as the Dedekind zeta function.

In the sequel, we suppose that there exists a $\mathbb{Z}$-linear map $\deg : G \to \mathbb{Z}$ which is positive on $\mathcal{B}$ called a degree map of G and satisfies $\deg \mathfrak{a} > 0$ if $\mathfrak{a} \in G^+ \setminus \{0\}$. We assume that $N(\mathfrak{a}) = q^{\deg \mathfrak{a}}$ for every $\mathfrak{a} \in G^+$ where q is a positive real number > 1. We put $t = q^{-s}$ and we set

$$Z_G(t) := \zeta_G(s).$$

We have

$$Z_G(t) = \sum_{n=0}^{\infty} b_n t^n, \quad |t| \ll 1,$$

where $b_n = \#\{\mathfrak{a} \in G^+ \mid \deg \mathfrak{a} = n\}$.

We put $\rho_\lambda = q^{-s_\lambda}$. We assume that Z_G admits a meromorphic continuation to the whole complex plane, which we denote by the same notation. We suppose that there exists $c_0 > 0$ and $\chi \in \mathbb{R}$ such that

$$|Z_G(t)| \sim c_0 R^{\chi} \tag{9}$$

as $R = |t| \to \infty$. We set for any $\lambda \neq 1$,

$$\mathcal{H}_G(n, \lambda) := H_G(q^n, \lambda).$$

So that,

$$\frac{\lambda - 1}{\lambda - \zeta_G(s)} = \frac{\lambda - 1}{\lambda - Z_G(t)} = \sum_{n \in \mathbb{N}} \mathcal{H}_G(n, \lambda) t^n,$$

and (8) becomes

$$\sum_{\mathfrak{a} \in G^+, \deg \mathfrak{a} \leq x} \mathcal{H}_G(\mathfrak{a}, \lambda) = \sum_{n \leq x} \mathcal{H}_G(n, \lambda).$$

For any R, we put $\mathrm{C}_R := \{z \in \mathbb{C} \mid |z| = R\}$.

Claim 4. *For any $n \in \mathbb{N}$,*

$$\mathcal{H}_G(n, \lambda) = \mathrm{Res}\left(\frac{\lambda - 1}{t^{n+1}(\lambda - Z_G(t))}, t = 0 \right)$$

and for $R \gg |\lambda|$,

$$\frac{1}{2\pi\sqrt{-1}} \sum_{|\rho| < R} \mathrm{Res}\left(\frac{\lambda - 1}{t^{n+1}(\lambda - Z_G(t))}, t = \rho \right) = \int_{C_R} \frac{\lambda - 1}{t^{n+1}(\lambda - Z_G(t))} dt, \tag{10}$$

where the sum runs over the zeros of $t^{n+1}(\lambda - Z_G(t))$.

Proof. This follows from the Cauchy theorem. □

Claim 5. *Fix λ. For any $R \gg |\lambda|$ and for any $n \in \mathbb{N}$, we have*

$$\left|\int_{C_R} \frac{\lambda-1}{t^{n+1}(\lambda - Z_G(t))}dt\right| \leq \frac{c_2}{R^n}\frac{|\lambda-1|}{||\lambda| - c_1 R^{\chi}|}$$

where c_1 and c_2 are two positive constants which do not depend on R.

Proof. Let $R \gg |\lambda|$, we have

$$\left|\int_{C_R} \frac{\lambda-1}{t^{n+1}(\lambda - Z_G(t))}dt\right| \leq \frac{1}{R^{n+1}}\int_{C_R} \frac{|\lambda-1|}{||\lambda| - |Z_G(t)||}|dt|.$$

By (9), we can find a constant c_1 such that $||\lambda|-|Z_G(t)|| \geq ||\lambda| - c_1 R^{\chi}|$ for $|t| \gg 1$. The claim follows easily.

□

Claim 6. *Let ρ be a zero of $\lambda - Z_G(t)$*[a]*. We have, for any $n \in \mathbb{N}$,*

$$\operatorname{Res}\Big(\frac{\lambda-1}{t^{n+1}(\lambda - Z_G(t))}, \rho\Big) = \frac{m_\rho!(-1)^{m_\rho-1}}{Z_G^{(m_\rho)}(\rho)}\rho^{-(n+m_\rho)}\binom{n+m_\rho-1}{m_\rho-1}$$

where m_ρ is the order of ρ and $Z_G^{(m_\rho)}(t)$ is the m_ρ-th derivative of $Z_G(t)$.

Proof. For $|t-\rho| \ll 1$, we have $\frac{1}{\lambda - Z_G(t)} = \frac{m_\rho!}{Z_G^{(m_\rho)}(\rho)}(t-\rho)^{-m_\rho} + (\text{an entire function})$ and $t^{-(n+1)} = \rho^{-(n+1)}\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}\frac{(t-\rho)^k}{\rho^k}$. Then

$$\operatorname{Res}\left(\frac{\lambda-1}{t^{n+1}(\lambda - Z_G(t))}, \rho\right) = (\lambda-1)\frac{m_\rho!(-1)^{m_\rho-1}}{Z_G^{(m_\rho)}(\rho)}\rho^{-(n+m_\rho)}\binom{n+m_\rho-1}{m_\rho-1}. \tag{11}$$

□

Corollary 7. *For any $n \geq \chi + 1$, we have*

$$\mathcal{H}_G(n,\lambda) = -(\lambda-1)\sum_{\rho|Z(\rho)=\lambda}\frac{m_\rho!(-1)^{m_\rho-1}}{Z_G^{(m_\rho)}(\rho)}\rho^{-(n+m_\rho)}\binom{n+m_\rho-1}{m_\rho-1},$$

Proof. The equation (10) does not depend on $R \gg 1$. The right hand side of the inequality of Claim 5 converges to 0 as R tends to ∞ for any $n \geq \chi+1$. The equality follows from Claim 4 and 6. □

Lemma 8. *Suppose that the set $\{n \in \mathbb{N} | b_n \neq 0\}$ contains an arithmetic progression of the following form $\{ck+d | k \in \mathbb{N}\}$ with $\gcd(c,d)=1$. If $\zeta_G(s) = \lambda$ with $\lambda > b_0$ then $\operatorname{Re}(s) \leq s_\lambda$. Moreover, if $\operatorname{Re}(s) = s_\lambda$ then $\operatorname{Im}(s) \in \frac{1}{\log q}\mathbb{Z}$.*

[a] Note that $\rho \neq 0$, because $\lambda > b_0$.

Proof. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$ and such that $\zeta(s) = \lambda$. We set $\rho = q^{-s}$. If $\mathrm{Re}(s) > s_\lambda$, then

$$\lambda = \zeta_G(s) \leq \zeta_G(\mathrm{Re}(s)) < \zeta_G(s_\lambda) = \lambda.$$

This is impossible. If $\mathrm{Re}(s) = s_\lambda$, we write $s = s_\lambda + it$ with $t \in \mathbb{R}$. We have

$$\lambda = Z_G(\rho) = \sum_{n=0}^{\infty} b_n q^{-ns_\lambda - int} = \sum_{n=0}^{\infty} b_n q^{-ns_\lambda} \cos(nt \log q),$$

then $\cos(nt \log q) = 1$ for any n such that $b_n \neq 0$. By assumption, we can show that $t \in \frac{2\pi}{\log q}\mathbb{Z}$.

□

Theorem 9. *There exists $c(\lambda)$ a constant, such that,*

$$\lim_{n\to\infty} \frac{\mathcal{H}_G(n,\lambda)}{\rho_\lambda^{-n} n^{m_{\rho_\lambda}-1}} = c(\lambda), \tag{12}$$

If we set $X = q^n$ and $H_G(X,\lambda) := \mathcal{H}_G(n,\lambda)$, then (12) *becomes*

$$\lim_{X\to\infty} \frac{H_G(X,\lambda)}{X^{s_\lambda}(\log X)^{m_{s_\lambda}-1}} = c(\lambda) \tag{13}$$

Remark 10. *Equation* (13) *can be seen as an analogue of [**10**, (5.1) p. 140]. Note that Hille deduced his equation (5.1) from (4.3) in [**10**].*

Proof of Theorem 9. For $n \gg 1$, and by Corollary 7, we have

$$\begin{aligned}\mathcal{H}_G(n,\lambda) =&(\lambda-1)\sum_{Z_G(\rho)=\lambda} \frac{m_\rho(-1)^{m_\rho-1}}{Z_G^{(m_\rho)}(\rho)}\rho^{-(n+m_\rho)}n^{m_\rho-1} + O(n^{m_\rho-2})\\ =&(\lambda-1)\sum_{Z_G(\rho)=\lambda,|\rho|<1} \frac{m_\rho(-1)^{m_\rho-1}}{Z_G^{(m_\rho)}(\rho)}\rho^{-(n+m_\rho)}n^{m_\rho-1} + O(n^{m_\rho-2}).\end{aligned}$$

By Lemma 8, if $Z_G(\rho) = \lambda$ with $|\rho| < 1$ and $\rho \neq \rho_\lambda$, then $|\rho| < \rho_\lambda$. We conclude the following

$$\lim_{n\to\infty} \frac{\mathcal{H}_G(n,\lambda)}{\rho_\lambda^{-n} n^{m_{\rho_\lambda}-1}} = (\lambda-1)\frac{m_{\rho_\lambda}(-1)^{m_\rho-1}}{\rho_\lambda^{m_{\rho_\lambda}} Z_G^{(m_{\rho_\lambda})}(\rho_\lambda)}.$$

□

Lemma 11. *Let f be a non-constant analytic function defined on an open neighborhood of 0 in $\mathbb{C}$. Suppose that $f^{(k)}(0) \geq 0$ for all $k \in \mathbb{N}$. Set $g(t) = \exp(f(t))$. Then, there exists $k_0 \in \mathbb{N}$ such that*

$$g^{(sk_0+k_0-1)}(0) \neq 0 \quad \forall s \in \mathbb{N}.$$

In particular, if we write $g(t) = \sum_{n=0}^{\infty} b_n t^n$ for $|t| \ll 1$, then $k_0\mathbb{N} + k_0 - 1 \subset \{n \in \mathbb{N} | b_n \neq 0\}$.

Proof. Let k_0 be the smallest integer satisfying $f^{(k_0)}(0) \neq 0$. For $k \geq k_0$, we have

$$g^{(k)}(0) = \binom{k-1}{k_0-1} f^{(k_0)}(0) g^{(k-k_0)}(0) + \sum_{i \neq k_0+1} \binom{k-1}{i} f^{(i+1)}(0) g^{(k-1-i)}(0).$$

From this equation, we can see that $g^{(k)}(0) \neq 0$ if $g^{(k-k_0)}(0) \neq 0$. Since $g^{(k_0-1)}(0) \neq 0$, we conclude that,

$$g^{(sk_0+k_0-1)}(0) \neq 0 \quad \forall s \in \mathbb{N}.$$

□

Lemma 12. *Let ρ be a real number. If $0 < \rho < 1$, then for any $x \in \mathbb{N}$ large enough,*

$$\sum_{1 \leq n \leq x} \rho^{-n} n^k = \frac{\rho^{-1}}{\rho^{-1} - 1} \rho^{-x} x^k (1 + o(x))$$

Proof. Let $x > 1$. By [**1**, Theorem 4.2],

$$\sum_{1 \leq n \leq x} \rho^{-n} n^k = x^k A(x) - A(1) - k \int_1^x A(t) t^{k-1} dt$$

with $A(x) = \sum_{1 \leq n \leq x} \rho^{-n}$. We put $\eta := \rho^{-1}$. We have,

$$\begin{aligned}
\int_0^x \rho^{-[t]} t^{k-1} dt =& \eta^x \int_0^x \eta^{[x-s]-x} (x-s)^{k-1} ds \\
=& \eta^x \int_0^x \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \eta^{[x-s]-x} x^j s^{k-1-j} ds \\
=& x^{k-1} \int_{[x]}^x \eta^{[t]} dt + x^{k-1} \int_0^{[x]} \eta^{[t]} dt \\
&+ \eta^x \sum_{j=0}^{k-2} (-1)^{k-1-j} \binom{k-1}{j} x^j \int_0^x \eta^{[x-s]-x} s^{k-1-j} ds \\
=& x^{k-1}(x - [x]) \eta^{[x]} + x^{k-1} \frac{\eta^{[x]} - 1}{\eta - 1} \\
&+ \eta^x \sum_{j=0}^{k-2} (-1)^{k-1-j} \binom{k-1}{j} x^j \int_0^x \eta^{[x-s]-x} s^{k-1-j} ds.
\end{aligned}$$

Observe that for any $0 \leq j \leq k-2$,

$$0 \leq \int_0^x \eta^{[x-s]-x} s^{k-1-j} ds \leq \frac{1}{(\log \eta)^{k-j}} \int_0^{x \log \eta} e^{-s} s^{k-1-j} ds \leq \frac{\Gamma(k-j)}{(\log \eta)^{k-j}}.$$

It follows that

$$\int_0^x \rho^{-[t]} t^{k-1} dt = \frac{\rho^{-x} x^{k-1}}{\rho^{-1}-1}(1+o(x)) \quad \forall x \gg 1, x \in \mathbb{N}.$$

Then,

$$\sum_{1 \leq n \leq x} \rho^{-n} n^k = \frac{\rho^{-x-1} x^k}{\rho^{-1}-1}(1+o(x)) \quad \forall x \in \mathbb{N}, x \gg 1.$$

□

Theorem 13. *Let* $\lambda > \#\{\mathfrak{a} \mid N(\mathfrak{a}) = 1\}$ *and assume that* $\{n \in \mathbb{N} \mid b_n \neq 0\}$ *contains an arithmetic progression of the form* $\{ck + d | k \in \mathbb{N}\}$ *with* $\gcd(c, d) = 1$. *For any* $\lambda \in \mathbb{R} \setminus \{1\}$, *we have*

$$\sum_{1 \leq n \leq x} \mathcal{H}_G(n, \lambda) = (1-\lambda) \frac{m_{\rho_\lambda}(-1)^{m_\rho - 1}}{\rho_\lambda^{m_{\rho_\lambda}} Z_G^{(m_{\rho_\lambda})}(\rho_\lambda)} \frac{\rho_\lambda^{-x} x^{m_\rho - 1}}{1-\rho_\lambda}(1+o(x)) \quad \forall x \gg 1, x \in \mathbb{N}.$$

Proof. Recall that, for any $m \geq k \in \mathbb{N}$, the unsigned Stirling numbers $\begin{bmatrix} m \\ k \end{bmatrix}$ are defined as follows

$$x(x+1)\cdots(x+m-1) = \sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} x^k.$$

Then, by Corollary 7, we obtain

$$\mathcal{H}_G(n, \lambda) = -(\lambda - 1) \sum_{\rho | Z_G(\rho) = \lambda} \frac{m_\rho (-1)^{m_\rho - 1}}{Z_G^{(m_\rho)}(\rho)} \rho^{-(n+m_\rho)} \sum_{k=0}^{m_\rho - 1} \begin{bmatrix} m_\rho \\ k+1 \end{bmatrix} n^k.$$

Then

$$\sum_{1 \leq n \leq x} \mathcal{H}_G(n, \lambda) = -(\lambda - 1) \sum_{Z_G(\rho) = \lambda} \frac{m_\rho (-1)^{m_\rho - 1} \rho^{-m_\rho}}{Z_G^{(m_\rho)}(\rho)} \sum_{k=0}^{m_\rho - 1} \begin{bmatrix} m_\rho \\ k+1 \end{bmatrix} \Big(\sum_{1 \leq n \leq x} \rho^{-n} n^k \Big).$$

By Lemmas 8 and 12, we get

$$\sum_{1 \leq n \leq x} \mathcal{H}_G(n, \lambda) = (1-\lambda) \frac{m_{\rho_\lambda}(-1)^{m_\rho - 1}}{Z_G^{(m_{\rho_\lambda})}(\rho_\lambda)} \rho_\lambda^{-m_{\rho_\lambda}} \frac{\rho_\lambda^{-x-1} x^{m_\rho - 1}}{\rho_\lambda^{-1} - 1}(1+o(x)) \quad \forall x \gg 1, x \in \mathbb{N}.$$

□

Remark 14. *We denote by* m_{s_λ} *the multiplicity of* s_λ. *An easy computation shows that* $\zeta^{(m_{s_\lambda})}(s_\lambda) = (-\log q)^{m_{s_\lambda}} \rho_\lambda^{m_{\rho_\lambda}} Z_G^{(m_{\rho_\lambda})}(\rho_\lambda)$, *and then* $m_{s_\lambda} = m_{\rho_\lambda}$. *So, the asymptotic expansion of Theorem 13 becomes*

$$\sum_{1 \leq n \leq x} \mathcal{H}_G(n, \lambda) = (1-\lambda) \frac{(\log q)^{m_{s_\lambda}} m_{s_\lambda}}{(q^{-s_\lambda} - 1)} \frac{q^{x s_\lambda}}{s_\lambda \zeta_G^{(m_{s_\lambda})}(s_\lambda)} x^{m_{s_\lambda} - 1}(1+o(x)).$$

Recall that, by definition, $H_G(N,\lambda) = \mathcal{H}_G(n,\lambda)$ *if* $N = q^n$ *and* $H_G(N,\lambda) = 0$ *otherwise. Then, for any integer* X *which is a power of* q*, we have*

$$\sum_{1\le N\le X} H_G(N,\lambda) = C(\lambda)X^{s_\lambda}(\log_q X)^{m_{s_\lambda}-1}(1+o(X)),$$

with $C(\lambda) = (1-\lambda)\frac{(\log q)^{m_{s_\lambda}} m_{s_\lambda}}{(q^{-s_\lambda}-1)s_\lambda \zeta_G^{(m_{s_\lambda})}(s_\lambda)}$.

3. APPLICATIONS

The theory developed in Section **2** can be applied to large class of zeta functions. In this section, we list three examples of such zeta functions which satisfy the conditions of Section **2**.

3.1 Function fields case

Let K be a global function field, that is a function field over constant finite field. We can define the notion of primes in K (see [**14**, p. 46]), for example, when $K = \mathbb{F}(T)$, the rational function field, then a monic irreducible polynomial in $\mathbb{F}[T]$ defines a prime in K, and T^{-1} in $\mathbb{F}[T^{-1}]$ defines a prime in K. We can show that the set of primes of $\mathbb{F}(T)$ is determined by the monic irreducible polynomials of $\mathbb{F}[T]$ and T^{-1}. We denote by G the free abelian group generated by the primes of K. An effective divisor $\mathfrak{a}$ is an element of G with non-negative coefficients. We use the notation $\mathfrak{a} \ge 0$ to denote an effective divisor $\mathfrak{a}$. We define the zeta function of K, which we denote by ζ_K instead of ζ_G as follows:

$$\zeta_K(s) = \sum_{\mathfrak{a}\ge 0} \frac{1}{(N\mathfrak{a})^s},$$

it is known that ζ_K converges absolutely for $\mathrm{Re}(s) > 1$ (see [**14**, p. 51]). We have

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{b_n}{q^{ns}},$$

with $b_n = \#\{a|\deg(a) = n\}$, and

$$\zeta_K(s) = \prod_P (1 - \frac{1}{NP^s})^{-1} = \prod_{n=1}^{\infty} (1 - \frac{1}{q^{ns}})^{-a_n},$$

where the first product runs over the set of primes, and a_n is the number of primes of degree n ([**14**, p. 52]). We put $t = q^{-s}$, we have

$$Z_K(t) := \zeta_K(s) = \exp(\sum_{m=1}^{\infty} \frac{N_n}{m} t^m),$$

with $N_m = \sum_{d|m} da_d$ (see [**14**, p. 57]). We see that $\sum_{m=1}^{\infty} \frac{N_n}{m} t^m$ satisfies Lemma 11 and then Lemma 8. It is known that there exists a polynomial $L_K(t) \in \mathbb{Z}[t]$ of degree $2g$, where g is the genus of K, such that

$$\zeta_K(s) = \frac{L_K(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

(see [**14**, Theorem 5.9]). So, it is clear that ζ_K admits a meromorphic extension to $\mathbb{C}$, and so is Z_K. It follows that

$$|Z_K(t)| \sim c_0 R^{2g-2}, \quad R = |t| \to \infty$$

where c_0 is a constant. Therefore, we can define the extended Eulerian numbers of K.

3.2 Higher dimension

Let X be a scheme of finite type over $\mathbb{Z}$. Let $|X|$ be the set of closed points in X. For $x \in |X|$, we set $N(x) = \#k(x)$ where $k(x)$ is the residue field of x, which is a finite field. The Hasse-Weil zeta function of X is by definition the following function

$$\zeta_X(s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}.$$

It is known that this product converges absolutely for $\mathrm{Re}(s)$ large enough. When $X = \mathrm{Spec}(\mathbb{Z})$, then $\zeta_X(s)$ is the Riemann zeta function.

In this paragraph, we use the notations of [**6**, (I.4)]. Let X_0 be a scheme of finite type over a finite field $\mathbb{F}_q$ of dimension N. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$, and let $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ be the corresponding scheme over $\overline{\mathbb{F}}_q$. We denote by N_m the number of points on the scheme X_0 with coefficients in $\mathbb{F}_{q^m}$, that is the cardinal of $X_0(\mathbb{F}_{q^m})$. For $x \in |X|$, set $\deg(x) = [k(x) : \mathbb{F}_q]$. So, $N(x) = q^{\deg(x)}$. We set $t = q^{-s}$. We consider

$$Z(X_0; t) = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}.$$

It is well know that this infinite product converges for $|t| \ll 1$. We have

$$\zeta_X(s) = Z(X_0; q^{-s}) \quad \forall\, \mathrm{Re}(s) \gg 1,$$

and,

$$Z(X_0; t) = \exp(\sum_{n=1}^{\infty} \frac{N_m}{m} t^m) \quad \forall |t| \ll 1.$$

Let $\mathrm{CH}_0(X)$ be the Chow group of 0-dimensional cycles [**8**, p. 10]. An element D of $\mathrm{CH}_0(X)$ is of the form $D = \sum_{i=1}^{r} a_i x_i$ with r is a positive integer, $x_i \in |X_0|$ and

$a_i \in \mathbb{Z}$ for $i = 1, \dots, r$. In other words, D is a formal sum of a finite set of closed points. We say that D is effective if $a_i \geq 0$ for any i. We define the degree and the norm of D as follows

$$\deg(D) = \sum_i a_i \deg(x_i), \quad N(D) = \prod_i N(x_i)^{a_i}.$$

So, by definition $N(D) = q^{\deg D}$.

It is known that (see [**6**])

$$Z(X_0; t) = \frac{P_1(t)P_3(t)\cdots P_{2N-1}(t)}{P_0(t)P_2(t)\cdots P_{2N}(t)},$$

where P_i are polynomials with coefficients in some $\mathbb{Q}$. They satisfy the following identities

$$P_{2N-i}(t) = (-1)^{b_i} \frac{q^{Nb_i} t^{b_i}}{\det(f^*; H^i)} P_i(\frac{1}{q^N t}), \quad i = 0, \dots, 2N,$$

where f is the Frobenius $X \to X$. These equalities can be seen as the functional equation of the zeta function $\zeta_X(s)$.

Theorem 15. *The polynomials $P_i(t) \in \mathbb{Z}[t]$, and*

$$P_i(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} t)$$

b_i (a known integer) where the α_{ij} are algebraic integers with $|\alpha_{ij}| = q^{i/2}$.

Proof. see [**6**, Theorem 1.6]. □

If we set $G = \mathrm{CH}_0(X)$ endowed with the degree and norm functions defined above, then the previous discussion allows us to apply the theory developed in Section **2**. We can consider $\mathcal{H}_X(n, \lambda)$, the extended Eulerian numbers associated to X,

$$\frac{\lambda - 1}{\lambda - \zeta_X(s)} = \frac{\lambda - 1}{\lambda - Z(X_0, t)} = \sum_{n=0}^{\infty} \mathcal{H}_X(n, \lambda) t^n.$$

To X, we associate a Möbius function denoted by $\mu_X(n)$, defined as follows: If $D = \sum_{i=1}^r a_i x_i$ (with $a_i \neq 0$ and $x_i \neq x_j$ for $(i \neq j) = 1, \dots, r$), then

$$\mu_X(D) = \begin{cases} 0 & \text{if } \exists i \text{ with } a_i \geq 2 \\ (-1)^r & \text{if } a_i = 1 \text{ for any } i. \end{cases}$$

For $\mathrm{Re}(s) \gg 1$, we have

$$\begin{aligned}\frac{1}{\zeta_X(s)} &= \prod_{x\in|X|}(1-N(x)^{-s})^{-1} \\ &=1-\sum_{x\in|X_0|}\frac{1}{N(x)^s}+\sum_{\substack{x_1,x_2\in|X|\\ x_1\neq x_2}}\frac{1}{N(x_1)^sN(x_2)^s} \\ &\quad -\sum_{\substack{x_1,x_2,x_3\in|X|\\ x_i\neq x_j,\text{if } i\neq j}}\frac{1}{N(x_1)^sN(x_2)^sN(x_3)^s}+\cdots \\ &=\sum_{D\geq 0}\frac{\mu_X(D)}{N(D)^s}.\end{aligned}$$

So, when $\lambda = 0$, we have

$$\mathcal{H}_X(n,0) = \sum_{D\geq 0,\deg D=n}\mu_X(D).$$

and

$$M_X(x) := \sum_{D\geq 0,\deg D\leq y}\mu_X(D) = \sum_{n\leq x}\mathcal{H}_X(n,0)$$

We call $M_X(x)$ the Mertens function of X.

For example, a graph $X = (V, E)$ is a finite nonempty set V of vertices and a finite multiset E of unordered pairs of vertices, called edges. To X, we can associate a zeta function called Ihara zeta function defined as follows:

$$Z_X(t) = \prod_{[c]}(1-t^{l(c)})^{-1}$$

where the product is over the prime cycles in X and $l(c)$ is the length of the cycle c in X. One can see that Z_X satisfies the conditions of Section **2**. Moreover, it is known that the reciprocal of $Z_X(t)$ is a polynomial of degree $\leq 2\#(E)$ (see [**2, 12, 11**]). As a consequence, the coefficients $\mathcal{H}_X(n,0)$ vanishes for all $n > 2\#(E)$.

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(Received 30. 09. 2021.)
(Revised 30. 04. 2022.)

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