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# TWO PARAMETERIZED SERIES REPRESENTATIONS FOR THE DIGAMMA FUNCTION

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Numerous series representations for various special functions and mathematical constants have been developed by many authors. The aim of this article is to establish two parameterized series representations for the digamma function that seem interesting due to their independence from the given parameters. Among many particular cases of our two main findings, some are covered in the examples.

## 1. INTRODUCTION AND PRELIMINARIES

The digamma (or Psi) function $\psi$ is defined as follows:

$$\psi(u) = \frac{d}{du}\{\log\,\Gamma(u)\} = \frac{\Gamma'(u)}{\Gamma(u)} \quad \text{or} \quad \log\,\Gamma(u) = \int_1^u \psi(x)\,\mathrm{d}x,$$

where $\Gamma$ denotes the well-known Gamma function (see, e.g., [**16**, Section 1.1]). The digamma function fulfills

$$\psi(u) = -\gamma - \frac{1}{u} + \sum_{n=1}^{\infty} \frac{u}{n(u+n)} \quad (u \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}), \tag{1}$$

where $\gamma$ denotes the Euler-Mascheroni constant (see, e.g., [**16**, Section 1.2]; see also [**10, 12, 15**]).

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Among numerous integral representations for the digamma function (see, e.g., [**16**, Section 1.3]), we recall the following identity which plays an important in the proof of our theorems (see, e.g., [**16**, p. 25, Eq. (13)]):

$$\psi(x+1) = -\gamma + \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t \quad (\Re(x) > -1). \tag{2}$$

The polygamma functions $\psi^{(n)}$ are given by

$$\psi^{(n)}(u) = \frac{d^{n+1}}{du^{n+1}} \log\Gamma(u) = \frac{d^n}{du^n}\psi(u) \quad (u \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0},\ n \in \mathbb{N}).$$

It is helpful to remember the following (well-known) relationship between the Hurwitz (or generalized) zeta function $\zeta(s,u)$ (see, e.g., [**16**, Section 2.2]) and the polygamma functions:

$$\psi^{(n)}(u) = (-1)^{n+1}\, n! \sum_{j=0}^{\infty} \frac{1}{(j+u)^{n+1}} = (-1)^{n+1}\, n!\, \zeta(n+1, u)$$

$$(u \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0},\ n \in \mathbb{N}).$$

Here $\zeta(s) = \zeta(s,1)$ is Riemann zeta function (see, e.g., [**16**, Section 2.3]).

Further, we recall the familiar Beta function (see, e.g., [**8**, Section 1.5], [**16**, p. 8])

$$B(\alpha,\,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t & (\Re(\alpha) > 0,\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}). \end{cases} \tag{3}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, and positive integers, respectively. Also let $\mathbb{Z}_{\leq \ell}$ be the set of integers which are less than or equal to some integer $\ell$.

The Pochhammer symbol $(\lambda)_n$ is defined (for $\lambda \in \mathbb{C}$) by (see, e.g., [**16**, p. 2 and pp. 4-6])

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$
$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}).$$

The hypergeometric series ${}_2F_1$ is defined by (see, e.g., [**16**, Section 1.5]):

$${}_2F_1\left[\begin{matrix}\alpha,\,\beta\,; \\ \gamma\,;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}\,\frac{z^n}{n!} = {}_2F_1(\alpha,\,\beta;\,\gamma;\,z).$$

For parameter and variable constraints and convergence conditions of ${}_2F_1$, one may refer to (for example) [**16**, Section 1.5]).

Numerous series representations for the digamma function (including (1)), for other special functions, and for mathematical constants have been developed. Take two examples. The Leibniz formula for $\pi$ is

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

whose alternating series is also called the Madhava-Leibniz series. Kórus [**11**, Formula 2.6] presented the following intriguing series representation for Apéry's constant $\zeta(3)$ in terms of the Beta function given in (3):

$$\zeta(3) = -\frac{1}{2}\,\psi^{(2)}(1) = 3\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{B\big(\frac{1}{2}n+\frac{\sqrt{3}}{2}ni,\ \frac{1}{2}n-\frac{\sqrt{3}}{2}ni\big)}{n^2} \quad (i=\sqrt{-1}).$$

The purpose of this paper is to establish two parameterized series representations for the digamma function, which seem intriguing owing to the provided parameters' independence. Our representations offer connections between the digamma function and the Beta function and between the digamma function and ${}_2F_1$, which we could not locate in the literature. Among the many particular cases of our main findings, some are addressed in the examples.

We should point out that the techniques of proof provided here are not novel. They were used in [**2, 3, 4, 5**] to discover series representations for a number of special functions and mathematical constants.

## 2. SERIES REPRESENTATIONS

The following theorem establishes a parameterized series representation for $\psi(x)$ in terms of the Beta function.

**Theorem 1.** *Let* $0 < a < 2$ *and* $\Re(x) > 0$. *Then*

$$\psi(x) = -\gamma + \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^{j+1}\Big\{2B(j+1,x) - B\Big(\frac{j+1}{2},x\Big)\Big\}. \tag{4}$$

*Proof.* Let $\Re(y) > -1$. Setting $t = u(2-u)$ in the integral in (2) gives

$$\frac{1}{2}\int_0^1 \frac{1-t^y}{1-t}\,\mathrm{d}t = \int_0^1 \frac{1-u^y(2-u)^y}{1-u}\,\mathrm{d}u. \tag{5}$$

Using (5) in (2), we find

$$
\begin{aligned}
\frac{1}{2}\left\{\psi(y+1)+\gamma\right\} &= \int_0^1 \frac{1-t^y}{1-t}\,\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{1-t^y}{1-t}\,\mathrm{d}t \\
&= \int_0^1 \frac{1-u^y}{1-u}\,\mathrm{d}u - \int_0^1 \frac{1-u^y(2-u)^y}{1-u}\,\mathrm{d}u \\
&= \int_0^1 \frac{u^y(2-u)^y-u^y}{1-u}\,\mathrm{d}u.
\end{aligned} \tag{6}
$$

By putting $u=1-t$ into the rightmost integral of (6), we get

$$
\frac{1}{2}\left\{\psi(y+1)+\gamma\right\} = \int_0^1 \frac{\left(1-t^2\right)^y-(1-t)^y}{t}\,\mathrm{d}t. \tag{7}
$$

Using the geometric series and the binomial theorem, we obtain that, for $a\in(0,2)$ and $t\in(0,1)$,

$$
\frac{1}{t} = \frac{a}{1-(1-at)} = a\sum_{k=0}^{\infty}(1-at)^k = a\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^j\,t^j. \tag{8}
$$

Let $x=y+1$. Setting (8) in (7) and integrating the resultant identity termwise, with the aid of the Beta function (3), we get

$$
\frac{1}{2}\left\{\psi(x)+\gamma\right\} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^{j+1}G_j(x), \tag{9}
$$

where

$$
G_j(x) = B(j+1,x) - \int_0^1 t^j(1-t^2)^{x-1}\,\mathrm{d}t. \tag{10}
$$

Taking $z=2$, $x=j+1$, and $y=x$ in Equation (17) in [**8**, p. 10] yields

$$
\int_0^1 t^j(1-t^2)^{x-1}\,\mathrm{d}t = \frac{1}{2}\,B\left(\frac{j+1}{2},x\right) \quad (\Re(x)>0). \tag{11}
$$

From (9), (10), and (11), we obtain the desired series representation (4). □

The following theorem provides an alternative parameterized series representation for $\psi(x)$ whose terms are involved in ${}_2F_1$.

**Theorem 2.** *Let $b>-\frac{1}{2}$ and $\Re(x)>0$. Then*

$$
\begin{aligned}
\psi(x) = &-\gamma + 2\sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}} \\
&\times \sum_{j=0}^{k}\binom{k}{j}\frac{b^{k-j}}{j+x}\left\{{}_2F_1\left(1,\,1-x\,;\,j+x+1\,;\,-1\right)-1\right\}.
\end{aligned} \tag{12}
$$

*Proof.* We have for $b > -1/2$ and $u \in (0,1)$,

$$-1 < \frac{u+b}{1+b} < 1.$$

Let $\Re(y) > -1$. Using the geometric series and the binomial theorem gives

$$\begin{aligned}\int_0^1 \frac{u^y(2-u)^y - u^y}{1-u}\,\mathrm{d}u &= \int_0^1 \frac{1}{1+b}\cdot\frac{1}{1-(u+b)/(1+b)}\big(u^y(2-u)^y - u^y\big)\,\mathrm{d}u \\ &= \int_0^1 \frac{1}{1+b}\sum_{k=0}^{\infty}\Big(\frac{u+b}{1+b}\Big)^k \big(u^y(2-u)^y - u^y\big)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}}\int_0^1 (u+b)^k\big(u^y(2-u)^y - u^y\big)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}}\int_0^1 \big(u^y(2-u)^y - u^y\big)\sum_{j=0}^{k}\binom{k}{j}u^j b^{k-j}\,\mathrm{d}u.\end{aligned}$$

Thus, we obtain

$$\int_0^1 \frac{u^y(2-u)^y - u^y}{1-u}\,\mathrm{d}u = \sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}b^{k-j}H_j(y), \tag{13}$$

where

$$H_j(y) = \int_0^1 u^{j+y}\big((2-u)^y - 1\big)\mathrm{d}u = \int_0^1 u^{j+y}(2-u)^y\mathrm{d}u - \frac{1}{j+y+1}. \tag{14}$$

Using a known integral formula [**9**, p. 313, Entry 3.196-1], we get

$$\int_0^1 u^{j+y}(2-u)^y\,\mathrm{d}u = \frac{1}{j+y+1}\,{}_2F_1\left(1,\,-y\,;\,j+y+2\,;\,-1\right). \tag{15}$$

From (14) and (15), we find

$$H_j(y) = \frac{1}{j+y+1}\left\{{}_2F_1\left(1,\,-y\,;\,j+y+2\,;\,-1\right) - 1\right\}. \tag{16}$$

Finally, we set $y = x - 1$ with $\Re(x) > 0$ and apply (6), (13), and (16). This leads to (12). □

**Remark 1.** *Let us consider the convergence of the double series on the right members of Equations* (4) *and* (12). *For this, under the respective restrictions of* $a$, $b$ *and* $x$, *as we trace back to the process of each proof, we return to the starting point, that is, the formula* (2)*:*

$$\psi(x+1) + \gamma = \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t \quad (\Re(x) > -1). \tag{17}$$

*Therefore, it may suffice to show that the improper integral on the right member of* (17) *converges.*

*To do this, for the moment, assume that* $x$ *is a real number bigger than* $-1$ *and maintain it fixed. Separate the following two cases:*

***Case 1.*** $x \geq 0$. *We have*

$$\int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t = \lim_{\epsilon \uparrow 1} \int_0^\epsilon \frac{1-t^x}{1-t}\,\mathrm{d}t. \tag{18}$$

*L'Hospital's rule says that*

$$\lim_{t\uparrow 1} \frac{1-t^x}{1-t} = \lim_{t \uparrow 1} x\, t^{x-1} = x.$$

*From this, the integrand* $(1-t^x)/(1-t)$ *may be looked upon as a continuous function on the closed bounded interval* $[0,1]$, *and so the improper integral in* (18) *exists as a finite real number.*

***Case 2.*** $-1 < x < 0$. *We write*

$$\int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t = -\lim_{\delta \downarrow 0} \int_\delta^{\frac{1}{2}} \frac{t^x-1}{1-t}\,\mathrm{d}t - \lim_{\epsilon \uparrow 1} \int_{\frac{1}{2}}^{\epsilon} \frac{t^x-1}{1-t}\,\mathrm{d}t.$$

*We find*

$$\int_\delta^{\frac{1}{2}} \frac{t^x-1}{1-t}\,\mathrm{d}t \leq 2\int_\delta^{\frac{1}{2}} (t^x-1)\,\mathrm{d}t < \frac{1}{x+1}\left(\frac{1}{2}\right)^{x+1}.$$

*Thus, holding* $-1 < x < 0$ *fixed, we observe that the set*

$$E := \left\{ \int_\delta^{\frac{1}{2}} \frac{t^x-1}{1-t}\,\mathrm{d}t \,\middle|\, 0 < \delta \leq \frac{1}{2} \right\}$$

*is a nonempty subset of* $\mathbb{R}$ *which is bounded above by* $\frac{1}{x+1}\left(\frac{1}{2}\right)^{x+1}$. *Then, by the Completeness Axiom,* $\sup E$ *exists as a finite real number. Moreover, we obtain*

$$\sup E = \lim_{\delta \downarrow 0} \int_\delta^{\frac{1}{2}} \frac{t^x-1}{1-t}\,\mathrm{d}t = \int_0^{\frac{1}{2}} \frac{t^x-1}{1-t}\,\mathrm{d}t.$$

*Likewise, we can show that*

$$\lim_{\epsilon \uparrow 1} \int_{\frac{1}{2}}^{\epsilon} \frac{t^x-1}{1-t}\,\mathrm{d}t = \int_{\frac{1}{2}}^{1} \frac{t^x-1}{1-t}\,\mathrm{d}t$$

*exists as a finite real number.*

*For the case* $x \in \mathbb{C}$ *with* $\Re(x) > -1$, *by using the principle of analytic continuation and the identity theorem, the formula* (17) *holds true in the half-plane* $\Re(x) > -1$.

*Moreover, in light of the assertion made in the first paragraph of this comment, it is observed that the parameter values a and b do not seem to affect the convergence of each double series.*

## 3. PARTICULAR CASES AND REMARKS

Among many particular cases of our central findings in Theorems 1 and 2, some are shown in the following instances, along with some potential comments.

**Example 1.** *In the two series in* (4) *and* (12)*, it is seen that* $\psi(x)+\gamma$ *is independent of the parameters* $a$ *and* $b$*. This implies that every pair of parameters* $a$ *and* $b$ *in either one of the two series in* (4) *or* (12) *results in the same series. Taking* $b=1$ *and* $b=0$ *as an example in* (12)*, we get*

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+x} \left\{ {}_2F_1\left(1,\, 1-x\,;\, j+x+1\,;\, -1\right) - 1 \right\}$$
$$= \sum_{k=0}^{\infty} \frac{1}{k+x} \left\{ {}_2F_1\left(1,\, 1-x\,;\, k+x+1\,;\, -1\right) - 1 \right\}.$$

**Example 2.** *Gauss devised a well-known formula for*

$$\psi\left(\frac{p}{q}\right) \quad (0<p<q,\ p,\ q \in \mathbb{N}),$$

*which is represented as a finite combination of elementary functions and has several variants (see, e.g., [**16**, Section 1.3]). By using this Gauss's formula in conjunction with the following well-known formula*

$$\psi(z+n) = \psi(z) + \sum_{k=1}^{n} \frac{1}{z+k-1} \quad (n \in \mathbb{N})$$

*to the identities in Theorems 1 and 2, we can provide as many specific instances as possible for the two series in* (4) *and* (12)*. Several of them are illustrated here.*

*We have*

$$\Psi\Big(\frac{1}{2}\Big) = -\gamma - 2\log 2, \quad \psi\Big(\frac{1}{3}\Big) = -\gamma - \frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}}, \quad \psi\Big(\frac{1}{4}\Big) = -\gamma - 3\log 2 - \frac{\pi}{2}.$$

*Applying these formulas and* (4) *with* $x=1/2$*,* $x=1/3$*,* $=1/4$*, we obtain for* $a \in (0,2)$*:*

$$\log 2 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-a)^{j+1} \Big\{ \frac{1}{2} B\Big(\frac{j+1}{2}, \frac{1}{2}\Big) - B\Big(j+1, \frac{1}{2}\Big) \Big\},$$

$$\frac{3}{2}\log 3+\frac{\pi}{2\sqrt{3}}=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^{j+1}\Big\{B\Big(\frac{j+1}{2},\frac{1}{3}\Big)-2B\Big(j+1,\frac{1}{3}\Big)\Big\},$$

$$3\log 2+\frac{\pi}{2}=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^{j+1}\Big\{B\Big(\frac{j+1}{2},\frac{1}{4}\Big)-2B\Big(j+1,\frac{1}{4}\Big)\Big\},$$

*and using* (12) *gives for* $b>-1/2$*:*

$$\log 2=\sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}\frac{b^{k-j}}{j+1/2}\Big\{1-{}_2F_1\Big(1,\frac{1}{2};j+\frac{3}{2};-1\Big)\Big\},$$

$$\frac{3}{4}\log 3+\frac{\pi}{4\sqrt{3}}=\sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}\frac{b^{k-j}}{j+1/3}\Big\{1-{}_2F_1\Big(1,\frac{2}{3};j+\frac{4}{3};-1\Big)\Big\},$$

$$\frac{3}{2}\log 2+\frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{1}{(1+b)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}\frac{b^{k-j}}{j+1/4}\Big\{1-{}_2F_1\Big(1,\frac{3}{4};j+\frac{5}{4};-1\Big)\Big\}.$$

**Example 3.** *Applying* (4) *with suitable numbers* $x_1$ *and* $x_2$*, we obtain for* $\psi(x_1)-\psi(x_2)$ *series representations involving simple rational expressions. We give two examples. Let*

$$R(j,x)=2B(j+1,x)-B\Big(\frac{j+1}{2},x\Big).$$

*Using*

$$R(j,4)-R(j,2)=2\,\frac{j^3+16j^2+41j-34}{(j+2)(j+3)(j+4)(j+5)(j+7)},$$

$$R(j,5)-R(j,4)=12\,\frac{7j^2+32j+1}{(j+2)(j+3)(j+4)(j+5)(j+7)(j+9)},$$

*and*

$$\psi(4)-\psi(2)=\frac{5}{6},\quad \psi(5)-\psi(4)=\frac{1}{4}$$

*we obtain from* (4) *for* $a\in(0,2)$*:*

$$\frac{5}{12}=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^{j+1}\frac{j^3+16j^2+41j-34}{(j+2)(j+3)(j+4)(j+5)(j+7)},$$

$$\frac{1}{48}=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-a)^{j+1}\frac{7j^2+32j+1}{(j+2)(j+3)(j+4)(j+5)(j+7)(j+9)}.$$

**Remark 2.** *There is optimism that a new series representation (even if it is more complicated and has a smaller domain of validity than existing ones) will make it feasible to demonstrate that certain mathematical constants are irrational or transcendent.*

**Remark 3.** *Since $\psi(1) = -\gamma$ (see, e.g., [**16**, p. 24, Eq. (4)]), the two series in* (4) *and* (12) *when $x = 1$ may vanish to be zero. This is readily apparent since we have*

$$2B(j+1,1) = B\Big(\frac{j+1}{2},1\Big) \quad \text{and} \quad {}_2F_1\,(1,\,0\,;\,j+2\,;\,-1) = 1.$$

**Remark 4.** *The digamma function and polygamma functions have diverse applications. They are particularly useful in evaluations of certain infinite series, such as*

$$\sum_{k=1}^{\infty} \frac{1}{(k+a)^p\,(k+b)^q} \qquad (a,\, b \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1},\, a \neq b,\, p,\, q \in \mathbb{N})$$

*(see, e.g., [**1**]). Interesting number theoretic properties of these functions can be found in [**13**] and [**14**]. Among others, the authors study the transcendence of the values $\psi(p/q)$, where $(p,q) = 1$, $1 \leq p < q$.*

**Remark 5.** *The Euler-Mascheroni constant $\gamma$ has a number of integral representations (see, e.g., [**7**], [**16**, Section 1.2]).*

**Remark 6.** *We see that the ${}_2F_1(-1)$ in* (12) *is a generalization of one of Kummer's three well-known theorems for ${}_2F_1$ (see, e.g., [**6**, Eqs. (4), (5), (6)]). Numerous authors have investigated various extensions of Kummer's summation theorems (see the references cited in [**6**]). Using Eq. (7) or Eq. (32) in [**6**], it is found that ${}_2F_1(-1)$ in* (12) *may be represented in terms of the Gamma function.*

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