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A q-DIRAC BOUNDARY VALUE PROBLEM WITH EIGENPARAMETER-DEPENDENT BOUNDARY CONDITIONS

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We study a boundary value problem for the q-Dirac equation and eigenvalue-dependent boundary conditions. We introduce a self-adjoint operator in a suitable Hilbert space and illustrate the boundary value problem as a spectral problem for this operator. We investigate the properties of the eigenvalues and vector-valued eigenfunctions. We construct Green's function.

1. INTRODUCTION

Recently there has been an increasing interest in the theory of q-Dirac systems which was introduced in [1]. In [22], the author deals with a q-Dirac system and investigate the asymptotic behavior of the eigenvalues and eigenfuctions. In [2] (also see [4]), the authors consider the symmetric q-Dirac operator, describe dissipative, accumulative, self-adjoint and other extensions of such operators with general boundary conditions. They construct a self-adjoint dilation of a dissipative operator, and thus determine the scattering matrix of dilation. They also construct a functional model of this operator and define its characteristic function and they prove that all root vectors of this operator are complete. In [23], the author derives the sampling representation for a q-Dirac system. In [3], the authors establish a Parseval equality and expansion formula in eigenfunctions for the q-Dirac operator on the whole line. In [29], the author proves the existence of a spectral function and

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establishes the Parseval equality for a singular q-Dirac system. In [5], the authors investigate the properties of the eigenvalues and the eigenfunctions of a q-fractional Dirac system. They give a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions by using a fixed point theorem.

In this paper, we study the boundary value problem

$$\begin{cases} -D_q y_2^\sigma(t) + p(t)y_1(t) = \lambda y_1(t), \\ D_q y_1(t) + r(t)y_2(t) = \lambda y_2(t), \end{cases} \quad t \in [0,a], \tag{1}$$

$$U_1(y) := \alpha_1 y_1(0) - \alpha_2 y_2^\sigma(0) + \lambda\big(\alpha_1' y_1(0) - \alpha_2' y_2^\sigma(0)\big) = 0, \tag{2}$$

$$U_2(y) := \beta_1 y_1(a) - \beta_2 y_2^\sigma(a) + \lambda\big(\beta_1' y_1(a) - \beta_2' y_2^\sigma(a)\big) = 0, \tag{3}$$

where $p, r \in \mathcal{L}_q^1(0,a)$ are real-valued functions defined on $[0,a]$ and continuous at zero, $q \in (0,1)$ is fixed, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $y^\sigma(t) := y(q^{-1}t)$, $\alpha_i, \alpha_i', \beta_i, \beta_i' \neq 0$ $(i = 1,2)$ are arbitrary real numbers and λ is a spectral parameter. The motivation behind this paper is twofold: to extend the findings of [16] (also see [17]) to q-Dirac problems and to form a basis for subsequent research. To the best of our knowledge, no work has considered the boundary value problem (1)–(3) in the context of q-derivative.

Boundary value problems with dependence on the eigenparameter in the boundary conditions have a long history and arise in various types of mathematical problems and their applications [8]. Fulton and Walter [19, 32] introduced the operator-theoretic formulation for the usual Sturm–Liouville problems with eigenparameter-dependent boundary conditions. Since then, there is quite an extensive literature concerning both classical Sturm–Liouville and classical Dirac problems with eigenparameter-dependent boundary conditions. For some of the latest works one can see [21, 26, 30, 31, 33] and the references therein. Studies of q-Sturm–Liouville and q-Dirac problems in the context of discrete and difference equations can be found in [9–15].

The structure of the paper is as follows. In Section 2, we introduce notations, definitions and preliminary facts which are used throughout the paper. In Section 3, we establish an operator-theoretic formulation for the boundary value problem (1)–(3) in the Hilbert space $\mathcal{H}_q = \mathcal{L}_{2,q}(0,a,\mathbb{C}^2) \oplus \mathbb{C}^2$ in the manner of Fulton and Walter [19,32]. We introduce a self-adjoint operator and we give some of the virtues of eigenvalues and vector-valued eigenfunctions. In Section 4, we construct Green's function for the boundary value problem (1)–(3).

2. PRELIMINARIES

In this section, we introduce some of the q-notations which will be used throughout the paper. We use the standard notations found in [6, 7, 27].

The set of nonnegative integers is denoted by $\mathbb{N}_0$, and the set of positive integers is denoted by $\mathbb{N}$. For $t>0$,

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad A^*_{q,t} := A_{q,t} \cup \{0\}.$$

When $t=1$, we simply use A_q, A^*_q to denote $A_{q,1}$, $A^*_{q,1}$, respectively.

A set $S \subseteq \mathbb{R}$ is called a q-geometric set if, for every $t \in S$, $qt \in S$. Let y be a real or complex valued function defined on a q-geometric set S. The q-difference operator is defined by

$$D_q y(t) := \frac{y(t) - y(qt)}{t(1-q)}, \quad t \neq 0.$$

If $0 \in S$, then the q-derivative of a function y at zero is defined as

$$D_q y(0) := \lim_{n\to\infty} \frac{y(tq^n) - y(0)}{tq^n}, \quad t \in S,$$

if the limit exists and does not depend on t.

Let us define the function $y^\rho(t) := y(qt)$. Note that $(y^\sigma)^\rho = (y^\rho)^\sigma = y$. The following lemma can be proven similar to (1.12) in [25].

Lemma 1. *The q-product rule holds:*

$$D_q(yz)(t) = y(t) D_q z(t) + z^\rho(t) D_q y(t). \tag{4}$$

Lemma 2. *The equation is always valid:*

$$D_q\big(y_1 \overline{z_2^\sigma} - y_2^\sigma \overline{z_1}\big) = y_1 \overline{D_q z_2^\sigma} + \overline{z_2} D_q y_1 - y_2 \overline{D_q z_1} - \overline{z_1} D_q y_2^\sigma. \tag{5}$$

Lemma 3. *The equation below holds:*

$$\frac{1}{q}\big(D_q y\big)^\sigma = D_q y^\sigma. \tag{6}$$

Proof. We have

$$D_q y^\sigma(t) = \frac{y^\sigma(t) - y^\sigma(qt)}{t - qt} = \frac{y(q^{-1}t) - y(t)}{q(q^{-1}t - t)} = \frac{y(q^{-1}t) - y(t)}{q(q^{-1}t - t)} = \frac{1}{q}\big(D_q y\big)^\sigma(t),$$

if $t>0$, and if $t=0$

$$D_q y^\sigma(0) = \lim_{n\to\infty} \frac{y^\sigma(tq^n) - y^\sigma(0)}{tq^n} = \lim_{n\to\infty} \frac{y(tq^{n-1}) - y(0)}{q(tq^{n-1})} = \frac{1}{q}\big(D_q y\big)^\sigma(0)$$

completing the proof. □

As a right inverse of the q-difference operator, q-integration is defined by Jackson [24] with

$$\int_0^t y(s)\,\mathrm{d}_q s := t(1-q) \sum_{n=0}^{\infty} q^n y(tq^n), \quad t \in S,$$

if the series converges. In general, we have

$$\int_a^b y(s)\,\mathrm{d}_q s := \int_0^b y(s)\,\mathrm{d}_q s - \int_0^a y(s)\,\mathrm{d}_q s, \quad a, b \in S.$$

There is no unique canonical choice for the q-integration over $[0, \infty)$. Hahn [20] defined the q-integration for a function y over $[0, \infty)$ by

$$\int_0^\infty y(s)\,\mathrm{d}_q s = (1-q) \sum_{n=-\infty}^{\infty} q^n y(q^n),$$

while Matsuo [28, (2.2)] defined q-integration on the interval $[0, \infty)$ by

$$\int_0^{b\infty} y(s)\,\mathrm{d}_q s := b(1-q) \sum_{n=-\infty}^{\infty} q^n y(bq^n), \quad b > 0,$$

provided that the series converges. Consequently, the q-integration of a function y defined on $\mathbb{R}$ can be defined as

$$\int_{-\infty/b}^{\infty/b} y(s)\,\mathrm{d}_q s = \frac{1-q}{b} \sum_{n=-\infty}^{\infty} q^n \left(y(q^n/b) + y(-q^n/b)\right), \quad b > 0$$

provided that the series converges absolutely.

Definition 4. Let y be a function defined on a q-geometric set S. We say that y is q-integrable on S if and only if $\int_0^t y(s)\,\mathrm{d}_q s$ exists for all $t \in S$.

Let S^* be a q-geometric set containing zero. A function y defined on S^* is called q-regular at zero if

$$\lim_{n\to\infty} y(tq^n) = y(0)$$

holds for all $t \in S^*$.

Let $\mathrm{C}(S^*)$ denote the space of all functions that are q-regular at zero and defined on S^* with values in $\mathbb{R}$. $\mathrm{C}(S^*)$, associated with the norm

$$\|y\| = \sup\left\{|y(tq^n)| : t \in S^*, n \in \mathbb{N}_0\right\},$$

is a normed space. The q-integration by parts rule [7] is

$$\int_a^b z(t) D_q y(t)\,\mathrm{d}_q t = (yz)|_a^b + \int_a^b y(qt) D_q z(t)\,\mathrm{d}_q t, \quad a, b \in S^*$$

for $y, z \in C(S^*)$.

Let y be function that is q-regular at zero and defined on the q-geometric set S^*. Define

$$Y(z) := \int_c^z y(t)\,\mathrm{d}_q t, \quad (z \in S^*),$$

where c is a fixed point in S^*. Then Y is q-regular at zero. Furthermore, $D_qY(z)$ exists for every $z \in S^*$ and $D_qY(z) = y(z)$ for every $z \in S^*$. Conversely, if a and b are two points in S^*, then

$$\int_a^b D_q y(t)\,\mathrm{d}_q t = y(b) - y(a). \tag{7}$$

For $p > 0$ and X equal to $A_{q,t}$ or $A^*_{q,t}$, the space $L^q_p(X)$ is the normed space of all functions defined on X such that

$$\|y\|_p := \left(\int_0^x |y(t)|^p\,\mathrm{d}_q t\right)^{1/p} < \infty.$$

If $p = 2$, then $L^2_q(X)$ associated with the inner product

$$\langle y, z\rangle := \int_0^x y(t)\overline{z(t)}\,\mathrm{d}_q t$$

is a Hilbert space.

The space of all q-absolutely functions on $A^*_{q,t}$ is denoted by $\mathrm{AC}_q(A^*_{q,t})$ and defined as the space of all functions y that are q-regular at zero and satisfy

$$\sum_{j=0}^{\infty} \left|y(tq^j) - y(tq^{j+1})\right| \le K$$

for all $t \in A^*_{q,t}$, and K is a constant depending on the function y (c.f., [7, Definiton 4.3.1, pg. 118]), i.e., $\mathrm{AC}_q(A^*_{q,t}) \subseteq \mathrm{C}_q(A^*_{q,t})$. The space $\mathrm{AC}^{(n)}_q(A^*_{q,t})$ $(n \in \mathbb{N})$ is the space of all functions y defined on S^* such that y, $D_q y$, $\cdots$, $D_q^{n-1} y$ are q-regular at zero and $D_q^{n-1} y \in \mathrm{AC}_q(A^*_{q,t})$ (c.f., [7, Definition 4.3.2, pg. 119]).

3. OPERATOR-THEORETIC FORMULATION

In this section, we establish the operator-theoretic formulation for the boundary value problem (1)–(3) in a special Hilbert space in the manner of Fulton and Walter [19,32]. We introduce a self-adjoint operator and we give some of the virtues of eigenvalues and vector-valued eigenfunctions.

In the Hilbert space $\mathcal{H}_q = \mathcal{L}_{2,q}(0, a, \mathbb{C}^2) \oplus \mathbb{C}^2$, let an inner product be defined by

$$\langle Y, Z\rangle := \int_0^a \left(y_1(t)\overline{z_1(t)} + y_2(t)\overline{z_2(t)}\right)\mathrm{d}_q t + \frac{y_3\overline{z_3}}{\chi_\alpha} + \frac{y_4\overline{z_4}}{\chi_\beta}$$

where $Y = (y_1, y_2, y_3, y_4)^T \in \mathcal{H}_q$, $Z = (z_1, z_2, z_3, z_4)^T \in \mathcal{H}_q$,

$$\chi_\alpha := \alpha_1\alpha_2' - \alpha_1'\alpha_2 > 0, \quad \chi_\beta := \beta_1'\beta_2 - \beta_1\beta_2' > 0.$$

We define the operator A

$$A(Y) := \begin{pmatrix} \ell(y) \\ -\alpha_1 y_1(0) + \alpha_2 y_2^\sigma(0) \\ -\beta_1 y_1(a) + \beta_2 y_2^\sigma(a) \end{pmatrix}$$

with the domain

$$\mathcal{D}(A) := \left\{ \begin{array}{c} Y \in \mathcal{H}_q : y_1, y_2 \in \mathrm{AC}_q[0, aq^n], \ell(y) \in \mathcal{L}_{2,q}(0, a, \mathbb{C}^2), \\ y_3 = \alpha_1' y_1(0) - \alpha_2' y_2^\sigma(0), y_4 = \beta_1' y_1(a) - \beta_2' y_2^\sigma(a) \end{array} \right\},$$

where

$$\ell(y) = \begin{pmatrix} -D_q y_2^\sigma + p(t) y_1 \\ D_q y_1 + r(t) y_2 \end{pmatrix}.$$

Thus, the boundary value problem (1)–(3) is equivalent to the equation

$$A(Y) = \lambda Y. \tag{8}$$

Theorem 5. *The operator A is formally self-adjoint in $\mathcal{H}_q$.*

Proof. For $Y, Z \in \mathcal{D}(A)$, we have

$$\begin{aligned} \langle AY, Z\rangle - \langle Y, AZ\rangle = &\int_0^a \big(-D_q y_2^\sigma + p y_1\big)(t)\overline{z_1(t)}\, \mathrm{d}_q t \\ &+ \int_0^a \big(D_q y_1 + r y_2\big)(t)\overline{z_2(t)}\, \mathrm{d}_q t \\ &- \frac{\big(\alpha_1 y_1(0) - \alpha_2 y_2^\sigma(0)\big)\overline{\big(\alpha_1' z_1(0) - \alpha_2' z_2^\sigma(0)\big)}}{\chi_\alpha} \\ &- \frac{\big(\beta_1 y_1(a) - \beta_2 y_2^\sigma(a)\big)\overline{\big(\beta_1' z_1(a) - \beta_2' z_2^\sigma(a)\big)}}{\chi_\beta} \\ &- \int_0^a y_1(t)\big(\overline{-D_q z_2^\sigma + p z_1}\big)(t)\, \mathrm{d}_q t \\ &- \int_0^a y_2(t)\big(\overline{D_q z_1 + r z_2}\big)(t)\, \mathrm{d}_q t \\ &+ \frac{\big(\alpha_1' y_1(0) - \alpha_2' y_2^\sigma(0)\big)\overline{\big(\alpha_1 z_1(0) - \alpha_2 z_2^\sigma(0)\big)}}{\chi_\alpha} \\ &+ \frac{\big(\beta_1' y_1(a) - \beta_2' y_2^\sigma(a)\big)\overline{\big(\beta_1 z_1(a) - \beta_2 z_2^\sigma(a)\big)}}{\chi_\beta} \\ = &\int_0^a \big(\overline{z_2} D_q y_1 - \overline{z_1} D_q y_2^\sigma + y_1 D_q \overline{z_2^\sigma} - y_2 D_q \overline{z_1}\big)(t)\, \mathrm{d}_q t \\ &+ \frac{\big(y_1(0)\overline{z_2^\sigma(0)} - y_2^\sigma(0)\overline{z_1(0)}\big)(\alpha_1\alpha_2' - \alpha_1'\alpha_2)}{\chi_\alpha} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\big(y_2^\sigma(a)\overline{z_1(a)}-y_1(a)\overline{z_2^\sigma(a)}\big)(\beta_1'\beta_2-\beta_1\beta_2')}{\chi_\beta}\\
&\overset{(5)}{=}\int_0^a D_q\big(y_1\overline{z_2^\sigma}-y_2^\sigma\overline{z_1}\big)(t)\,\mathrm{d}_qt\\
&\quad+y_1(0)\overline{z_2^\sigma(0)}-y_2^\sigma(0)\overline{z_1(0)}+y_2^\sigma(a)\overline{z_1(a)}-y_1(a)\overline{z_2^\sigma(a)}\\
&\overset{(7)}{=}\big(y_1\overline{z_2^\sigma}-y_2^\sigma\overline{z_1}\big)(a)-\big(y_1\overline{z_2^\sigma}-y_2^\sigma\overline{z_1}\big)(0)\\
&\quad+\big(y_1\overline{z_2^\sigma}-y_2^\sigma\overline{z_1}\big)(0)-\big(y_1\overline{z_2^\sigma}-y_2^\sigma\overline{z_1}\big)(a)\\
&=0,
\end{aligned}
$$

completing the proof. □

Let $\varphi(\cdot,\lambda)=\begin{pmatrix}\varphi_1(\cdot,\lambda)\\ \varphi_2(\cdot,\lambda)\end{pmatrix}$ and $\psi(\cdot,\lambda)=\begin{pmatrix}\psi_1(\cdot,\lambda)\\ \psi_2(\cdot,\lambda)\end{pmatrix}$ be the solutions of (1) satisfying the initial conditions

$$
\begin{cases}\varphi_1(0,\lambda)=-\alpha_2-\lambda\alpha_2', & \varphi_2^\sigma(0,\lambda)=-\alpha_1-\lambda\alpha_1',\\ \psi_1(a,\lambda)=-\beta_2-\lambda\beta_2', & \psi_2^\sigma(a,\lambda)=-\beta_1-\lambda\beta_1'.\end{cases}\tag{9}
$$

Clearly,

$$
U_1(\varphi)=U_2(\psi)=0.\tag{10}
$$

Denote

$$
\Delta(\lambda)=\varphi_2^\sigma(\cdot,\lambda)\psi_1(\cdot,\lambda)-\varphi_1(\cdot,\lambda)\psi_2^\sigma(\cdot,\lambda).\tag{11}
$$

The function $\Delta(\lambda)$ is the characteristic function of the boundary value problem (1)–(3). It is an entire function with respect to λ and thus, the boundary value problem (1)–(3) has an at most countable set of eigenvalues $\{\lambda_n\}$ with no finite limit points. It can be easily seen that the characteristic function does not depend on t. Indeed, taking ψ for y and φ for z in (5) yields

$$
\begin{aligned}
D_q\Delta(\lambda)&\overset{(5)}{=}\psi_1(\cdot,\lambda)D_q\varphi_2^\sigma(\cdot,\lambda)+\varphi_2(\cdot,\lambda)D_q\psi_1(\cdot,\lambda)\\
&\quad-\psi_2(\cdot,\lambda)D_q\varphi_1(\cdot,\lambda)-\varphi_1(\cdot,\lambda)D_q\psi_2^\sigma(\cdot,\lambda)\\
&=\psi_1(\cdot,\lambda)\big(p(\cdot)-\lambda\big)\varphi_1(\cdot,\lambda)+\varphi_2(\cdot,\lambda)\big(\lambda-r(\cdot)\big)\psi_2\\
&\quad-\psi_2(\cdot,\lambda)\big(\lambda-r(\cdot)\big)\varphi_2(\cdot,\lambda)-\varphi_1\big(p(\cdot)-\lambda\big)\psi_1(\cdot,\lambda)\\
&=0.
\end{aligned}
$$

Substituting $t=0$ and $t=a$ into (11) and taking (9) into account, we get

$$
\Delta(\lambda)=-U_1(\psi)=U_2(\varphi).\tag{12}
$$

Lemma 6. *All eigenvalues of the boundary value problem* (1)–(3) *are real.*

Proof. For the self-adjoint operator A, we have $\langle AY, Z\rangle = \langle Y, AZ\rangle$, in particular $\langle AY, Y\rangle = \langle Y, AY\rangle$. Hence, $\overline{\langle AY, Y\rangle} = \langle Y, AY\rangle = \langle AY, Y\rangle \in \mathbb{R}$. Now, let λ be an eigenvalue of the operator A and Y be a vector-valued eigenfunction corresponding to λ. Equation (8) implies $\langle AY, Y\rangle = \lambda \langle Y, Y\rangle$. Since $\langle Y, Y\rangle > 0$ and $\langle AY, Y\rangle \in \mathbb{R}$, we obtain $\lambda = \dfrac{\langle AY, Y\rangle}{\langle Y, Y\rangle} \in \mathbb{R}$. □

Theorem 7. *The eigenvalues of the boundary value problem* (1)–(3) *coincide with the simple zeros of* $\Delta(\lambda)$. *The functions* $\varphi(\cdot, \lambda_n)$ *and* $\psi(\cdot, \lambda_n)$ *are eigenfunctions and there exists a sequence* $\{k_n\}$ *such that*

$$\psi(t, \lambda_n) = k_n \varphi(t, \lambda_n), \quad k_n \neq 0.$$

Proof. The proof can be done similar to [18, Theorem 1.1.1, pg. 6]. Indeed, let λ_0 be a zero of the characteristic function $\Delta(\lambda)$, i.e., $\Delta(\lambda_0) = 0$. Then, by virtue of (10)–(12), $\psi(\cdot, \lambda_0) = k_0 \varphi(\cdot, \lambda_0)$ and the functions $\psi(\cdot, \lambda_0)$ and $\varphi(\cdot, \lambda_0)$ satisfy the boundary conditions (2), (3). Hence, λ_0 is an eigenvalue, and $\psi(\cdot, \lambda_0)$ and $\varphi(\cdot, \lambda_0)$ are eigenfunctions related to λ_0. Now, let λ_0 be an eigenvalue of the boundary value problem (1)–(3) and $y_0 = \begin{pmatrix} y_{01}(\cdot, \lambda_n) \\ y_{02}(\cdot, \lambda_n) \end{pmatrix} \neq 0$ be a corresponding eigenfunction. Then $U_1(y_0) = U_2(y_0) = 0$. Without loss of generality, we put $y_{01}(0) = -\alpha_2 - \lambda\alpha_2'$. Then $y_{02}(0) = -\alpha_1 - \lambda\alpha_1'$, and consequently $y_0(t) \equiv \varphi(t, \lambda_0)$. Therefore (12) yields $\Delta(\lambda_0) = U_2(\varphi(t, \lambda_0)) = U_2(y_0(t)) = 0$. We have also proved that for each eigenvalue, there exists only one (up to a multiplicative constant) eigenfunction, and therefore eigenvalues of the boundary value problem (1)–(3) are simple from the geometric point of view. □

The vector-valued eigenfunctions of the operator A are in the form of

$$Y(\cdot, \lambda_n) = Y_n := \begin{pmatrix} y_1(\cdot, \lambda_n) \\ y_2(\cdot, \lambda_n) \\ \alpha_1' y_1(0, \lambda_n) - \alpha_2' y_2^{\sigma}(0, \lambda_n) \\ \beta_1' y_1(a, \lambda_n) - \beta_2' y_2^{\sigma}(a, \lambda_n) \end{pmatrix}.$$

Lemma 8. *The vector-valued eigenfunctions* Y_1 *and* Y_2 *corresponding to different eigenvalues* $\lambda_1 \neq \lambda_2$ *are orthogonal, i.e.,* $\langle Y_1, Y_2\rangle = 0$.

Proof. Since Y_1 and Y_2 are vector-valued eigenfunctions corresponding to λ_1 and λ_2, respectively, from (8), we have $AY_1 = \lambda_1 Y_1$ and $AY_2 = \lambda_2 Y_2$. Hence $\langle AY_1, Y_2\rangle = \langle \lambda_1 Y_1, Y_2\rangle = \lambda_1 \langle Y_1, Y_2\rangle$ and $\langle Y_1, AY_2\rangle = \langle Y_1, \lambda_2 Y_2\rangle = \lambda_2 \langle Y_1, Y_2\rangle$. Since A is selfadjoint, we have $\langle AY_1, Y_2\rangle - \langle Y_1, AY_2\rangle = 0$, which therefore yields $(\lambda_1 - \lambda_2)\langle Y_1, Y_2\rangle = 0$. As $\lambda_1 \neq \lambda_2$, we conclude $\langle Y_1, Y_2\rangle = 0$. □

4. CONSTRUCTION of GREEN'S FUNCTION

The q-type Green function appears when we pursue a solution of the inhomogeneous boundary value problem

$$\left\{ \begin{array}{l} -D_q y_2^\sigma(t) + \{p(t) - \lambda\} y_1(t) = f_1(t), \\ D_q y_1(t) + \{r(t) - \lambda\} y_2(t) = f_2(t), \end{array} \right. \qquad t \in [0, a], \tag{13}$$

$$U_1(y) = f_3, \quad U_2(y) = f_4, \tag{14}$$

where $p, r \in \mathcal{L}_q^1(0, a)$ are real-valued functions defined on $[0, a]$ and continuous at zero, $q \in (0, 1)$ is fixed, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $y^\sigma(t) = y(q^{-1}t)$, $\alpha_i, \alpha_i', \beta_i, \beta_i' \neq 0$ $(i = 1, 2)$ are arbitrary real numbers, λ is a spectral parameter, and $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{D}(A)$.

We shall search the solution of the inhomogeneous boundary value problem (13)–(14) in the form of

$$y(t, \lambda) = c_\varphi(t, \lambda)\varphi(t, \lambda) + c_\psi(t, \lambda)\psi(t, \lambda). \tag{15}$$

Assume (15) solves (13)–(14). Then,

$$\begin{aligned} f_1(t) &= -D_q y_2^\sigma(t, \lambda) + (p(t) - \lambda) y_1(t, \lambda) \\ &= -D_q\big(c_\varphi^\sigma(t, \lambda)\varphi_2^\sigma(t, \lambda) + c_\psi^\sigma(t, \lambda)\psi_2^\sigma(t, \lambda)\big) \\ &\quad + (p(t) - \lambda)\big(c_\varphi(t, \lambda)\varphi_1(t, \lambda) + c_\psi(t, \lambda)\psi_1(t, \lambda)\big) \\ &\overset{(4)}{=} -\big\{c_\varphi(t, \lambda) D_q \varphi_2^\sigma(t, \lambda) + \varphi_2^\sigma(t, \lambda) D_q c_\varphi^\sigma(t, \lambda) \\ &\quad + c_\psi(t, \lambda) D_q \psi_2^\sigma(t, \lambda) + \psi_2^\sigma(t, \lambda) D_q c_\psi^\sigma(t, \lambda)\big\} \\ &\quad + (p(t) - \lambda)(c_\varphi(t, \lambda)\varphi_1(t, \lambda) + c_\psi(t, \lambda)\psi_1(t, \lambda)) \\ &= c_\varphi(t, \lambda)\big(-D_q \varphi_2^\sigma(t, \lambda) + (p(t) - \lambda)\varphi_1(t, \lambda)\big) \\ &\quad + c_\psi(t, \lambda)\big(-D_q \psi_2^\sigma(t, \lambda) + (p(t) - \lambda)\psi_1(t, \lambda)\big) \\ &\quad - \big(\varphi_2^\sigma(t, \lambda) D_q c_\varphi^\sigma(t, \lambda) + \psi_2^\sigma(t, \lambda) D_q c_\psi^\sigma(t, \lambda)\big) \\ &= -\big(\varphi_2^\sigma(t, \lambda) D_q c_\varphi^\sigma(t, \lambda) + \psi_2^\sigma(t, \lambda) D_q c_\psi^\sigma(t, \lambda)\big) \\ &\overset{(6)}{=} -\frac{1}{q}\big(\varphi_2(t, \lambda) D_q c_\varphi(t, \lambda) + \psi_2(t, \lambda) D_q c_\psi(t, \lambda)\big)^\sigma, \end{aligned}$$

and thus, we have

$$f_1^\rho(t, \lambda) = -\frac{1}{q}\big(\varphi_2(t, \lambda) D_q c_\varphi(t, \lambda) + \psi_2(t, \lambda) D_q c_\psi(t, \lambda)\big).$$

Also,

$$f_2(t) = D_q y_1(t, \lambda) + (r(t) - \lambda) y_2(t, \lambda)$$

$$
\begin{aligned}
&= D_q\big(c_\varphi(t,\lambda)\varphi_1(t,\lambda)+c_\psi(t,\lambda)\psi_1(t,\lambda)\big)\\
&\quad+(r(t)-\lambda)\big(c_\varphi(t,\lambda)\varphi_2(t,\lambda)+c_\psi(t,\lambda)\psi_2(t,\lambda)\big)\\
&\overset{(4)}{=} c_\varphi(t,\lambda)D_q\varphi_1(t,\lambda)+\varphi_1^\rho(t,\lambda)D_qc_\varphi(t,\lambda)\\
&\quad+c_\psi(t,\lambda)D_q\psi_1(t,\lambda)+\psi_1^\rho(t,\lambda)D_qc_\psi(t,\lambda)\\
&\quad+(r(t)-\lambda)(c_\varphi(t,\lambda)\varphi_2(t,\lambda)+c_\psi(t,\lambda)\psi_2(t,\lambda))\\
&= c_\varphi(t,\lambda)\big(D_q\varphi_1(t,\lambda)+(r(t)-\lambda)\varphi_2(t,\lambda)\big)\\
&\quad+c_\psi(t,\lambda)\big(D_q\psi_1(t,\lambda)+(r(t)-\lambda)\psi_2(t,\lambda)\big)\\
&\quad+\varphi_1^\rho(t,\lambda)D_qc_\varphi(t,\lambda)+\psi_1^\rho(t,\lambda)D_qc_\psi(t,\lambda)\\
&= \varphi_1^\rho(t,\lambda)D_qc_\varphi(t,\lambda)+\psi_1^\rho(t,\lambda)D_qc_\psi(t,\lambda).
\end{aligned}
$$

Together,

$$
\begin{pmatrix} f_1^\rho(t)\\ f_2(t)\end{pmatrix}=\begin{pmatrix}-\frac{1}{q}\varphi_2(t,\lambda) & -\frac{1}{q}\psi_2(t,\lambda)\\ \varphi_1^\rho(t,\lambda) & \psi_1^\rho(t,\lambda)\end{pmatrix}\begin{pmatrix} D_qc_\varphi(t,\lambda)\\ D_qc_\psi(t,\lambda)\end{pmatrix}
$$

and

$$
\det\begin{pmatrix}-\frac{1}{q}\varphi_2(t,\lambda) & -\frac{1}{q}\psi_2(t,\lambda)\\ \varphi_1^\rho(t,\lambda) & \psi_1^\rho(t,\lambda)\end{pmatrix}=-\frac{\Delta(\lambda)}{q}.
$$

Hence,

$$
\begin{pmatrix} D_qc_\varphi(t,\lambda)\\ D_qc_\psi(t,\lambda)\end{pmatrix}=\frac{1}{\Delta(\lambda)}\begin{pmatrix}-q\psi_1^\rho(t,\lambda) & -\psi_2(t,\lambda)\\ q\varphi_1^\rho(t,\lambda) & \varphi_2(t,\lambda)\end{pmatrix}\begin{pmatrix} f_1^\rho(t)\\ f_2(t)\end{pmatrix}.
$$

Thus,

$$
\left\{\begin{aligned}
D_qc_\varphi(t,\lambda)&=-\frac{1}{\Delta(\lambda)}\big(q(\psi_1(t,\lambda)f_1(t))^\rho+\psi_2(t,\lambda)f_2(t)\big),\\
D_qc_\psi(t,\lambda)&=\frac{1}{\Delta(\lambda)}\big(q(\varphi_1(t,\lambda)f_1(t))^\rho+\varphi_2(t,\lambda)f_2(t)\big).
\end{aligned}\right. \tag{16}
$$

Now, we integrate the first equation in (16) from t to a and the second equation from 0 to t:

$$
\left\{\begin{aligned}
c_\varphi(t,\lambda)&=c_\varphi(a,\lambda)+\frac{1}{\Delta(\lambda)}\textstyle\int_t^a\big(q\psi_1(\rho(s),\lambda)f_1(\rho(s))+\psi_2(s,\lambda)f_2(s)\big)\,\mathrm{d}_qs,\\
c_\psi(t,\lambda)&=c_\psi(0,\lambda)+\frac{1}{\Delta(\lambda)}\textstyle\int_0^t\big(q\varphi_1(\rho(s),\lambda)f_1(\rho(s))+\varphi_2(s,\lambda)f_2(s)\big)\,\mathrm{d}_qs.
\end{aligned}\right.
$$

Substituting these equations into (15) we have

$$
\begin{aligned}
y(t,\lambda)=&\left(c_\varphi(a,\lambda)+\frac{1}{\Delta(\lambda)}\int_t^a\Big(q\psi_1(\rho(s),\lambda)f_1(\rho(s))+\psi_2(s,\lambda)f_2(s)\Big)\,\mathrm{d}_qs\right)\varphi(t,\lambda)\\
&+\left(c_\psi(0,\lambda)+\frac{1}{\Delta(\lambda)}\int_0^t\Big(q\varphi_1(\rho(s),\lambda)f_1(\rho(s))+\varphi_2(s,\lambda)f_2(s)\Big)\,\mathrm{d}_qs\right)\psi(t,\lambda)
\end{aligned}
$$

$$= c_\psi(0,\lambda)\psi(t,\lambda) + c_\varphi(a,\lambda)\varphi(t,\lambda) + \int_0^a G(t,s;\lambda)\begin{pmatrix} f_1^\rho(s) \\ f_2(s)\end{pmatrix} \mathrm{d}_q s,$$

where

$$G(t,s;\lambda) = \frac{1}{\Delta(\lambda)}\begin{cases} q\varphi_1^\rho(s,\lambda)\psi_1(t,\lambda) + \varphi_2(s,\lambda)\psi_2(t,\lambda), & s < t, \\ q\psi_1^\rho(s,\lambda)\varphi_1(t,\lambda) + \psi_2(s,\lambda)\varphi_2(t,\lambda), & s > t. \end{cases} \tag{17}$$

Substituting (15) into (14), and taking (9) into account, we obtain

$$c_\psi(0,\lambda) = \frac{f_3 + \delta_0}{\Delta(\lambda)} \quad \text{and} \quad c_\varphi(a,\lambda) = \frac{f_4 - \delta_a}{\Delta(\lambda)},$$

where

$$\delta_0 = \delta_{0\varphi} + \delta_{0\psi}, \quad \delta_a = \delta_{a\varphi} + \delta_{a\psi}$$

with

$$\begin{aligned} \delta_{0\varphi} &:= \varphi_1(0,\lambda)\varphi_2^\sigma(0,\lambda)[c_\varphi^\sigma(0,\lambda) - c_\varphi(0,\lambda)], \\ \delta_{0\psi} &:= \varphi_1(0,\lambda)\psi_2^\sigma(0,\lambda)[c_\psi^\sigma(0,\lambda) - c_\psi(0,\lambda)], \\ \delta_{a\varphi} &:= \psi_1(a,\lambda)\varphi_2^\sigma(a,\lambda)[c_\varphi^\sigma(a,\lambda) - c_\varphi(a,\lambda)], \\ \delta_{a\psi} &:= \psi_1(a,\lambda)\psi_2^\sigma(a,\lambda)[c_\psi^\sigma(a,\lambda) - c_\psi(a,\lambda)]. \end{aligned}$$

Thus, we have proved the following theorem.

Theorem 9. *Assume λ is not an eigenvalue of the boundary value problem* (1)–(3). *Then the boundary value problem* (13)–(14) *has the solution*

$$y(t,\lambda) = \int_0^a G(t,s;\lambda)\begin{pmatrix} f_1^\rho(s) \\ f_2(s)\end{pmatrix} \mathrm{d}_q s + \frac{f_3+\delta_0}{\Delta(\lambda)}\psi(t,\lambda) + \frac{f_4-\delta_a}{\Delta(\lambda)}\varphi(t,\lambda),$$

where $G(t,s;\lambda)$ given in (17) *is Green's function of the boundary value problem* (12)–(14).

Remark 10. Equation (1) is equivalent to

$$\begin{cases} -\dfrac{1}{q}D_{q^{-1}}y_2(t) + p(t)y_1(t) = \lambda y_1(t), \\ D_q y_1(t) + r(t)y_2(t) = \lambda y_2(t), \end{cases} \quad t \in [0,a]. \tag{18}$$

Proof. We only need to show that the first equations in (1) and in (18) correspond to each other. It is easy to see that

$$\begin{aligned} D_q y^\sigma(t) &= \frac{y^\sigma(t) - y^\sigma(qt)}{t(1-q)} = \frac{y(q^{-1}t) - y(t)}{t(1-q)} = \frac{y(q^{-1}t) - y(t)}{qt(q^{-1}-1)} \\ &= \frac{y(t) - y(q^{-1}t)}{qt(1-q^{-1})} = \frac{1}{q}D_{q^{-1}}y(t). \end{aligned}$$

One can refer to [1–4, 22, 23] for boundary value problems that consist of (18) instead of (1). The similarity between the first equation of (1) and the equations of the boundary value problems in [16, 17] is also worth noting. ☐

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