

## AN EQUIVALENT PROPERTY OF A HILBERT-TYPE INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Making use of complex analytic techniques as well as methods involving weight functions, we study a few equivalent conditions of a Hilbert-type integral inequality with nonhomogeneous kernel and parameters. In the form of applications we deduce a few equivalent conditions of a Hilbert-type integral inequality with homogeneous kernel, and we additionally consider operator expressions.

### 1. INTRODUCTION

In 1925, Hardy [6] proved the following result, which is now very well known as the classical Hardy-Hilbert integral inequality. This states that for positive real numbers  $p, q$  with  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and functions  $f(x), g(y) \geq 0$ , with

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have

$$(1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},$$

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2020 Mathematics Subject Classification. 26D15 47A05.

Keywords and Phrases. Hilbert-type integral inequality, Weight function, Equivalent form, Operator, Norm.

where the constant factor

$$\frac{\pi}{\sin(\pi/p)}$$

is the best possible.

For  $p = q = 2$ , (1) yields the well known Hilbert integral inequality. Both (1), as well as Hilbert's integral inequality play an important role in analysis and its applications (cf. [7], [16]).

In 1934, Hardy et al. established the following extension of (1):

If  $k_1(x, y)$  is a nonnegative homogeneous function of degree  $-1$ , and one defines

$$k_p = \int_0^\infty k_1(u, 1)u^{-\frac{1}{p}} du \in \mathbb{R}_+ := (0, \infty),$$

then we have the following Hardy-Hilbert-type integral inequality:

$$(2) \quad \int_0^\infty \int_0^\infty k_1(x, y)f(x)g(y)dx dy < k_p \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},$$

where the constant factor  $k_p$  is the best possible (cf. [7], Theorem 319).

Additionally, the following Hilbert-type integral inequality with nonhomogeneous kernel holds true:

If  $h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1} du \in \mathbb{R}_+$ , then

$$(3) \quad \int_0^\infty \int_0^\infty h(xy)f(x)g(y)dx dy < \phi\left(\frac{1}{p}\right) \left( \int_0^\infty x^{p-2} f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},$$

where the constant factor  $\phi\left(\frac{1}{p}\right)$  is the best possible (cf. [7], Theorem 350).

In 1998, by introducing an independent parameter  $\lambda > 0$ , Yang established an extension of Hilbert's integral inequality, namely the following (cf. [19], [20]):

$$(4) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left( \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty y^{1-\lambda} g^2(y)dy \right)^{\frac{1}{2}},$$

where the constant factor  $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$  is the best possible ( $B(u, v)$  is the beta function).

In 2004, by introducing two pairs of conjugate exponents  $(p, q)$  and  $(r, s)$ , Yang [21] proved the following extension of (1):

If  $\lambda > 0, p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$ , and  $f(x), g(y) \geq 0$ , satisfy

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy < \infty,$$

then we have

$$(5) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin(\pi/r)} \left[ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right]^{\frac{1}{q}},$$

where the constant factor

$$\frac{\pi}{\lambda \sin(\pi/r)}$$

is the best possible. For  $\lambda = 1, r = q, s = p$ , (5) reduces to (1).

In 2005, the paper [22] also provided an extension of (1) and (4) with the kernel  $\frac{1}{(x+y)^\lambda}$  and two pairs of conjugate exponents. Krnić et al. [1], [2], [3], [8], [10], [11], [14], [18], [30], proved some extensions and particular cases of (1), (2) and (3) with parameters. In 2009, Yang established an extension of (2) and (5), namely the following (cf. [23], [24]):

If  $\lambda_1 + \lambda_2 = \lambda \in \mathbb{R}$ ,  $k_\lambda(x, y)$  is a nonnegative homogeneous function of degree  $-\lambda$ , satisfying

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y) \quad (u, x, y > 0),$$

and

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1-1} du \in \mathbb{R}_+,$$

then we have

$$(6) \quad \int_0^\infty \int_0^\infty k_\lambda(x, y) f(x)g(y) dx dy < k(\lambda_1) \left[ \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

where the constant factor  $k(\lambda_1)$  is the best possible.

For  $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ , (6) reduces to (2), while for  $\lambda > 0, \lambda_1 = \frac{\lambda}{r}, \lambda_2 = \frac{\lambda}{s}$ ,  $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$ , (6) reduces to (5).

Additionally, the following extension of (3) was proved:

$$(7) \quad \int_0^\infty \int_0^\infty h(xy) f(x)g(y) dx dy < \phi(\sigma) \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

where the constant factor  $\phi(\sigma)$  is the best possible (cf. [25]).

For  $\sigma = \frac{1}{p}$ , (7) reduces to (3). Some equivalent inequalities of (6) and (7) were constructed in [24]. In 2013, Yang [25] also studied the equivalence of (6) and (7) by adding a condition  $h(u) = k_\lambda(u, 1)$ . In 2017, Hong [9] studied an equivalent

condition for (6) involving certain parameters, and some further related results were established in [4], [5], [15], [29], [28].

In the present paper, making use of complex analytic techniques as well as methods involving weight functions, we study a few equivalent conditions of a Hilbert-type integral inequality with the nonhomogeneous kernel

$$\frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \quad (c_k > 0)$$

and a best possible constant factor. In the form of applications we deduce a few equivalent conditions of a Hilbert-type integral inequality with homogeneous kernel. We also consider some operator expressions.

### 2. SOME LEMMAS

**Lemma 1.** (cf. [26]) *If  $\mathbb{C}$  is the set of complex numbers and  $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ ,*

$$z_k \in \mathbb{C} \setminus \{z \mid \operatorname{Re}(z) \geq 0, \operatorname{Im}(z) = 0\} \quad (k = 1, 2, \dots, n)$$

*are different points, the function  $f(z)$  is analytic in  $\mathbb{C}_\infty$  except for  $z_i$  ( $i = 1, 2, \dots, n$ ), and  $z = \infty$  is a zero point of  $f(z)$  whose order is not less than 1, then for  $\alpha \in \mathbb{R}$ , we have*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{k=1}^n \operatorname{Res}(s)[f(z)z^{\alpha-1}, z_k],$$

*where  $0 < \operatorname{Im}(\ln z) = \arg z < 2\pi$ . In particular, if  $z_k$  ( $k = 1, \dots, n$ ) are all poles of order 1, setting*

$$\varphi_k(z) = (z - z_k)f(z) \quad (\varphi_k(z_k) \neq 0),$$

*then*

$$(8) \quad \int_0^\infty f(x)x^{\alpha-1}dx = \frac{\pi}{\sin \pi\alpha} \sum_{k=1}^n (-z_k)^{\alpha-1} \varphi_k(z_k).$$

**Example 2.** *For  $s \in \mathbb{N} = \{1, 2, \dots\}$  and  $0 < c_1 \leq \dots \leq c_s$ ,  $0 < \sigma < s\lambda, \varepsilon > 0$ , we set*

$$h(u) := \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)}, \quad (u > 0),$$

*and*

$$\tilde{c}_k = c_k + (k - 1)\varepsilon \quad (k = 1, \dots, s).$$

*By (8), for  $z_k = -\tilde{c}_k$ , we derive that*

$$\begin{aligned} \tilde{k}_s(\sigma) &= \int_0^\infty \frac{1}{\prod_{k=1}^s (t^\lambda + \tilde{c}_k)} t^{\sigma-1} dt \\ &= \frac{1}{\lambda} \int_0^\infty \frac{1}{\prod_{k=1}^s (u + \tilde{c}_k)} u^{\frac{\sigma}{\lambda}-1} du \\ &= \frac{\pi}{\lambda \sin \frac{\pi\sigma}{\lambda}} \sum_{k=1}^s \frac{\tilde{c}_k^{\frac{\sigma}{\lambda}-1}}{\prod_{j=1(j \neq k)}^s (\tilde{c}_j - \tilde{c}_k)}. \end{aligned}$$

Setting  $\mu = s\lambda - \sigma (> 0)$ , we obtain that

$$\begin{aligned} 0 < \tilde{k}_s(\sigma) &= \frac{1}{\lambda} \int_0^\infty \frac{1}{\prod_{k=1}^s (u + \tilde{c}_k)} u^{\frac{\sigma}{\lambda}-1} du \\ &\leq \frac{1}{\lambda} \int_0^\infty \frac{1}{(u + c_1)^s} u^{\frac{\sigma}{\lambda}-1} du \\ &= \frac{1}{\lambda c_1^{\mu/\lambda}} \int_0^\infty \frac{1}{(v+1)^s} v^{\frac{\sigma}{\lambda}-1} dv \\ &= \frac{1}{\lambda c_1^{\mu/\lambda}} B\left(\frac{\sigma}{\lambda}, \frac{\mu}{\lambda}\right) < \infty, \end{aligned}$$

and by Levi's theorem (cf. [12]), it follows that

$$\begin{aligned} k_s(\sigma) &= \int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^s (t^\lambda + c_k)} dt = \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^s (t^\lambda + \tilde{c}_k)} dt \\ (9) \quad &= \lim_{\varepsilon \rightarrow 0^+} \tilde{k}_s(\sigma) = \frac{\pi}{\lambda \sin \frac{\pi\sigma}{\lambda}} \sum_{k=1}^s \frac{c_k^{\frac{\sigma}{\lambda}-1}}{\prod_{j=1(j \neq k)}^s (c_j - c_k)} \in \mathbb{R}_+. \end{aligned}$$

In particular:

(i) for  $s = 1$ , we obtain

$$k_1(\sigma) = \frac{1}{\lambda} \int_0^\infty \frac{u^{(\sigma/\lambda)-1}}{u + c_1} du = \frac{\pi}{\lambda c_1^{\mu/\lambda} \sin\left(\frac{\pi\sigma}{\lambda}\right)};$$

(ii) for  $s = 2$ , we get that

$$\begin{aligned} k_2(\sigma) &= \int_0^\infty \frac{1}{(t^\lambda + c_1)(t^\lambda + c_2)} t^{\sigma-1} dt \\ &= \frac{\pi}{\lambda \sin \frac{\pi\sigma}{\lambda}} \frac{c_1^{\frac{\sigma}{\lambda}-1} - c_2^{\frac{\sigma}{\lambda}-1}}{c_2 - c_1}; \end{aligned}$$

(iii) for  $c_s = \dots = c_1$  in (9), we have

$$k(\sigma) := \int_0^\infty \frac{t^{\sigma-1}}{(t^\lambda + c_1)^s} dt = \frac{s}{\lambda c_1^{\mu/\lambda}} B\left(\frac{\sigma}{\lambda}, \frac{\mu}{\lambda}\right).$$

If  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $s \in \mathbb{N}$ ,  $0 < c_1 \leq \dots \leq c_s$ ,  $0 < \sigma < s\lambda$ ,  $\sigma_1 \in \mathbb{R}$ , then for  $n \in \mathbb{N}$ , we define the following two expressions:

$$(10) \quad I_1 := \int_1^\infty \left\{ \int_0^1 \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} x^{\sigma + \frac{1}{pn} - 1} dx \right\} y^{\sigma_1 - \frac{1}{qn} - 1} dy,$$

$$(11) \quad I_2 := \int_0^1 \left\{ \int_1^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} x^{\sigma - \frac{1}{pn} - 1} dx \right\} y^{\sigma_1 + \frac{1}{qn} - 1} dy.$$

Setting  $u = xy$  in (10) and (11), by Fubini's theorem (cf. [12]), we obtain

$$\begin{aligned}
 I_1 &= \int_1^\infty \left[ \int_0^y \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} \left(\frac{u}{y}\right)^{\sigma + \frac{1}{pn} - 1} \frac{1}{y} du \right] y^{\sigma_1 - \frac{1}{qn} - 1} dy \\
 &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} \left[ \int_0^y \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \right] dy \\
 &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \\
 &\quad + \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} \int_1^y \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du dy \\
 &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \\
 (12) \quad &\quad + \int_1^\infty \left[ \int_u^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du,
 \end{aligned}$$

$$\begin{aligned}
 I_2 &= \int_0^1 \left\{ \int_y^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} \left(\frac{u}{y}\right)^{\sigma - \frac{1}{pn} - 1} \frac{1}{y} du \right\} y^{\sigma_1 + \frac{1}{qn} - 1} dy \\
 &= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} \left[ \int_y^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du \right] dy \\
 &= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_y^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du \\
 &\quad + \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} \int_1^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du dy \\
 &= \int_0^1 \left[ \int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du \\
 (13) \quad &\quad + \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du.
 \end{aligned}$$

In what follows we suppose that  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, s \in \mathbb{N}, 0 < c_1 \leq \dots \leq c_s, \sigma, \mu > 0, \sigma + \mu = s\lambda, \sigma_1 \in \mathbb{R}$ .

**Lemma 3.** *If there exists a constant  $M$ , such that for any nonnegative measurable functions  $f(x)$  and  $g(y)$  in  $(0, \infty)$ , the following inequality*

$$\begin{aligned}
 I &:= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \\
 (14) \quad &\leq M \left[ \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1) - 1} g^q(y) dy \right]^{\frac{1}{q}}
 \end{aligned}$$

holds true, then we have  $\sigma_1 = \sigma$ . In this case, it follows that  $M \geq k_s(\sigma)$ .

*Proof.* If  $\sigma_1 < \sigma$ , then for  $n > \frac{1}{\sigma - \sigma_1}$  ( $n \in \mathbb{N}$ ), we set two functions

$$f_n(x) := \begin{cases} 0, & 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, & x \geq 1 \end{cases}, \quad g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, & 0 < y \leq 1 \\ 0, & y > 1 \end{cases}.$$

Hence, we obtain that

$$\begin{aligned} J_2 &:= \left[ \int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left( \int_1^\infty x^{-\frac{1}{n} - 1} dx \right)^{\frac{1}{p}} \left( \int_0^1 y^{\frac{1}{n} - 1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

By (13) and (14), we have

$$\begin{aligned} &\int_0^1 \left[ \int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du \\ (15) \quad &\leq I_2 = \int_0^\infty \int_0^\infty \frac{f_n(x) g_n(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \leq M J_2 = Mn. \end{aligned}$$

Since  $(\sigma_1 - \sigma) + \frac{1}{n} < 0$ , it follows that for any  $u \in (0, 1)$ ,

$$\int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \infty.$$

By (15), in view of

$$\frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} > 0, \quad u \in (0, 1),$$

we deduce that  $\infty \leq Mn < \infty$ , which is a contradiction.

If  $\sigma_1 > \sigma$ , then for  $n > \frac{1}{\sigma_1 - \sigma}$  ( $n \in \mathbb{N}$ ), we set

$$\tilde{f}_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, & 0 < x \leq 1 \\ 0, & x > 1 \end{cases}, \quad \tilde{g}_n(y) := \begin{cases} 0, & 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, & y \geq 1 \end{cases}.$$

Hence, we derive that

$$\begin{aligned} \tilde{J}_2 &:= \left[ \int_0^\infty x^{p(1-\sigma)-1} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} \tilde{g}_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left( \int_0^1 x^{\frac{1}{n} - 1} dx \right)^{\frac{1}{p}} \left( \int_1^\infty y^{-\frac{1}{n} - 1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

By (12) and (14), we have

$$\begin{aligned} &\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \\ (16) \quad &\leq I_1 = \int_0^\infty \int_0^\infty \frac{\tilde{f}_n(x) \tilde{g}_n(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \leq M \tilde{J}_2 = Mn. \end{aligned}$$

Since  $(\sigma_1 - \sigma) - \frac{1}{n} > 0$ , it follows that

$$\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy = \infty.$$

By (16), in view of

$$\int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du > 0,$$

we have  $\infty \leq Mn < \infty$ , which is a contradiction.

Hence, we conclude that  $\sigma_1 = \sigma$ .

For  $\sigma_1 = \sigma$ , we reduce (12) and then apply (16) as follows:

$$\begin{aligned} \frac{1}{n} I_1 &= \frac{1}{n} \left[ \int_1^\infty y^{-\frac{1}{n} - 1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \right. \\ &\quad \left. + \int_1^\infty \left( \int_u^\infty y^{-\frac{1}{n} - 1} dy \right) \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \right] \\ &= \int_0^1 \frac{u^{\sigma + \frac{1}{pn} - 1}}{\prod_{k=1}^s (u^\lambda + c_k)} du + \int_1^\infty \frac{u^{\sigma - \frac{1}{qn} - 1}}{\prod_{k=1}^s (u^\lambda + c_k)} du \\ &\leq \frac{1}{n} M \tilde{J}_2 = M. \end{aligned}$$

Since the sequence

$$\left\{ \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} \right\}_{n=1}^\infty \quad \left( \text{resp.} \quad \left\{ \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{qn} - 1} \right\}_{n=1}^\infty \right)$$

is nonnegative and increasing in  $(0, 1)$  (resp.  $(1, \infty)$ ), by Levi's theorem (cf. [12]), we deduce that

$$\begin{aligned} k_s(\sigma) &= \int_0^1 \lim_{n \rightarrow \infty} \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \\ &\quad + \int_1^\infty \lim_{n \rightarrow \infty} \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{qn} - 1} du \\ &= \lim_{n \rightarrow \infty} \left[ \int_0^1 \frac{u^{\sigma + \frac{1}{pn} - 1}}{\prod_{k=1}^s (u^\lambda + c_k)} du + \int_1^\infty \frac{u^{\sigma - \frac{1}{qn} - 1}}{\prod_{k=1}^s (u^\lambda + c_k)} du \right] \leq M < \infty. \end{aligned}$$

This completes the proof of the lemma. □



### 3. MAIN RESULTS

**Theorem 4.** *The following conditions are equivalent:*

(i) *There exists a constant  $M$ , such that for any  $f(x) \geq 0$ , satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we have the following inequality:*

$$(17) \quad \begin{aligned} J &:= \left\{ \int_0^\infty y^{p\sigma_1-1} \left[ \int_0^\infty \frac{f(x)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned}$$

(ii) *there exists a constant  $M$ , such that for any  $f(x), g(y) \geq 0$ , satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*and*

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

*we have the following Hilbert-type integral inequality with nonhomogeneous kernel:*

$$(18) \quad \begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \\ &< M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned}$$

(iii)  $\sigma_1 = \sigma$ .

*If Condition (iii) is satisfied, then  $M \geq k_s(\sigma)$  and the constant factor  $M = k_s(\sigma)$  in (17) and (18) is the best possible.*

*Proof.* (i)  $\Rightarrow$  (ii). By Hölder's inequality (cf. [13]), we have

$$(19) \quad \begin{aligned} I &= \int_0^\infty \left\{ y^{\sigma_1 - \frac{1}{p}} \int_0^\infty \frac{f(x)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx \right\} \left( y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \\ &\leq J \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned}$$

Then by (17), we derive (18).

(ii)  $\Rightarrow$  (iii). By Lemma 1, we have  $\sigma_1 = \sigma$ .

(iii)  $\Rightarrow$  (i). Setting  $u = xy$  for  $y > 0$ , we obtain the following weight function

$$\begin{aligned}
 \omega(\sigma, y) &:= y^\sigma \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} x^{\sigma-1} dx \\
 (20) \qquad &= \int_0^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-1} du = k_s(\sigma).
 \end{aligned}$$

By Hölder's weighed inequality and (20), we have

$$\begin{aligned}
 &\left\{ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} f(x) dx \right\}^p \\
 &= \left\{ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \left[ \frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[ \frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\
 &\leq \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
 &\quad \times \left\{ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right\}^{p/q} \\
 &= \left[ \frac{\omega(\sigma, y)}{y^{q(\sigma-1)+1}} \right]^{p-1} \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
 (21) \qquad &= \frac{(k_s(\sigma))^{p-1}}{y^{p\sigma-1}} \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx
 \end{aligned}$$

If (21) assumes the form of equality for some  $y \in (0, \infty)$ , then (cf. [13]) there exist constants  $A$  and  $B$ , such that they are not both zero, and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \quad \text{a.e. in } \mathbb{R}_+.$$

We suppose that  $A \neq 0$  (otherwise  $B = A = 0$ ). Then it follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \quad \text{a.e. in } \mathbb{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty.$$

Hence, (21) assumes the form of strict inequality.

For  $\sigma_1 = \sigma$ , by Fubini's theorem, we have

$$\begin{aligned} J &< (k_s(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\ &= (k_s(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (k_s(\sigma))^{\frac{1}{q}} \left[ \int_0^\infty \omega(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= k_s(\sigma) \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Setting  $M \geq k_s(\sigma)$ , then (17) follows.

Therefore, the conditions (i), (ii) and (iii) are equivalent.

When Condition (iii) is satisfied, if there exists a constant  $M < k_s(\sigma)$ , such that (18) is valid, then by Lemma 3, we have  $M \geq k_s(\sigma)$ . By this contradiction it follows that the constant factor  $M = k_s(\sigma)$  in (18) is the best possible. The constant factor  $M = k_s(\sigma)$  in (17) is still the best possible. Otherwise, by (19) (for  $\sigma_1 = \sigma$ ), we would conclude that the constant factor  $M = k_s(\sigma)$  in (18) is not the best possible.  $\square$

Setting  $y = \frac{1}{Y}$ ,  $G(Y) = Y^{s\lambda-2} g(\frac{1}{Y})$ ,  $\mu_1 = s\lambda - \sigma_1$  in Theorem 4, then replacing  $Y$  (respectively  $G(Y)$ ) by  $y$  (respectively  $g(y)$ ), we deduce the following result.

**Corollary 5.** *The following conditions are equivalent:*

(i) *There exists a constant  $M$ , such that for any  $f(x) \geq 0$ , satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we have the following integral inequality:*

$$\begin{aligned} &\left\{ \int_0^\infty y^{p\mu_1-1} \left[ \int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx \right]^p dy \right\}^{\frac{1}{p}} \\ (22) \quad &< M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned}$$

(ii) *There exists a constant  $M$ , such that for any  $f(x), g(y) \geq 0$ , satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*and*

$$0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following Hilbert-type integral inequality with homogeneous kernel:

$$(23) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx dy < M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy \right]^{\frac{1}{q}};$$

(iii)  $\mu_1 = \mu$ .

If Condition (iii) holds, then we have  $M \geq k_s(\sigma)$ , and the constant factor  $M = k_s(\sigma)$  in (22) and (23) is the best possible.

**Remark 6.** On the other hand, setting  $y = \frac{1}{Y}$ ,  $G(Y) = Y^{s\lambda-2} g(\frac{1}{Y})$ ,  $\sigma_1 = s\lambda - \mu_1$ , in Corollary 5, then replacing  $Y$  (resp.  $G(Y)$ ) by  $y$  (resp.  $g(y)$ ), we deduce Theorem 4. Hence, Theorem 4 and Corollary 5 are equivalent.

#### 4. OPERATOR EXPRESSIONS

We set the following functions:

$$\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma)-1}, \phi(y) := y^{q(1-\mu)-1}, \text{ wherefrom,}$$

$$\psi^{1-p}(y) = y^{p\sigma-1}, \phi^{1-p}(y) = y^{p\mu-1} \quad (x, y \in \mathbb{R}_+).$$

Define the following real normed linear spaces:

$$\begin{aligned} L_{p,\varphi}(\mathbb{R}_+) &= \left\{ f : \|f\|_{p,\varphi} := \left( \int_0^\infty \varphi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\psi}(\mathbb{R}_+) &= \left\{ g : \|g\|_{q,\psi} := \left( \int_0^\infty \psi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{q,\phi}(\mathbb{R}_+) &= \left\{ g : \|g\|_{q,\phi} := \left( \int_0^\infty \phi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{p,\psi^{1-p}}(\mathbb{R}_+) &= \left\{ h : \|h\|_{p,\psi^{1-p}} = \left( \int_0^\infty \psi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\phi^{1-p}}(\mathbb{R}_+) &= \left\{ h : \|h\|_{q,\phi^{1-p}} = \left( \int_0^\infty \phi^{1-p}(y) |h(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 4 (setting  $\sigma_1 = \sigma$ ), for  $f \in L_{p,\varphi}(\mathbb{R}_+)$ , setting

$$h_1(y) := \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} f(x) dx \quad (y \in \mathbb{R}_+),$$

by (17), we have

$$(24) \quad \|h_1\|_{p,\psi^{1-p}} = \left( \int_0^\infty \psi^{1-p}(y) h_1^p(y) dy \right)^{\frac{1}{p}} < M \|f\|_{p,\varphi} < \infty.$$

**Definition 7.** Define a Hilbert-type integral operator with nonhomogeneous kernel  $T^{(1)} : L_{p,\varphi}(\mathbb{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbb{R}_+)$  as follows: For any  $f \in L_{p,\varphi}(\mathbb{R}_+)$ , there exists a unique representation  $T^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbb{R}_+)$ , satisfying  $T^{(1)}f(y) = h_1(y)$ , for any  $y \in \mathbb{R}_+$ .

In view of (24), it follows that

$$\|T^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and then the operator  $T^{(1)}$  is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq 0) \in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|T^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of  $T^{(1)}f$  and  $g$  as follows:

$$(T^{(1)}f, g) := \int_0^\infty \left\{ \int_0^\infty \frac{f(x)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx \right\} g(y) dy,$$

then we can rewrite Theorem 4 as follows:

**Theorem 8.** The following conditions are equivalent:

(i) There exists a constant  $M$ , such that for any  $f(x) \geq 0$ ,  $f \in L_{p,\varphi}(\mathbb{R}_+)$ ,  $\|f\|_{p,\varphi} > 0$ , we have the following inequality:

$$\|T^{(1)}f\|_{p,\psi^{1-p}} < M\|f\|_{p,\varphi};$$

(ii) there exists a constant  $M$ , such that for any  $f(x), g(y) \geq 0$ ,  $f \in L_{p,\varphi}(\mathbb{R}_+)$ ,  $g \in L_{q,\psi}(\mathbb{R}_+)$ ,  $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$ , we have the following inequality:

$$(T^{(1)}f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\psi}.$$

We still have  $\|T^{(1)}\| = k_s(\sigma) \leq M$ .

(b) In view of Corollary 5 (with  $\mu_1 = \mu$ ), for  $f \in L_{p,\varphi}(\mathbb{R}_+)$ , setting

$$h_2(y) := \int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx$$

defined for every  $y \in \mathbb{R}_+$ , by (22) we have

$$(25) \quad \|h_2\|_{p,\phi^{1-p}} = \left( \int_0^\infty \phi^{1-p}(y) h_2^p(y) dy \right)^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty.$$

**Definition 9.** Define a Hilbert-type integral operator with the homogeneous kernel  $T^{(2)} : L_{p,\varphi}(\mathbb{R}_+) \rightarrow L_{p,\phi^{1-p}}(\mathbb{R}_+)$  as follows: For any  $f \in L_{p,\varphi}(\mathbb{R}_+)$ , there exists a unique representation  $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbb{R}_+)$ , satisfying  $T^{(2)}f(y) = h_2(y)$ , for any  $y \in \mathbb{R}_+$ .

In view of (25), it follows that

$$\|T^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and then the operator  $T^{(2)}$  is bounded satisfying

$$\|T^{(2)}\| = \sup_{f(\neq 0) \in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|T^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of  $T^{(2)}f$  and  $g$  as follows:

$$(T^{(2)}f, g) := \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx \right] g(y) dy,$$

then we can rewrite Corollary 5 as below:

**Corollary 10.** *The following conditions are equivalent:*

(i) *There exists a constant  $M$ , such that for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbb{R}_+)$ ,  $\|f\|_{p,\varphi} > 0$ , we have the following inequality:*

$$\|T^{(2)}f\|_{p,\phi^{1-p}} < M\|f\|_{p,\varphi};$$

(ii) *there exists a constant  $M$ , such that for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbb{R}_+)$ ,  $g \in L_{q,\phi}(\mathbb{R}_+)$ ,  $\|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$ , we have the following inequality:*

$$(T^{(2)}f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\phi}.$$

We still have  $\|T^{(2)}\| = k_s(\sigma) \leq M$ .

**Remark 11.** *Theorem 8 and Corollary 10 are equivalent.*

**Acknowledgments.** B. Yang: This work is supported by the National Natural Science Foundation (No. 61772140), and Characteristic innovation project of Guangdong Provincial Colleges and universities in 2020 (No. 2020KTSCX088). We are grateful for this support.

### REFERENCES

1. V. ADIYASUREN, T. BATBOLD, M. KRNIĆ: *Half-discrete Hilbert-type inequalities with mean operators, the best constants, and applications*, Applied mathematics and computation, 231(2014), 148-159 doi:10.1016/j.amc.2014.01.011
2. B. ARPAD, O. CHOONGHONG: *Best constant for certain multi linear integral operator*, Journal of Inequalities and Applications, 2006, no. 28582.
3. I. BRNETIĆ, M. KRNIĆ, J. PEČARIĆ: *Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters*, Bulletin of the Australian Mathematical Society, 71 (2005), 447-457

4. Q. CHEN, B. HE, Y. HONG, L. C. ZHEN: *Equivalent parameter conditions for the validity of half-discrete Hilbert-type multiple integral inequality with generalized homogeneous kernel*, Journal of Function Spaces, Volume 2020, Article ID 7414861, 6 pages.
5. P. GAO: *On weight Hardy inequalities for non-increasing sequence*, Journal of Mathematical Inequalities, 2018, 12(2), 551-557.
6. G. H. HARDY: *Note on a theorem of Hilbert concerning series of positive terms*, Proceedings London Math. Soc., 1925, 23(2), Records of Proc. xlv-xlvi.
7. G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA: *Inequalities*, Cambridge University Press, Cambridge, USA, 1934.
8. Y. HONG: *On Hardy-Hilbert integral inequalities with some parameters*, J. Ineq. in Pure & Applied Math., 2005, 6(4), Art. 92: 1-10.
9. Y. HONG: *On the structure character of Hilbert's type integral inequality with homogeneous kernel and applications*, Journal of Jilin University (Science Edition), 2017, 55(2), 189-194.
10. M. KRNIĆ, J. E. PEČARIĆ: *Hilbert's inequalities and their reverses*, Publ. Math. Debrecen, 2005, 67(3-4), 315-331.
11. M. KRNIĆ, P. VUKOVIĆ: *Multidimensional Hilbert-type inequalities obtained via local fractional calculus*, Acta applicandae mathematicae, 169 (2020), 1; 667-680 doi:10.1007/s10440-020-00317-x
12. J. C. KUANG: *Real and functional analysis (Continuation)(second volume)*, Higher Education Press, Beijing, 2015.
13. J. C. KUANG: *Applied inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.
14. Y. J. LI, B. HE: *On inequalities of Hilbert's type*, Bulletin of the Australian Mathematical Society, 2007, 76(1), 1-13.
15. Q. LIU: *A Hilbert-type integral inequality under configuring free power and its applications*, Journal of Inequalities and Applications (2019), 2019:91.
16. D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK: *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic, Boston, USA, 1991.
17. Y. L. PAN, H. T. WANG, F. T. WANG: *On complex functions*, Science Press, Beijing, China (2006).
18. J. S. XU: *Hardy-Hilbert's Inequalities with two parameters*, Advances in Mathematics, 2007, 36(2), 63-76.
19. B. C. YANG: *On Hilbert's integral inequality*, Journal of Mathematical Analysis and Applications, 1998, 220, 778-785.
20. B. C. YANG: *A note on Hilbert's integral inequality*, Chinese Quarterly Journal of Mathematics, 1998, 13(4), 83-86.
21. B. C. YANG: *On an extension of Hilbert's integral inequality with some parameters*, The Australian Journal of Mathematical Analysis and Applications, 2004, 1(1), Art.11,1-8.
22. B. C. YANG, I. BRNETIĆ, M. KRNIĆ, J. E. PEČARIĆ: *Generalization of Hilbert and Hardy-Hilbert integral inequalities*, Math. Ineq. and Appl., 2005, 8(2), 259-272.

23. B. C. YANG: *The norm of operator and Hilbert-type inequalities*, Science Press, Beijing, China, 2009.
24. B. C. YANG: *Hilbert-type integral inequalities*, Bentham Science Publishers Ltd., The United Emirates, 2009.
25. B. C. YANG: *On Hilbert-type integral inequalities and their operator expressions*, Journal of Guangaong University of Education, 2013, 33(5), 1-17.
26. B. C. YANG: *On a more accurate multidimensional Hilbert-type inequality with parameters*, Mathematical Inequalities and Applications, 2015, 18(2),429-441.
27. B. C. YANG, D. ANDRICA, O. BAGDASAR AND M. TH. RASSIAS: *On Two Kinds of the Hardy-Type Integral Inequalities in the Whole Plane with the Equivalent Forms*, In: Mathematical Analysis in Interdisciplinary Research, Springer (to appear).
28. M. H. YOU, Y. GUAN: *On a Hilbert-type integral inequality with non-homogeneous kernel of mixed hyperbolic functions*, Journal of Mathematical Inequalities, 2019, 13(4), 1197 -1208.
29. C. J. ZHAO, WINGSUM CHEUNG: *On Hilbert's inequalities with alternating signs*, Journal of Mathematical Inequalities, 2018, 12(1), 191-200.
30. W. Y. ZHONG, B. C. YANG: *On multiple Hardy-Hilbert's integral inequality with kernel*, Journal of Inequalities and Applications, Vol. 2007, Art.ID 27962, 17 pages.

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(Received 14. 05. 2022.)

(Revised 19. 09. 2022.)

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