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NOTE ON AN INEQUALITY OF M. A. MALIK

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Let $P(z) := \sum_{v=0}^{n} a_v z^v$ be a univariate complex coefficient polynomial of degree n. It was shown by Malik [J London Math Soc, **1** (1969), 57–60] that if $P(z)$ has all its zeros in $|z| \leq k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|.$$

In this paper, we prove an inequality for the polar derivative of a polynomial which besides give extensions and refinements of the above inequality also produce various inequalities that are sharper than the previous ones known in very rich literature on this subject.

1. INTRODUCTION

Let $P(z) := \sum_{v=0}^{n} a_v z^v$ be a univariate complex coefficient polynomial of degree n and $P'(z)$ its derivative. Various inequalities in both directions relating the norm of the derivative and the polynomial itself play a key role in the literature for proving the inverse theorems in approximation theory and, of course have their own intrinsic interest. The Bernstein and Turán-type inequalities are known for various norms and for many classes of functions such as polynomials with various constraints, and on various regions of the complex plane. One basic result of Bernstein [**2**] that relates the uniform-norm of the derivative of a polynomial on the unit circle to that of the polynomial itself on the same circle states that: if $P(z)$ is a polynomial

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of degree n, then it is true that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin. A natural question to ask is, with the restriction on the zeros of $P(z)$, how big or small can $|P'(z)|$ be on $|z| = 1$? There are interesting inequalities in both directions, but the present paper only considers lower estimates for $|P'(z)|$ on $|z| = 1$. It turns out that to have any hope of a lower bound for $|P'(z)|$, one must have some control over the location of zeros of $P(z)$. One of the first results along these lines is by P. Turán [**18**], who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

Polynomials whose derivatives actually achieve this lower estimate have all their zeros on $|z| = 1$. More generally, if the polynomial $P(z)$ has all its zeros in $|z| \leq k,\ k \leq 1$, it was proved by Malik [**8**] that the inequality (2) can be replaced by

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{3}$$

In the last few decades, various generalizations and refinements of (3) have appeared in the literature. In fact, in 1991, Govil [**5**] proved that if $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)| + \frac{n}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)|. \tag{4}$$

For more results of similar nature, and a survey of their various extensions and refinements, we refer the reader to the comprehensive books of Marden [**9**], Milovanović et al. [**10**] and Rahman and Schmeisser [**17**].
For a polynomial $P(z)$ of degree n, we define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point α (see [**9**]). The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \left\{ \frac{D_\alpha P(z)}{\alpha} \right\} = P'(z),$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.
In the literature, we see that inequalities have been extended from ordinary derivative to polar derivative of polynomials, and for some of the recent research and developments in this direction, one can consult the papers ([**6**], [**11**]-[**16**], [**19**],

[**20**]). As a polar derivative generalization of (3), Aziz and Rather [**1**] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for every complex number α with $|\alpha| \ge k$,

$$\max_{|z|=1} |D_\alpha P(z)| \ge n \left(\frac{|\alpha| - k}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{5}$$

The corresponding polar derivative analogue of (4) and a refinement of (5), was given by Dewan et al. [**3**]. They proved that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for every complex number α with $|\alpha| \ge k$,

$$\max_{|z|=1} |D_\alpha P(z)| \ge n \left(\frac{|\alpha| - k}{1+k} \right) \max_{|z|=1} |P(z)| + n \left(\frac{|\alpha| + 1}{k^{n-1}(1+k)} \right) \min_{|z|=k} |P(z)|. \tag{6}$$

In this paper, we have obtained an inequality relating the uniform-norm on the unit circle of the polar derivative and the underlying polynomial. The obtained result provides refinements and generalizations of (5) and (6). Besides, it gives strengthening of (3), (4) and related results as well.

Theorem 1. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,*

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \ge{}& n \left(\frac{|\alpha| - k}{1+k} \right) \max_{|z|=1} |P(z)| + n \left(\frac{|\alpha| + 1}{k^{n-1}(1+k)} \right) m \\ &+ n \left(\frac{k - A_1}{1+k} \right) \max_{|z|=1} |P(z)| + n \left(\frac{A_1 - k}{k^{n}(1+k)} \right) m \\ &+ k \left(\frac{|\alpha| - A_1}{1+k} \right) \left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|} \right) \left(\max_{|z|=1} |P(z)| - \frac{m}{k^n} \right), \end{aligned} \tag{7}$$

where

$$A_1 = \frac{n\left(|a_n| - \frac{m}{k^n}\right)k^2 + |a_{n-1}|}{n\left(|a_n| - \frac{m}{k^n}\right) + |a_{n-1}|} \quad \text{and } m = \min_{|z|=k} |P(z)|.$$

Equality in (7) holds for $P(z) = (z + k)^n$, with real $\alpha \ge k$.

Remark 2. *It is easy to show that*

$$n \left(\frac{k - A_1}{1+k} \right) \max_{|z|=1} |P(z)| + \frac{n(A_1 - k)m}{k^n(1+k)} \ge 0,$$

which is equivalent to showing that

$$n \left(\frac{k - A_1}{1+k} \right) \max_{|z|=1} |P(z)| \ge \frac{n(k - A_1)m}{k^n(1+k)}.$$

In view of Lemma 4 (for $\mu = 1$), the above inequality becomes equivalent to

$$\max_{|z|=1} |P(z)| \geq \frac{m}{k^n}. \tag{8}$$

Now, using (1) in Lemma 5 (for $\mu = 1$), we get

$$\begin{aligned} |Q'(z)| &\leq nk \max_{|z|=1} |P(z)| - \frac{nk}{k^n} m \\ &= nk \left\{ \max_{|z|=1} |P(z)| - \frac{m}{k^n} \right\}, \end{aligned}$$

which is true and hence inequality (8) holds.

Remark 3. *Since* $P(z) = \sum_{v=0}^{n} a_v z^v \neq 0$ *in* $|z| > k,\ k \leq 1$, *and if* $z_1, z_2, ..., z_n$, *are the zeros of* $P(z)$, *then*

$$\left| \frac{a_0}{a_n} \right| = |z_1 z_2 ... z_n| \leq k^n. \tag{9}$$

Here, we show that

$$m + |a_0| \leq k^n |a_n|. \tag{10}$$

We can assume without loss of generality that $P(z)$ has no zeros on $|z| = k$, for otherwise (10) holds trivially by (9). Now, as in the proof of Theorem 1 (given in next section), we have for every β with $|\beta| < 1$, the polynomial

$$P(z) - \frac{\beta m z^n}{k^n} = \left(a_n - \frac{\beta m}{k^n} \right) z^n + \sum_{v=0}^{n-1} a_v z^v,$$

has all its zeros in $|z| < k,\ k \leq 1$, hence

$$\left| \frac{a_0}{a_n - \frac{\beta m}{k^n}} \right| \leq k^n. \tag{11}$$

If in (11), we choose the argument of β suitably, so that

$$\left| a_n - \frac{\beta m}{k^n} \right| = |a_n| - \frac{|\beta| m}{k^n},$$

which is possible by Lemma 8, we get

$$|a_0| \leq k^n |a_n| - |\beta| m. \tag{12}$$

The inequality (10) follows by letting $|\beta| \to 1$ in (12).
Using this and the fact from Lemma 4 that $A_1 \leq k$, we get from Theorem 1 the following improved version of (6).

Corollary 4. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$, we have*

$$
\begin{aligned}
\max_{|z|=1} |D_\alpha P(z)| \ge n \left(\frac{|\alpha| - k}{1+k}\right) \max_{|z|=1} |P(z)| + n \left(\frac{|\alpha| + 1}{k^{n-1}(1+k)}\right) m \\
+ k \left(\frac{|\alpha| - k}{1+k}\right) \left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|}\right) \left(\max_{|z|=1} |P(z)| - \frac{m}{k^n}\right), \qquad (13)
\end{aligned}
$$

where

$$m = \min_{|z|=k} |P(z)|.$$

Equality in (13) holds for $P(z) = (z+k)^n$ with real $\alpha \ge k$.
Dividing both sides of inequality (13) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following refinement of (4).

Corollary 5. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then*

$$
\begin{aligned}
\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k} \max_{|z|=1} |P(z)| + \frac{nm}{k^{n-1}(1+k)} \\
+ \frac{k}{1+k} \left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|}\right) \left(\max_{|z|=1} |P(z)| - \frac{m}{k^n}\right), \qquad (14)
\end{aligned}
$$

where

$$m = \min_{|z|=k} |P(z)|.$$

Equality in (14) holds for $P(z) = (z+k)^n$.

Remark 6. *It may be remarked that $A_1 = 1$ for $k = 1$, and by Remark 2 for $0 < k < 1$, the bound obtained in (7) is sharper than the bound obtained from (6). One can also observe that the inequality (14) improves inequality (4) considerably when $|a_n|k^n - |a_0| - m \ne 0$ and $k \ne 1$.*

2. AUXILIARY RESULTS

For the proof of the theorem, we shall make use of the following lemmas. The following lemma is due to Govil and McTume [**7**].

Lemma 7. *If $P(z) = \sum_{v=0}^{n} a_\nu z^v$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then on $|z| = 1$, we have*

$$|Q'(z)| \le S_1 |P'(z)|,$$

where here and throughout

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} \text{ and } S_1 = \frac{nk^2|a_n| + |a_{n-1}|}{|a_{n-1}| + n|a_n|}.$$

Lemma 8. *If $x_v,\ v=1,2,...,n,$ is a sequence of real numbers such that for all $v\in\mathbb{N}$, we have $0\le x_v\le 1$, then for all $n\in\mathbb{N}$*

$$\sum_{v=1}^{n}\frac{1-x_v}{1+x_v}\ge\frac{1-\prod\limits_{v=1}^{n}x_v}{1+\prod\limits_{v=1}^{n}x_v}.$$

The above lemma easily follows by induction. We omit the details.

Lemma 9. *If $P(z)=\sum_{v=0}^{n}a_\nu z^v$ is a polynomial of degree n having all its zeros in $|z|\le k,\ k\le 1$, then for any complex number α with $|\alpha|\ge S_1$ and $|z|=1$, we have*

$$|D_\alpha P(z)|\ge n\left(\frac{|\alpha|-S_1}{1+k}\right)\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{15}$$

Proof of Lemma 9. By hypothesis, the polynomial $P(z)=\sum\limits_{v=0}^{n}a_vz^v$ has all its zeros in $|z|\le k,\ k\le 1$. If $z_1,z_2,...,z_n$, are the zeros of $P(z)$, then $|z_v|\le k,\ k\le 1$, and we can write

$$P(z)=a_n\prod_{v=1}^{n}(z-z_v).$$

This gives

$$\frac{zP'(z)}{P(z)}=\sum_{v=1}^{n}\frac{z}{z-z_v}.$$

Now for points $e^{i\theta},\ 0\le\theta<2\pi$, with $P(e^{i\theta})\ne 0$, we have

$$\begin{aligned}\text{Re}\left(\frac{zP'(z)}{P(z)}\right)\bigg|_{z=e^{i\theta}}&=\sum_{v=1}^{n}\text{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-z_v}\right)\\&\ge\sum_{v=1}^{n}\frac{1}{1+|z_v|}\\&=\frac{n}{1+k}+\frac{k}{1+k}\sum_{v=1}^{n}\frac{k-|z_v|}{k+k|z_v|}\\&\ge\frac{n}{1+k}+\frac{k}{1+k}\sum_{v=1}^{n}\frac{k-|z_v|}{k+|z_v|}\ \text{(because } k\le 1)\\&\ge\frac{n}{1+k}+\frac{k}{1+k}\left(\frac{1-\prod\limits_{v=1}^{n}\frac{|z_v|}{k}}{1+\prod\limits_{v=1}^{n}\frac{|z_v|}{k}}\right)\ \text{(by Lemma 2)}\\&=\frac{n}{1+k}+\frac{k}{1+k}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right),\end{aligned}$$

which gives for all z on $|z|=1$ for which $P(z)\neq 0$,

$$\left|\frac{zP'(z)}{P(z)}\right| \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}.$$

This gives for $|z|=1$, that

$$|P'(z)| \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}|P(z)|. \tag{16}$$

If $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then $P(z)=z^n\overline{Q\left(\frac{1}{\bar{z}}\right)}$ and it is easy to verify that for $|z|=1$,

$$|Q'(z)|=|nP(z)-zP'(z)|. \tag{17}$$

Also, for every complex number α with $|\alpha|\geq S_1$, we have

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z)+(\alpha-z)P'(z)|\\ &\geq |\alpha||P'(z)|-|nP(z)-zP'(z)|,\end{aligned}$$

which gives by (17) for $|z|=1$, that

$$|D_\alpha P(z)| \geq |\alpha||P'(z)|-|Q'(z)|. \tag{18}$$

Inequality (18) when combined with Lemma 1, gives for $|z|=1$, that

$$|D_\alpha P(z)| \geq (|\alpha|-S_1)|P'(z)|,$$

which in conjunction with (16) gives (15). This completes the proof of Lemma 9.

□

The following two lemmas are due to Dewan et al. [**4**].

Lemma 10. *If $P(z)=a_nz^n+\sum_{v=\mu}^{n}a_{n-v}z^{n-v}$, $1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then*

$$A_\mu=\frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})k^{\mu-1}+\mu|a_{n-\mu}|}\leq k^\mu,$$

where

$$m=\min_{|z|=k}|P(z)|.$$

Lemma 11. *If $P(z)=a_nz^n+\sum_{v=\mu}^{n}a_{n-v}z^{n-v}$, $1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then*

$$|Q'(z)|\leq k^\mu|P'(z)|-\frac{n}{k^{n-\mu}}\min_{|z|=k}|P(z)|.$$

Lemma 12. *(See Marden ([**9**], p.49)) If all the zeros of a polynomial of degree n lie in* $|z| \leq k$*, then for* $|\alpha| \geq k$*, the polar derivative* $D_\alpha P(z)$ *of* $P(z)$ *at the point* α *also has all its zeros in* $|z| \leq k$*.*

Lemma 13. *The function*

$$S(x) = \frac{nxk^2 + |a_{n-1}|}{nx + |a_{n-1}|},$$

where $k \leq 1$ *is a non-increasing function of* x*.*

The proof follows by considering the first derivative test for $S(x)$.

Lemma 14. *If* $P(z) = \sum_{v=0}^{n} a_\nu z^v$ *is a polynomial of degree* n *having all its in* $|z| \leq k,\ k > 0$*, then* $|Q(z)| \geq \frac{m}{k^n}$ *for* $|z| \leq \frac{1}{k}$*, and in particular*

$$|a_n| > \frac{m}{k^n},$$

where $m = \min_{|z|=k} |P(z)|$*.*

Proof of Lemma 14. Recall that $P(z) = \sum_{v=0}^{n} a_\nu z^v$ has all its zeros in $0 < |z| \leq k$, then $Q(z) \neq 0$ for $|z| \leq \frac{1}{k}$. We can assume without loss of generality that $Q(z)$ has no zeros on $|z| = \frac{1}{k}$, for otherwise the result holds trivially. Since $Q(z)$ is analytic in $|z| \leq \frac{1}{k}$ and has no zeros in $|z| \leq \frac{1}{k}$, by the Minimum Modulus Principle

$$\min_{|z|=\frac{1}{k}} |Q(z)| \leq |Q(z)| \text{ for } |z| \leq \frac{1}{k},$$

which implies

$$\frac{1}{k^n} \min_{|z|=k} |P(z)| \leq |Q(z)| \text{ for } |z| \leq \frac{1}{k},$$

which in particular implies

$$\frac{m}{k^n} < |Q(0)| = |a_n|.$$

This proves Lemma 14.

□

3. PROOF OF THE THEOREM

Proof of Theorem 1. Recall that the polynomial $P(z) = \sum_{v=0}^{n} a_v z^v$ of degree n has all its zeros in $|z| \leq k,\ k \leq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Lemma 3 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < k, k \leq 1$, so that $m > 0$. Now $m \leq |P(z)|$ for $|z| = k$, therefore, if β is any complex number with $|\beta| < 1$, then

$$\left|\frac{m\beta z^n}{k^n}\right| < |P(z)| \text{ for } |z| = k.$$

It follows by Rouché's theorem that all the zeros of $P(z) - \frac{m\beta z^n}{k^n}$ also lie in $|z| < k$. Hence by Lemma 6 for $|\alpha| \geq k$, the polynomial

$$D_\alpha\left(P(z) - \frac{m\beta z^n}{k^n}\right) = D_\alpha P(z) - \frac{\beta mn\alpha z^{n-1}}{k^n} \tag{19}$$

also has all its zeros in $|z| < k,\ k \leq 1$. This implies

$$|D_\alpha P(z)| \geq \frac{mn|\alpha||z|^{n-1}}{k^n} \text{ for } |z| \geq k \text{ and } |\alpha| \geq k. \tag{20}$$

Because if (20) is not true, then there is a point $z = z_0$ with $|z_0| \geq k$, such that

$$|D_\alpha P(z_0)| < \left|\frac{mn\alpha z_0^{n-1}}{k^n}\right|.$$

We choose

$$\beta = \frac{k^n D_\alpha P(z_0)}{mn\alpha z_0^{n-1}},$$

so that $|\beta| < 1$, and with this choice of β, from (19), we get

$$D_\alpha\left(P(z_0) - \frac{m\beta z_0^n}{k^n}\right) = 0,$$

where $|z_0| \geq k$, which contradicts the fact that all the zeros of

$$D_\alpha\left(P(z) - \frac{m\beta z^n}{k^n}\right),$$

lie in $|z| < k,\ k \leq 1$, for every β with $|\beta| < 1$. Now, we can apply Lemma 3 to the polynomial $P(z) - \frac{\beta m z^n}{k^n}$ and obtain for $|\alpha| \geq k \geq S_1'$ that

$$\left|D_\alpha\left(P(z) - \frac{\beta m z^n}{k^n}\right)\right| \geq n\left(\frac{|\alpha| - S_1'}{1+k}\right)\left\{1 + \frac{k}{n}\left(\frac{\left|a_n - \frac{\beta m}{k^n}\right|k^n - |a_0|}{\left|a_n - \frac{\beta m}{k^n}\right|k^n + |a_0|}\right)\right\}$$

$$\times\left|P(z) - \frac{m\beta z^n}{k^n}\right| \text{ for } |z| = 1, \tag{21}$$

where

$$S_1' = \frac{n\left|a_n - \frac{\beta m}{k^n}\right|k^2 + |a_{n-1}|}{n\left|a_n - \frac{\beta m}{k^n}\right| + |a_{n-1}|}. \tag{22}$$

Since for every β with $|\beta| < 1$, we have

$$\left|a_n - \frac{\beta m}{k^n}\right| \geq |a_n| - \frac{|\beta| m}{k^n} > |a_n| - \frac{m}{k^n}, \tag{23}$$

and $|a_n| > \frac{m}{k^n}$ by Lemma 8. Now combining (22), (23) and Lemma 7, we have for every β with $|\beta| < 1$,

$$S_1^{'} = \frac{n\left|a_n - \frac{\beta m}{k^n}\right| k^2 + |a_{n-1}|}{n\left|a_n - \frac{\beta m}{k^n}\right| + |a_{n-1}|} \leq \frac{n\left(|a_n| - \frac{m}{k^n}\right) k^2 + |a_{n-1}|}{n\left(|a_n| - \frac{m}{k^n}\right) + |a_{n-1}|} = A_1.$$

Hence from (19), (21) and the fact that

$$x \mapsto \frac{k^n x - |a_0|}{k^n x + |a_0|} \ (x \geq 0),$$

is a non-decreasing function of x, we get for $|z| = 1$,

$$\begin{aligned}\left|D_\alpha P(z) - \frac{\beta m n \alpha z^{n-1}}{k^n}\right| &\geq n\left(\frac{|\alpha| - A_1}{1+k}\right)\left\{1 + \frac{k}{n}\left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|}\right)\right\} \\ &\quad \times \left|P(z) - \frac{m\beta z^n}{k^n}\right| \\ &\geq n\left(\frac{|\alpha| - A_1}{1+k}\right)\left\{1 + \frac{k}{n}\left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|}\right)\right\} \\ &\quad \times \left(|P(z)| - \frac{m|\beta|}{k^n}\right).\end{aligned} \tag{24}$$

If in the left hand side of (24), we choose the argument of β such that

$$\left|D_\alpha P(z) - \frac{\beta m n \alpha z^{n-1}}{k^n}\right| = |D_\alpha P(z)| - \frac{mn|\beta||\alpha||z|^{n-1}}{k^n},$$

which is possible by (20), we get for $|z| = 1$,

$$\begin{aligned}|D_\alpha P(z)| - \frac{mn|\beta||\alpha|}{k^n} &\geq n\left(\frac{|\alpha| - A_1}{1+k}\right)\left\{1 + \frac{k}{n}\left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|}\right)\right\} \\ &\quad \times \left(|P(z)| - \frac{m|\beta|}{k^n}\right).\end{aligned}$$

The above inequality in particular gives for $|z| = 1$, that

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| &\geq n\left(\frac{|\alpha| - A_1}{1+k}\right)\max_{|z|=1}|P(z)| + \frac{mn|\beta|}{k^n}\left(\frac{|\alpha|k + A_1}{1+k}\right) \\ &\quad + k\left(\frac{|\alpha| - A_1}{1+k}\right)\left(\frac{|a_n|k^n - m - |a_0|}{|a_n|k^n - m + |a_0|}\right)\left(\max_{|z|=1}|P(z)| - \frac{m|\beta|}{k^n}\right).\end{aligned} \tag{25}$$

Now, if in (25) we make $|\beta| \to 1$, we get

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-A_1}{1+k}\right)\max_{|z|=1}|P(z)| + \frac{mn}{k^n}\left(\frac{|\alpha|k+A_1}{1+k}\right)\\ + k\left(\frac{|\alpha|-A_1}{1+k}\right)\left(\frac{|a_n|k^n-m-|a_0|}{|a_n|k^n-m+|a_0|}\right)\left(\max_{|z|=1}|P(z)|-\frac{m}{k^n}\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|P(z)| + n\left(\frac{|\alpha|+1}{k^{n-1}(1+k)}\right)m\\ + n\left(\frac{k-A_1}{1+k}\right)\max_{|z|=1}|P(z)| + n\left(\frac{A_1-k}{k^n(1+k)}\right)m\\ + k\left(\frac{|\alpha|-A_1}{1+k}\right)\left(\frac{|a_n|k^n-m-|a_0|}{|a_n|k^n-m+|a_0|}\right)\left(\max_{|z|=1}|P(z)|-\frac{m}{k^n}\right),\end{aligned}$$

which is (7) and this completes the Proof of Theorem 1. □

REFERENCES

1. A. Aziz and N. A. Rather: *A refinement of a theorem of Paul Turán concerning polynomials.* Math. Inequal. Appl., **1** (1998), 231–238.

2. S. Bernstein: *Sur l'ordre de la meilleure approximation des functions continues par des polynômes de degré donné.* Mem. Acad. R. Belg., **4** (1912), 1–103.

3. K. K. Dewan, N. Singh and R. Lal: *Inequalities for the polar derivatives of polynomials.* Int. J. Pure Appl. Math., **33** (2006), 109–117.

4. K. K. Dewan, N. Singh and A. Mir: *Extensions of some polynomial inequalities to the polar derivative.* J. Math. Anal. Appl., **352** (2009), 807–815.

5. N. K. Govil: *Some inequalities for derivative of polynomials.* J. Approx. Theory, **66** (1991), 29–35.

6. N. K. Govil and P. Kumar: *On sharpening of an inequality of Turán.* Appl. Anal. Discrete Math., **13** (2019), 711-720.

7. N. K. Govil and G. N. McTume: *Some generalizations involving the polar derivative for an inequality of Paul Turán.* Acta Math. Hungar., **104** (2004), 115–126.

8. M. A. Malik: *On the derivative of a polynomial.* J. Lond. Math. Soc., **1** (1969), 57–60.

9. M. Marden: *Geometry of Polynomials.* Math. Surveys, **3** (1966).

10. G. V. Milovanović, D. S. Mitrinović and TH. M. Rassias: *Topics in Polynomials, Extremal Problems, Inequalities, Zeros.* World Scientific, Singapore (1994).

11. G. V. Milovanović and A. Mir: *On the Erdös-Lax inequality concerning polynomials.* Math. Inequal. Appl., **23** (2020) 1499–1508.

12. A. Mir: *Integral norm estimates for the polar derivative of lacunary-type complex polynomials.* Appl. Anal. Discrete Math., **14** (2020), 472–489.

13. A. Mir: *Bernstein-type integral inequalities for a certain class of polynomials.* Mediterranean J. Math., **16** (2019), (Art. 143) pp. 1-11.

14. A. Mir: *Generalizations of Bernstein and Turán-type inequalities for the polar derivative of a complex polynomial.* Mediterranean J. Math., **17** (2020), (Art. 14) pp. 1-12.

15. A. Mir and I. Hussain: *On the Erdös-Lax inequality concerning polynomials.* C. R. Acad. Sci. Paris Ser. I, **355** (2017), 1055–1062.

16. A. Mir and A. Wani: *A note on two recent results about polynomials with restricted zeros.* J. Math. Inequal., **14** (2020), 47–52.

17. Q. I. Rahman and G. Schmeisser: *Analytic Theory of Polynomials.* Oxford University Press (2002).

18. P. Turán: *Über die Ableitung von Polynomen.* Compositio Math., **7** (1939), 89-95.

19. A. Zireh: *On the polar derivative of a polynomial.* Bull. Iranian Math. Soc., **40** (2014), 967–976.

20. A. Zireh and M. Bidkham: *Inequalities for the polar derivative of a polynomial with restricted zeros.* Kragjevac J. Math., **40** (2016), 113–124.

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