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BIER SPHERES OF EXTREMAL VOLUME AND GENERALIZED PERMUTOHEDRA

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We study hidden geometry of Bier spheres $Bier(K) = K *_\Delta K^\circ$ by describing their natural geometric realizations, compute their volume, describe an effective criterion for their polytopality, and associate to $Bier(K)$ a natural coarsening $Fan(K)$ of the braid fan. We also establish a connection of Bier spheres of maximal volume with recent generalizations of the classical Van Kampen-Flores theorem and clarify the role of Bier spheres in the theory of generalized permutohedra.

1. INTRODUCTION

The problem of deciding if a given triangulation of a sphere is realizable as the boundary sphere of a simplicial, convex polytope is known as the "Simplicial Steinitz problem" [**11**]. This is an example of a problem of geometric combinatorics which links together areas of mathematics as distant as combinatorial optimization [**21**], convex polytopes [**24**], algebraic geometry [**11**], topological combinatorics [**17**], discrete and computational geometry [**25**], etc.

It is known by an indirect and non-constructive argument that a vast majority of triangulated spheres are "non-polytopal", in the sense that they are not combinatorially isomorphic to the boundary of a convex polytope. This holds, in particular, for *Bier spheres* $Bier(K)$ (named after T. Bier, see [**17**, Section 5.6]), the $(n-2)$-dimensional, combinatorial spheres on $2n$-vertices, constructed with the aid of simplicial complexes $K \subsetneq 2^{[n]}$.

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Combinatorics of Bier spheres and their generalizations have been studied in numerous publications [**2, 5, 6, 13, 14, 15, 16, 22**]. In this paper we put more emphasis on the interplay of geometry and combinatorics of Bier spheres, in particular, continuing the research from [**13**], we study the connection with *generalized permutohedra*, one of the most important and well studied classes of convex polytopes in recent years.

2. AN OVERVIEW OF THE PAPER AND NEW RESULTS

One of the main new results of [**13**] was the observation ([**13**, Theorem 3.1]) that each Bier sphere $Bier(K)$, defined as a canonical triangulation of a $(n-2)$ sphere S^{n-2} associated to an abstract simplicial complex $K \subsetneq 2^{[n]}$, admits a starshaped embedding in $\mathbb{R}^{n-1}$.

It turns out that the radial fan $Fan(K)$ of the starshaped embedding of the Bier sphere $Bier(K)$, described in the proof of this result, is a coarsening of the *braid arrangement fan.* This fact was not emphasized in [**13**], however it is interesting in itself and certainly deserves further study.

Recall that the braid arrangement fan is the normal fan of the standard permutohedron [**24**] and that the coarsening of the braid fan leads to an important and well studied class of *generalized permutohedra* [**19, 20, 4, 7, 23, 18, 9, 12**], or deformed permutohedra, as they are called by some authors.

In this paper we take a closer look at the fan $Fan(K)$ (the *canonical* or *Bier fan* of a simplicial complex K), with the goal to clarify the role of Bier spheres in the theory of generalized permutohedra and to study other geometric properties of Bier spheres arising from this construction.

The main new results of the paper are the following.

In Section 4 (see also Sections 3 and 5) we give a combinatorial proof that $Fan(K)$ is refined by the braid fan, relying on the *preposets-braid cones* dictionary from [**20**]. In particular we show that the maximal cones of $Fan(K)$ are associated with *tree posets* which have precisely one node which is not a leaf.

In Section 6 we study Bier spheres (or rather the associated starshaped sets $Star(K)$) of extremal volume. In particular we show (Proposition 6) that Bier spheres of maximal volume are closely related to the class of *nearly neighborly Bier spheres*, studied in [**2**], and *balanced simplicial complexes* [**15**], which provide a natural class of examples extending the classical Van Kampen-Flores theorem, see [**15**, Theorem 3.5].

One of the consequences of Propositions 5 and 6 is that all starshaped sets $Star(K)$ of maximal volume coincide with one and the same, universal $(n-1)$-dimensional convex set (convex polytope), denoted by Ω_n and referred to as the *Van Kampen-Flores polytope.* The structure of the Van Kampen-Flores polytope is clarified (and its name explained) in Sections 6 and 7, in particular we show (Theorem 14) that the polar dual of Ω_n is affine-isomorphic to a *median hypersimplex.*

In Section 8 we prove a K-submodularity theorem which for polytopal Bier spheres plays the role similar to the role of classical submodular functions (polymatroids) in the theory of generalized permutohedra. With the aid of this result we obtain a useful criterion for a Bier sphere to be polytopal.

For the reader's convenience here is a glossary with brief descriptions of the main objects studied in this paper (see Section 5 for more complete exposition).

$Bier(K) = K *_\Delta K^\circ$, the Bier sphere of K, is a combinatorial object (simplicial complex), defined as a deleted join of two simplicial complexes (K and its Alexander dual K°).

$Fan(K) = BierFan(K)$, the *canonical* or the *Bier fan* of K, is a complete, simplicial fan in $H_0 \cong \mathbb{R}^{n-1}$, associated to a simplicial complex $K \subsetneq 2^{[n]}$.

$\mathcal{R}_{\pm\delta}(Bier(K))$ is the *canonical starshaped realization* of $Bier(K)$ described in [**13**, Theorem 3.1].

$Star(K)$ is the starshaped body whose boundary is the sphere $\mathcal{R}_{\pm\delta}(Bier(K))$.

Ω_n is a universal, $(n-1)$-dimensional convex polytope (the Van Kampen-Flores polytope) which is equal, as a convex body, to $Star(K)$ for each Bier sphere of maximal volume.

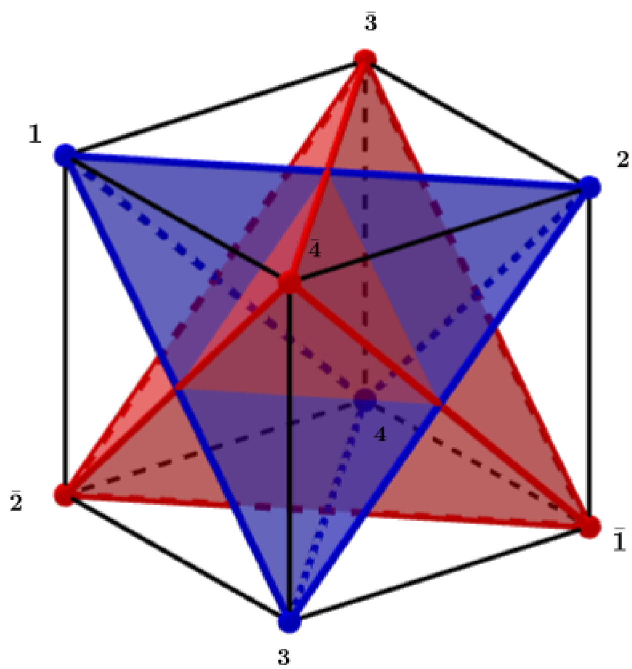


Figure 1: The 3-dimensional cube as the Van Kampen-Flores polytope Ω_4.

Example 1. The geometric objects listed in the glossary can be in the case $n = 4$ easily described with the aid of a 3-dimensional cube (Figure 1)).

The tetrahedron $\Delta_{[4]}$ with vertices $\{1, 2, 3, 4\}$ (colored in blue) is the ambient simplex of the simplicial complex K. Similarly, the ambient simplex of K° (the Alexander dual of K) is the red tetrahedron $\Delta_{[\bar{4}]}$ with vertices $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

For illustration let K be the 1-dimensional skeleton of $\Delta_{[4]}$. Then K° is the 0-dimensional skeleton of $\Delta_{[\bar{4}]}$ and the maximal simplices of the associated Bier sphere $Bier(K)$ are triangles (3-element sets) $\{i, j, \bar{k}\}$, for distinct elements $i, j, k \in [4]$.

Strictly speaking what we have just described (see [**17**, Definition 5.6.1]) is $Bier(K)$, as an abstract simplicial complex with vertices in $S = \{1, 2, 3, 4, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. By interpreting the elements of S as vertices of two centrally symmetric tetrahedra, $\Delta_{[4]}$ and $\Delta_{[\bar{4}]} = -\Delta_{[4]}$, depicted in Figure 1, each abstract 2-simplex $\{i, j, \bar{k}\}$ turns into a genuine, geometric triangle.

The union of these triangles is precisely the canonical starshaped realization $\mathcal{R}_{\pm\delta}(Bier(K))$ of $Bier(K)$, described in [**13**, Theorem 3.1] and listed in the glossary.

The cones over these triangles (with the apex in the center of the cube) form a complete, simplicial fan $Fan(K)$ which is referred to as the Bier fan of K.

$Star(K)$ is the starshaped body inside $Fan(K)$, whose boundary is the sphere $\mathcal{R}_{\pm\delta}(Bier(K))$.

The reader is invited to experiment with other subcomplexes $K \subsetneq \Delta_{[4]}$ and to determine the corresponding objects $\mathcal{R}_{\pm\delta}(Bier(K)), BierFan(K)$ and $Star(K)$. For example if $K = \partial\Delta_{[4]} = 2^{[4]} \setminus \{[4]\}$ then $Star(K) = \Delta_{[4]}$, etc.

3. BIER FANS OF SIMPLICIAL COMPLEXES

Let $K \subsetneq 2^{[n]}$ be a simplicial complex and $K^\circ := \{A \subset [n] \mid [n] \setminus A \notin K\}$ its Alexander dual. By definition, see [**17**, Section 5.6], the associated Bier sphere is the deleted join,

$$Bier(K) := K *_\Delta K^\circ. \tag{1}$$

As in Example 1 (see also [**17**, Section 5.5]) the vertices of the deleted join (1) are $[n] \cup [\bar{n}] = \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ and a simplex $\tau \in Bier(K)$ is described as the union $\tau = A_1 \cup \overline{A_2}$, where A_1 and A_2 are subsets of $[n]$ such that $A_1 \in K, A_2 \in K^\circ, A_1 \cap A_2 = \emptyset$ and (by definition) $\overline{C} := \{\bar{i} \mid i \in C\} \subseteq [\bar{n}]$.

Caveat: For convenience here we use (as in [**15**]) an extended $\tau = (A_1, A_2; B)$ notation for simplices in the Bier sphere, where $B := [n] \setminus (A_1 \cup A_2)$. Hence, an ordered partition $A_1 \sqcup A_2 \sqcup B = [n]$ corresponds to a simplex $\tau \in Bier(K)$ if and only if $A_1 \in K$, $A_2 \in K^\circ$ (which implies $\emptyset \neq B \neq [n]$). In the "interval notation", used in [**2**], the simplex $\tau = (A_1, A_2; B)$ is recorded as the pair $(A_1, [n] \setminus A_2)$ while the same simplex is denoted in [**17**, Section 5.6] by $\tau = A_1 \uplus A_2$.

For example the facets of $Bier(K)$ are triples $\tau = (A_1, A_2; B)$ where $B = \{\nu\}$ is a singleton. In this case τ is (in the interval notation) determined by the pair $A \subsetneq C$, where $A = A_1 \in K$ and $C = A_1 \cup \{\nu\} \notin K$.

The braid arrangement is the arrangement of hyperplanes $Braid_n = \{H_{i,j}\}_{1\leq i<j\leq n}$ in H_0 where $H_0 := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^n/(1,\ldots,1)\mathbb{R}$ and $H_{i,j} := \{x \mid x_i - x_j = 0\}$. The hyperplanes $H_{i,j}$ subdivide the space H_0 into the polyhedral cones

$$C_\pi := \{x \in H_0 \mid x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}\}$$

labeled by permutations $\pi \in S_n$. The cones C_π, together with their faces, form a complete simplicial fan in H_0, called the *braid arrangement fan.*

4. PREPOSETS AND BIER FANS

A binary relation $R \subseteq [n] \times [n]$ is a *preposet* on $[n]$ if it is both *reflexive* and *transitive*. Following [**20**], in explicit calculations we often write $\preccurlyeq_R$, instead of R, and $x \preccurlyeq_R y$, instead of $(x, y) \in R$ or xRy. Given a preposet $\preccurlyeq_R$ we write $x \prec_R y$ if $\preccurlyeq_R$ and $x \neq y$, and $x \equiv_R y$ if both $x \preccurlyeq_R y$ and $y \preccurlyeq_R x$.

For a more detailed account and, in particular, the *preposet-braid cone dictionary*, which describes the geometry of braid cones in the language of preposets, the reader is refereed to [**20**] (Sections 3.3 and 3.4).

Let $\tau = (A_1, A_2; B) \in Bier(K)$. The associated preposet $\preccurlyeq_\tau$ is the binary relation defined as the reflexive and transitive closure of the relation

$$\rho_\tau := (A_1 \times B) \cup (B \times B) \cup (B \times A_2) \subseteq [n] \times [n]\,.$$

Following [**20**] (Section 3.4), the associated *braid cone* is

$$Cone(\preccurlyeq_\tau) = Cone(\tau) = \{x \in H_0 \mid x_i \leq x_j \text{ for each } (i,j) \in \rho_\tau\}\,. \tag{2}$$

In other words $Cone(\preccurlyeq_\tau)$ is described by all inequalities $x_i \leq x_j$, where either $(i,j) \in A_1 \times B$ or $(j,i) \in A_2 \times B$, and all equalities $x_i = x_j$ for all pairs $(i,j) \in B \times B$.

The original proof (and the statement) of the following theorem is more geometric, emphasising the starshaped embedding $\mathcal{R}_{\pm\delta}(Bier(K))$ of the sphere $Bier(K)$. Here we give a different and more combinatorial proof, which uses the preposet-braid cone dictionary.

Theorem 2. ([**13**, Theorem 3.1]) *Let $K \subsetneq 2^{[n]}$ be a simplicial complex. Then the collection of convex cones*

$$Fan(K) = \{Cone(\preccurlyeq_\tau)\}_{\tau\in Bier(K)} \tag{3}$$

is a complete simplicial fan in $H_0 = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$, referred to as the canonical fan *associated to K. Moreover, the face poset $FaceFan(K)$ is isomorphic to the (extended) face poset $FaceBier_0(K) := FaceBier(K) \cup \{\emptyset\}$ of the Bier sphere $Bier(K)$. The construction of the canonical fan is faithful in the sense that if $Fan(K_1) = Fan(K_2)$ then $K_1 = K_2$.*

Proof. The faithfulness of the construction is quite immediate, since one can recover both K and K° from the preposets corresponding to maximal cones in $Fan(K)$. Moreover, the structure of the face poset of $Fan(K)$ is easily recovered from (3).

Let us begin the proof that $Fan(K)$ is a complete, simplicial fan by showing that for each permutation $\pi \in S_n$ there exists exactly one facet $\tau = (A_1, A_2; B) = (A_1, A_2; \{\nu\})$ of the Bier sphere $Bier(K)$ such that,

$$C_\pi = \{x \in H_0 \mid x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}\} \subseteq Cone(\tau)\,.$$

Since $[n] \notin K$ we know that $\{k \mid \{\pi(j)\}_{j\leq k} \notin K\} \neq \emptyset$. Let $p = \min\{k \mid \{\pi(j)\}_{j\leq k} \notin K\}$ and let $\nu = \pi(p)$. By construction $A_1 := \{\pi(j)\}_{j<p} \in K$ and $A_2 := \{\pi(j)\}_{j>p} \in K^\circ$, and it immediately follows that $C_\pi \subseteq Cone(\tau)$ where $\tau = (A_1, A_2; \{\nu\})$.

Conversely, let $Int(C_\pi) \cap Cone(\tau') \neq \emptyset$ where $\tau' = (A_1', A_2', \{\nu'\}) \in Bier(K)$. In other words there exists $x \in Cone(\tau')$ such that

$$x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(n)}\,.$$

Let $p' := \pi^{-1}(\nu')$. Then the condition $x \in Cone(\tau')$ implies that $\{\pi(j)\}_{j<p'} \subseteq A_1' \in K$ and $\{\pi(j)\}_{j>p'} \subseteq A_2' \in K^\circ$, which immediately implies $p = p'$ and $\tau = \tau'$.

If $\tau = (A_1, A_2; B)$ and $\tau' = (A_1', A_2'; B')$ are two, not necessarily maximal, faces of $Bier(K)$, then $Cone(\tau) \cap Cone(\tau') = Cone(\tau'')$ where $\tau'' = (A_1'', A_2''; B'')$ is the simplex determined by the conditions $A_1'' = A_1 \cap A_1'$ and $A_2'' = A_2 \cap A_2'$. Indeed, this follows from the preposet-braid cone dictionary, see [**20**, Proposition 3.5], and the following lemma.

Lemma 3. *The reflexive and transitive closure of the relation $\preccurlyeq_\tau \cup \preccurlyeq_{\tau'}$ coincides with the relation $\preccurlyeq_{\tau''}$. Moreover, $\preccurlyeq_{\tau''}$ is a* contraction *(in the sense of* [**20**], *Section 3.3) of both $\preccurlyeq_\tau$ and $\preccurlyeq_{\tau'}$.*

Proof of Lemma 3: Since $\rho_\tau \cup \rho_{\tau'} \subseteq \rho_{\tau''}$, it is sufficient to show that the reflexive/transitive closure $\preccurlyeq$ of $\preccurlyeq_\tau \cup \preccurlyeq_{\tau'}$ contains the relation $\rho_{\tau''}$. This will follow if we prove that

(4) $\quad i \preccurlyeq j$ for each pair of elements in $B'' = (A_1 \Delta A_1') \cup (A_2 \Delta A_2') \cup B \cup B'$.

As a first step in the proof of (4), let us show that $B \cup B' \subseteq B''$.

As an immediate consequence of the definition of the Alexander dual K° of a simplicial complex K, we obtain the implication

$$X \notin K \text{ and } Y \notin K^\circ \quad \Rightarrow \quad X \cap Y \neq \emptyset\,.$$

From here, in light of $A_1 \cup B \notin K$ and $A_2' \cup B' \notin K^\circ$, we deduce $(A_1 \cup B) \cap (A_2' \cup B') \neq \emptyset$. Choose $s \in B$ and $t \in B'$ and assume $z \in (A_1 \cup B) \cap (A_2' \cup B')$. Then, directly from the definition of preposets $\preccurlyeq_\tau$ and $\preccurlyeq_{\tau'}$, we obtain the relation $t \preccurlyeq_{\tau'} z \preccurlyeq_\tau s$ and, as a consequence, $t \preccurlyeq s$.

Similarly, from $A_1' \cup B' \notin K$ and $A_2 \cup B \notin K^\circ$, we deduce that $z' \in (A_1' \cup B) \cap (A_2 \cup B)$ for some z'. If $s \in B$ and $t \in B'$ then $s \preccurlyeq_\tau z' \preccurlyeq_{\tau'} t$, and as a consequence $s \preccurlyeq t$. The relations $s \preccurlyeq t$ and $t \preccurlyeq s$ together imply that $s \equiv_\preccurlyeq t$, which completes the proof of the inclusion $B \cup B' \subseteq B''$.

For the completion of the proof of (4) let us begin with the case $z \in A_1 \setminus A_1'$. Then $B' \preccurlyeq_{\tau'} z \preccurlyeq_\tau B$ and as a consequence $z \in B''$. Similarly, if $z \in A_2 \setminus A_2''$ then $B \preccurlyeq_{\tau'} z \preccurlyeq_\tau B'$ and again $z \in B''$. The other two cases $A_1' \setminus A_1 \neq \emptyset$ and $A_2' \setminus A_2 \neq \emptyset$ are treated analogously.

For the completion of the proof of Lemma 3 we need to show that both $A_1 \cap A_1'$ and $A_2 \cap A_2'$ are disjoint from B''. This is obvious since if $z \in A_1 \cap A_1'$ ($z \in A_2 \cap A_2'$) then z is never a right hand side (respectively left hand side) of a relation involving $\preccurlyeq_\tau$ or $\preccurlyeq_{\tau'}$ (except for the trivial relations $z \preccurlyeq_\tau z$ and $z \preccurlyeq_{\tau'} z$). $\square$

5. COMPARISON OF TWO DEFINITIONS OF BIER FANS

The definition of the Bier fan $Fan(K)$, used in the previous section (see (3)), uses the language of preposets. From this definition it is almost straightforward, in light of the results from [**20**], that $Fan(K)$ is a coarsening of the braid arrangement fan.

It turns out (Proposition 4) that the fan $Fan(K)$ is isomorphic to the radial fan associated to the starshaped realization $\mathcal{R}_{\pm\delta}(Bier(K))$ of the Bier sphere $Bier(K)$, constructed in [**13**], Theorem 3.1. For the reader's convenience here we provide some details of this construction.

For each set $b = \{b_i\}_{i=1}^n$ of affinelly independent vectors, there is an associated geometric simplex $\Delta_b = \mathrm{Conv}\{b_i\}_{i=1}^n$. For instance $\Delta_e \subset \mathbb{R}^n$ is the standard $(n-1)$-dimensional simplex associated to the standard basis $e = \{e_i\}_{i=1}^n$ in $\mathbb{R}^n$.

Moreover, if $S \in K \subseteq 2^{[n]}$ is a simplex in an abstract simplicial complex, then the associated b-realization is the geometric simplex $R_b(S) = \mathrm{Conv}\{b_i\}_{i\in S}$. For example, if $\delta = (\delta_1, \dots, \delta_n)$ is defined by $\delta_i = e_i - \frac{1}{n}(e_1 + \cdots + e_n)$ then

$$\Delta_\delta = \mathrm{Conv}\{\delta_i\}_{i=1}^n \quad \text{ and } \quad \Delta_{-\delta} = \mathrm{Conv}\{-\delta_i\}_{i=1}^n\,.$$

(Note that Δ_δ and $\Delta_{-\delta} = -\Delta_\delta$ are (for $n = 4$)) precisely the simplices $\Delta_{[4]}$ and $\Delta_{[\bar{4}]}$ used in Example 1 and depicted in Figure 1.) If $T \subseteq [n]$ then $\overline{T}$ is the corresponding subset of $[\bar{n}] = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. The *Starshaped geometric realization* of the abstract simplicial complex $Bier(K) = K *_\Delta K^\circ \subset 2^{[n]} * 2^{[\bar{n}]}$, described in [**13**, Theorem 3.1], is the geometric simplicial complex

$$\mathcal{R}_{\pm\delta}(Bier(K)) = \{R_\delta(S) * R_{-\delta}(T) \mid (S,T) \in K *_\Delta K^\circ\} \tag{5}$$

where $R_\delta(S) * R_{-\delta}(T) := \mathrm{Conv}(R_\delta(S) \cup R_{-\delta}(T))$ is the geometric join of simplices.

Let $\mathrm{Cone}(C) = \cup_{\lambda \geq 0} \lambda C$ be the convex cone with the apex at the origin generated by a convex set $C \subset H_0$. The collection of convex cones

$$\mathrm{Cone}_{\pm\delta}(K) = \{\mathrm{Cone}(R_\delta(S) * R_{-\delta}(T)) \mid (S,T) \in K *_\Delta K^\circ\}$$

is by [**13**, Theorem 3.1] a complete simplicial fan in H_0 (the radial fan of the starshaped set $\mathcal{R}_{\pm\delta}(Bier(K))$). The associated starshaped body is

$$\mathrm{Star}(K) = \{\lambda x \in H_0 \mid x \in \mathcal{R}_{\pm\delta}(Bier(K)) \text{ and } 0 \leq \lambda \leq 1\}\,. \tag{6}$$

Proposition 4. *The fan $Fan(K)$ coincides with the negative of the radial fan of the starshaped set $\mathcal{R}_{\pm\delta}(Bier(K)$. More explicitly,*

$$Fan(K) = \mathrm{Cone}_{\mp\delta}(K) := RadialFan(\mathcal{R}_{\mp\delta}(Bier(K)))\,.$$

Proof. Extremal rays of the simplicial cone $Cone(R_{-\delta}(S) * R_\delta(T))$ are generated by the vectors $\{\hat{\delta}_i\}_{i\in S} \cup \{\delta_j\}_{j\in T}$, where $\hat{\delta}_j$ is the barycenter of the facet $\Delta_i \subset \Delta_\delta := \mathrm{Conv}\{\delta_k\}_{k\in[n]}$, opposite to the vertex $\delta_i \in \Delta_\delta$.

Let us show that the extremal rays of the cone $Cone(\tau)$, where $\tau = (S,T;\{\nu\})$, have the same representation. In this case the preposet $\preccurlyeq_\tau$ (the reflexive/transitive closure of $\rho_\tau = S \times \{\nu\} \cup \{(\nu,\nu)\} \cup \{\nu\} \times T$) is a *tree-poset*, in the sense of [**20**], Section 3.3, meaning that the associated Hasse diagram is a spanning tree on $[n]$. The corresponding simplicial cone is described by inequalities listed in (2), and the associated extremal rays are obtained if all inequalities, with one exception, are turned into equalities.

If $x_i \leq x_\nu$ is the excepted inequality (where $i \in S$), then the corresponding ray has a parametric representation $x_k = t$ for $k \neq i$ and $x_i = -(n-1)t \leq x_\nu = t$. From here it immediately follows that this ray is spanned by $\hat{\delta}_i$. If $x_\nu \leq x_j$ is the excepted inequality (where $j \in T$), then the corresponding ray has a parametric representation $x_k = t$ for $k \neq j$ and $x_j = -(n-1)t \geqslant x_\nu = t$. In this case the spanning vector is δ_j. □

6. VOLUME AND FACE NUMBERS OF BIER SPHERES

Bier sphere $Bier(K)$, being an abstract simplicial complex, must be realized as a geometric sphere in order to discuss the volume of its inner region. A natural choice is the starshaped body $Star(K)$ introduced in Section 5 (equation (6)) whose boundary $\partial Star(K) = \mathcal{R}_{\pm\delta}(Bier(K))$ is the starshaped embedding of the Bier sphere originally described in [**13**] (see equation (5) for an explicit definition). Recall that (as a subset of H_0)

$$\mathcal{R}_{\pm\delta}(Bier(K)) = \bigcup\{R_\delta(S) * R_{-\delta}(T) \mid (S,T) \in K *_\Delta K^\circ\}\,. \tag{7}$$

Let $\tau = (S, T; \{i\})$ be a facet of $Bier(K)$. Then $R(\tau) = R_\delta(S) * R_{-\delta}(T)$, the corresponding geometric simplex from (7), contributes to the volume of $Star(K)$ the quantity Vol_τ where

$$(n-1)!\, Vol_\tau := |\mathrm{Det}(\tau)| = |\epsilon_1 b_1 \dots \hat{b}_i \dots \epsilon_n b_n|$$

($\epsilon_i = +1$ if $i \in S$ and $\epsilon_i = -1$ if $i \in T$) and the volume of $Star(K)$ is

$$Vol(Star(K)) = \sum_\tau Vol_\tau \, . \tag{8}$$

Notice that $Vol_0 = Vol_\tau$ is a constant, independent of the facet $\tau \in Bier(K)$. Let

$$m_i(K) = m_i = |\{S \in K \mid S \cup \{i\} \notin K\}| \, .$$

In light of (8) the volume of $Star(K)$ can be calculated as

$$Vol(Star(K)) = Vol_0 \sum_{i=1}^{n} m_i = Vol_0 f_{n-2}(Bier(K)) \tag{9}$$

where $f_{n-2}(Bier(K))$ is the number of facets of the Bier sphere $Bier(K)$.

The following proposition allows us to compare the volumes of Bier spheres which are obtained one from the other by a *bistellar operation*, see [**17**, Section 5.6].

Proposition 5. *Assume that $K \subsetneq 2^{[n]}$ is a simplicial complex and let $Star(K) \subset H_0$ be the associated starshaped body. Let $B \notin K$ be a minimal non-face of K in the sense that $(\forall i \in B)\, B \setminus \{i\} \in K$, and let $K' = K \cup \{B\}$. Let $C = [n] \setminus B$ the complement of B. Then*

$$Vol\,(Star(K')) - Vol\,(Start(K)) = V(K', K) = (|C| - |B|) Vol_0 \, .$$

The following relations are an immediate consequence

$$\begin{aligned} V(K', K) > 0, &\ \ \textit{if } |B| < \frac{n}{2} \\ V(K', K) = 0, &\ \ \textit{if } |B| = \frac{n}{2} \\ V(K', K) < 0, &\ \ \textit{if } |B| > \frac{n}{2} \end{aligned}$$

Proof. Let $\Sigma = R_\delta(B) * R_{-\delta}(C)$ be the (possibly degenerate) simplex in H_0 which has $R_\delta(B)$ and $R_{-\delta}(C)$ as two "complementary faces". (Note that Σ is degenerate precisely if $|B| = |C| = n/2$ in which case the simplices $R_\delta(B)$ and $R_{-\delta}(C)$ intersect in a common barycenter.)

If Σ is non-degenerate its boundary $\partial\Sigma$ is the union of two discs

$$\partial\Sigma = \partial(R_\delta(B) * R_{-\delta}(C)) = (\partial(R_\delta(B)) * R_{-\delta}(C)) \cup (R_\delta(B) * \partial(R_{-\delta}(C))) = \Sigma_1 \cup \Sigma_2$$

where $\Sigma_1 \subseteq Bier(K')$ and $\Sigma_2 \subseteq Bier(K)$. If Σ is degenerate then $\Sigma = \Sigma_1 = \Sigma_2$ (as sets), more precisely Σ_1 and Σ_2 are two different triangulations of Σ.

Note that $Bier(K') \setminus \Sigma_1 = Bier(K) \setminus \Sigma_2$ and $Cone(\Sigma_1) = Cone(\Sigma_2) = Cone(\Sigma)$. From here we observe that

1. $Star(K) = Star(K')$ if and only if $|B| = |C| = \frac{n}{2}$;
2. $Star(K) \subsetneq Star(K')$ if and only if $|B| > |C|$;
3. $|V(K', K)| = ||C| - |B||Vol_0 = (n - 2|B|)Vol_0 = Vol(\Sigma)$.

For example the third relation is a consequence of (9) or can be deduced directly by a similar argument. $\square$

Proposition 6. *If $n = 2m+1$ is odd the unique Bier sphere of maximal volume is $Bier(K)$ where*

$$K = \binom{[n]}{\leq m} = \{S \subset [n] \mid |S| \leq m\}\,. \tag{10}$$

If $n = 2m$ is even a Bier sphere $Bier(K)$ is of maximal volume if and only if

$$\binom{[n]}{\leq m-1} \subseteq K \subseteq \binom{[n]}{\leq m}\,. \tag{11}$$

A Bier sphere $Bier(K)$ is of minimal volume if and only if either $K = \{\emptyset\}$ or K is the boundary of the simplex $\Delta_{[n]}$, $K = \partial\Delta_{[n]} = 2^{[n]} \setminus \{[n]\}$.

Proof. The first half of proposition, describing the Bier spheres of maximal volume, is an immediate consequence of Proposition 5. The second, describing the Bier spheres of minimal volume, is an immediate consequence of the formula (9), since the unique triangulation of a sphere S^{m-1} with the minimum number of facets is the boundary of an m-dimensional simplex. $\square$

Corollary 7. *For all Bier spheres $Bier(K)$ of maximal volume, the convex body $\Omega_n = Star(K)$ is unique and independent of K. The body Ω_n is centrally symmetric. More explicitly $\Omega_n = \mathrm{Conv}(\Delta_\delta \cup \nabla_\delta)$ where $\Delta_\delta \subset H_0$ is the simplex spanned by vertices $\delta_i := e_i - \frac{1}{n}(e_1 + \cdots + e_n)$ and $\nabla_\delta := -\Delta_\delta = \Delta_{\bar{\delta}}$ is the simplex spanned by $\bar{\delta}_i = -\delta_i$. The centrally symmetric $(n-1)$-dimensional convex body Ω_n is from here on referred to as the* Van Kampen-Flores polytope *in dimension $n-1$.*

Proof. The body Ω_n is centrally symmetric since the sphere centrally symmetric to the Bier sphere $Bier(K)$ is the sphere $Bier(K^\circ)$ and $\Omega_n = Star(K) = Star(K^\circ)$ if K is one of the complexes described in equations (10) and (11). More precisely $\Omega_n = \mathrm{Conv}(\Delta_\delta \cup \nabla_\delta)$ since

$$\bigcup Star(L) = \mathrm{Conv}(\Delta_\delta \cup \nabla_\delta)$$

where the union on the left is taken over all simplicial complexes $L \subsetneq 2^{[n]}$. $\square$

We call Ω_n the *Van Kampen-Flores body* (polytope) (in dimension $n-1$) for the following reason. The Bier sphere of the simplicial complex (10) is precisely the simplicial triangulation of the $(n-2)$-sphere, used in the standard proof of the classical Van Kampen-Flores theorem, which claims that the $(m-1)$-dimensional complex $\binom{[2m+1]}{\leq m}$ is not embeddable in $\mathbb{R}^{2m-2}$ (see [**17**, Section 5.6]).

Moreover, the complexes $\binom{[2m]}{\leq m-1}$ and $\binom{[2m]}{\leq m}$ (the boundary complexes mentioned in (11)) appear in the *"sharpened Van Kampen-Flores theorem"* (Theorem 6.8 from [**3**]).

Finally all complexes mentioned in (10) and (11) appeared under the name *balanced complexes* in the following theorem, which unifies and extends previously known results.

Theorem 8. ([**15**, Theorem 3.5]) *Let $K \subset 2^{[n]}$ be a simplicial complex and let K° be its Alexander dual. Assume that K is balanced in the sense that either (10) or (11) is satisfied. Then for each continuous map $f : \Delta^{n-1} \to \mathbb{R}^{n-3}$ there exist disjoint faces $F_1 \in K$ and $F_2 \in K^\circ$ such that $f(F_1) \cap f(F_2) \neq \emptyset$.*

The importance of complexes listed in equations (10) and (11) in Proposition 6 was noted even earlier. In [**17**, Section 5.6] they were used as a source of examples of non-polytopal triangulations of spheres while in [**2**] they provided examples of *nearly neighborly Bier spheres*.

7. VAN KAMPEN-FLORES POLYHEDRA AND MEDIAN HYPERSIMPLICES

The Van Kampen-Flores polytope was introduced in the previous section as the convex hull

$$\Omega_n = \mathrm{Conv}(\Delta \cup \nabla) = \mathrm{Conv}\,\{\delta_1, \delta_2, \ldots, \delta_n, -\delta_1, -\delta_2, \ldots, -\delta_n\}$$

where $\Delta = \mathrm{Conv}\{\delta_i\}_{i=1}^n \subset H_0 \cong \mathbb{R}^{n-1}$ and $\delta_i := e_i - \frac{1}{n}(e_1 + \cdots + e_n)$.

In this section, for convenience, we use a slightly more general representation $\Omega_n = \mathrm{Conv}(\Delta \cup \nabla)$ where $\Delta = \Delta_u = \mathrm{Conv}(\{\mathrm{u_i}\})_{\mathrm{i=1}}^{\mathrm{n}}$ is a regular simplex centered at the origin, where $\Delta^\circ = -\Delta =: \nabla$ is the polar dual of Δ. Note that the vertices of Δ form a *circuit* in $\mathbb{R}^{n-1}$ in the sense that the linear map

$$\mathbb{R}^n \xrightarrow{\Lambda} \mathbb{R}^{n-1},\ \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \Lambda(\lambda) := \lambda_1 u_1 + \cdots + \lambda_n u_n \tag{12}$$

is an epimorphism with the kernel generated by $\mathbb{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

The polytope Ω_n must have been well-known, in this or equivalent form, in classical theory of convex polytopes, although, perhaps, without a specific name. In [**13**, Theorem 2.2] it originally appeared as a member of the family $Q_{L,\alpha} =$

$\mathrm{Conv}(\Delta_L \cup -\alpha\Delta_L)$ of polytopes where $\Delta_L = \mathrm{Conv}\{l_1u_1, \dots, l_nu_n\}$ is a radial perturbation of Δ (for some positive weight vector $L = (l_1, \dots, l_n)$) and $\alpha > 0$.

The results from Section 6 provide sufficient evidence that this polytope may deserve some independent interest. For this reason, and for future reference, we collect here some basic information about the facial structure of the Van Kampen-Flores polytope and its polar dual.

Proposition 9. *The vertex set of the polytope Ω_n is the set*

$$Vert(\Omega_n) = \{u_1, u_2, \dots, u_n, -u_1, -u_2, \dots, -u_n\}\,.$$

If a subset $\{u_i\}_{i\in I} \cup \{-u_j\}_{j\in J} \subset Vert(\Omega_n)$ corresponds to a proper face of Ω_n then

$$I \cap J = \emptyset \quad \textit{and} \quad |I|, |J| \leq \frac{n}{2}. \tag{13}$$

Conversely, this condition is also sufficient if n is an odd number. If n is even, then a pair (I, J) corresponds to a proper face of Ω_n if in addition to (13) either (a) $|I| = |J| = \frac{n}{2}$, or (b) both $|I|$ and $|J|$ are strictly less than $\frac{n}{2}$.

Proof. Let $z : \mathbb{R}^{n-1} \to \mathbb{R}$ be a non-zero linear form such that the associated hyperplane $H_z := \{x \in \mathbb{R}^{n-1} \mid \langle z, x\rangle = 1\}$ is a supporting hyperplane of Ω_n. The corresponding face of the polytope Ω_n is described by a pair (I, J) of subsets of $[n]$ recording which vertices of the polytope Ω_n belong to the hyperplane H_z

$$\Omega_n \cap H_z = \mathrm{Conv}(\{u_i\}_{i\in I} \cup \{-u_j\}_{j\in J})\,.$$

The ordered pair (I, J) of subsets of $[n]$ must satisfy the following

$$(\forall i \in I)\, \langle z, u_i\rangle = 1 \qquad (\forall j \in J)\, \langle z, -u_j\rangle = 1 \tag{14}$$

$$(\forall k \notin I)\, \langle z, u_k\rangle < 1 \qquad (\forall k \notin J)\, \langle z, -u_k\rangle < 1 \tag{15}$$

Conversely, one can prescribe in advance numbers $\{a_i\}_{i=1}^n$ such that $a_i = \langle z, u_i\rangle$ for some z, provided $a_1 + \dots + a_n = 0$. Moreover, if these numbers satisfy the conditions (14) and (15), then the pair $I = \{i \in [n] \mid a_i = 1\}$ and $J = \{j \in [n] \mid a_j = -1\}$ clearly corresponds to a face of Ω_n. From here the necessary conditions (13) is easily deduced.

The rest of the proposition is also an immediate consequence. If n is odd or both $|I|$ and $|J|$ are strictly less than $\frac{n}{2}$, then there are no obstacles for constructing a sequence $\{a_i\}_{i=1}^n$ with desired properties.

On the other hand if n is even and, say, $|I| = \frac{n}{2}$, then $|J| = \frac{n}{2}$ as well. If this is satisfied then such a pair (I, J) clearly corresponds to a facet of Ω_n. $\square$

We turn our attention now to the polar polytope Ω_n° of the Van Kampen-Flores polytope. As visible from Figure 1, in the case $n = 4$ the polytope Ω_4 is the three dimensional cube while Ω_n° is the octahedron.

Recall that the *Minkowski functional* μ_P of a convex body $P \subseteq \mathbb{R}^{n-1}$ (which contains the origin in its interior) is the convex function $\mu_P : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$\mu_P(x) = d(0,x)/d(0,x_0) = \mathrm{Inf}\{r > 0 \mid x \in rP\}$$

where $d(\cdot,\cdot)$ is the Euclidean distance function and x_0 is the intersection of the positive ray through x and the boundary of P.

The following proposition determines the polar dual of a convex body K, from the Minkowski functional μ_P, as the set $P^\circ = \{x \mid \mu_P(x) \leq 1\}$.

Proposition 10. *Minkowski functional of a convex body P is equal to the support functional of its polar dual*

$$\mu_P = h_{P^\circ}\,.$$

The following relation for two convex bodies P and Q (containing the origin in their interior) follows directly from the definition

$$\mu_{P\cap Q} = \max\{\mu_P, \mu_Q\}\,. \tag{16}$$

Let us calculate the Minkowski functional of the polytope Ω_n°. Since

$$(\mathrm{Conv}(P \cup Q))^\circ = P^\circ \cap Q^\circ$$

and $\Delta^\circ = \nabla, \nabla^\circ = \Delta$ we observe that

$$\Omega_n^\circ = (\mathrm{Conv}(\Delta \cup \nabla))^\circ \cong \nabla \cap \Delta\,.$$

We use basic properties of the functions $x^+ = \max\{0,x\}$ and $x^- = \max\{0,-x\} = (-x)^+$, which satisfy the well-known elementary relations

$$\begin{array}{cc} x = x^+ - x^- & |x| = x^+ + x^- \\ x^+ = \frac{1}{2}(|x| + x) & x^- = \frac{1}{2}(|x| - x)\,. \end{array}$$

Each vector $x \in \mathbb{R}^{n-1}$, in agreement with (12), has a unique representation

$$x = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n$$

where $\lambda_1 + \cdots + \lambda_n = 0$.

Proposition 11. *The Minkowski functionals of simplices Δ and ∇, and of their intersection $\Omega_n^\circ = \Delta \cap \nabla$ are the following*

$$\mu_\Delta(x) = n\max\{\lambda_i^-\}_{i=1}^n \qquad \mu_\nabla(x) = n\max\{\lambda_i^+\}_{i=1}^n \qquad \mu_{\Omega_n^\circ}(x) = n\max\{|\lambda_i|\}_{i=1}^n\,.$$

Proof. Assuming that $x = \lambda_1 u_1 + \cdots + \lambda_n u_n \neq 0$, let us calculate the corresponding point $x_0 \in \partial(\Delta) \cap \mathrm{Ray}(0,x)$, defined as the intersection point of the boundary of Δ with the ray emanating from the origin 0, passing through the point x.

If $\lambda := \max\{\lambda_i^-\}_{i=1}^n$ then

$$x = (\lambda + \lambda_1)u_1 + \cdots + (\lambda + \lambda_n)u_n$$

where $\lambda+\lambda_i \geq 0$ for each $i \in [n]$ and $\lambda+\lambda_j = 0$ for at least one $j \in [n]$. A moment's reflection shows

$$x_0 = \frac{x}{n\lambda} \in \partial(\Delta)$$

which immediately implies that $\mu_\Delta(x) = n\lambda = n\max\{\lambda_i^-\}_{i=1}^n$.

Since $\mu_{-K}(x) = \mu_K(-x)$ we observe that

$$\mu_\nabla(x) = \mu_\Delta(-x) = n\max\{(-\lambda_i)^-\}_{i=1}^n = n\max\{(\lambda_i)^+\}_{i=1}^n\,.$$

The third formula $\mu_{\Omega_n^\circ}(x) = n\max\{|\lambda_i|\}_{i=1}^n$ is an immediate consequence of (16) and the relation $\max\{\lambda_i^+, \lambda_i^-\} = |\lambda_i|$. □

Since $K = \{x \in \mathbb{R}^{n-1} \mid \mu_K(x) \leq 1\}$, as a corollary of Proposition 11 we obtain the following result.

Corollary 12.

$$\Omega_n^\circ = \Delta \cap \nabla = \{x = \lambda_1 u_1 + \dots \lambda_n u_n \,|\, \lambda_1 + \cdots + \lambda_n = 0 \text{ and } (\forall i)\, |\lambda_i| \leq 1\}\,.$$

Definition 13. *A hypersimplex $\Delta_{n,r}$ with parameters n, r is defined as the convex hull of all n-dimensional vectors, vertices of the n-dimensional cube $[0,1]^n$, which belong to the hyperplane $x_1 + \cdots + x_n = r$. Alternatively $\Delta_{n,r} = \mathrm{Newton}(\sigma_r)$ can be described as the Newton polytope of the elementary symmetric function σ_r of degree r in n variables.*

Theorem 14. *If $n = 2k$ is even then $\Omega_{2k}^\circ = \Delta \cap \nabla$ is affine isomorphic to the hypersimplex $\Delta_{2k,k}$. If $n = 2k+1$ then Ω_n° is affine isomorphic to the convex hull*

$$\Omega_{2k+1}^\circ \cong \mathrm{Conv}\{\lambda \in [0,1]^{2k+1} \mid (\forall i)\, \lambda_i \in \{0, \tfrac{1}{2}, 1\} \text{ and } |Z(\lambda)| = |W(\lambda)| = k\} \tag{17}$$

where $Z(\lambda) = \{j \mid \lambda_j = 0\}$ and $W(\lambda) = \{j \mid \lambda_j = 1\}$.

Proof. By Corollary 12 the polytope Ω_n° is affine isomorphic to the intersection of the hyperplane $\lambda_1 + \cdots + \lambda_n = 0$ with the n-cube $[-1,+1]^n$. The (inverse of the) affine transformation $\lambda_i = 2x_i - 1\,(i = 1, \dots, n)$ maps this to the intersection of the hypercube $[0,1]^n$ with the hyperplane $x_1 + \cdots + x_n = n/2$.

If $n = 2k$ we obtain the hypersimplex $\Delta_{2k,k}$. If $n = 2k+1$ we obtain the polytope (17). □

8. K-SUBMODULAR FUNCTIONS

In this section we return to the question of polytopality of Bier spheres. The main result is the *K-submodularity theorem* (Theorem 19) which for polytopal Bier spheres plays the role similar to the role of classical submodular functions (polymatroids) in the theory of generalized permutohedra.

Recall [**10, 21**] that a function $f : 2^{[n]} \to \mathbb{R}$ is submodular if

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) \quad \text{for each } X, Y \in 2^{[n]}\,.$$

Proposition 15. ([**1, 8**]) *Let $\mathcal{F}$ be an essential complete simplicial fan in $\mathbb{R}^n$ and* $\mathbf{G}$ *be the $N \times n$ matrix whose rows are the rays of $\mathcal{F}$. Then the following are equivalent for any vector $\mathbf{h} \in \mathbb{R}^N$.*

(I) *The fan $\mathcal{F}$ is the normal fan of the polytope $P_{\mathbf{h}} := \{x \in \mathbb{R}^n \mid \mathbf{G}x \leq \mathbf{h}\}$.*

(II) *For any two adjacent chambers $\mathbb{R}_{\geqslant 0}\mathbf{R}$ and $\mathbb{R}_{\geqslant 0}\mathbf{S}$ of $\mathcal{F}$ with $\mathbf{R} \setminus \{r\} = \mathbf{S} \setminus \{s\}$,*

$$\alpha \mathbf{h_r} + \beta \mathbf{h_s} + \sum_{\mathbf{t} \in \mathbf{R} \cap \mathbf{S}} \gamma_{\mathbf{t}} \mathbf{h_t} > 0, \tag{18}$$

where

$$\alpha \mathbf{r} + \beta \mathbf{s} + \sum_{\mathbf{t} \in \mathbf{R} \cap \mathbf{S}} \gamma_{\mathbf{t}} \mathbf{t} = 0 \tag{19}$$

is the unique (up to scaling) linear dependence with $\alpha, \beta > 0$ between the rays of $\mathbf{R} \cup \mathbf{S}$.

The inequalities in Proposition 15 are sometimes referred to as the wall-crossing inequalities.

Definition 16. *Given a (proper) simplicial complex $K \subsetneq 2^{[n]}$, an element $A \in K$ is a* boundary simplex *if $(\exists c \in [n])\, A \cup \{c\} \notin K$. Similarly $B \notin K$ is a* boundary non-simplex *if $(\exists c \in [n])\, B \setminus \{c\} \in K$. A pair $(A, B') \in (K, 2^{[n]} \setminus K)$ is a* boundary pair *if $B' = A \cup \{c\}$ for some $c \in [n]$.*

We already know (Section 3) that boundary pairs (A, B') correspond to maximal simplices in $Bier(K)$. In the following proposition we describe the ridges, i.e. the codimension one simplices in the Bier sphere $Bier(K)$.

Proposition 17. *The ridges (codimension one simplices) $\tau \in Bier(K)$ have one of the following three forms, exhibited in Figure 2. Here we use the* interval notation *$\tau = (X, Y)$ (Section 3) where $X \subsetneq Y, X \in K, Y \notin K$ and $(X, Y) \neq (\emptyset, [n])$.*

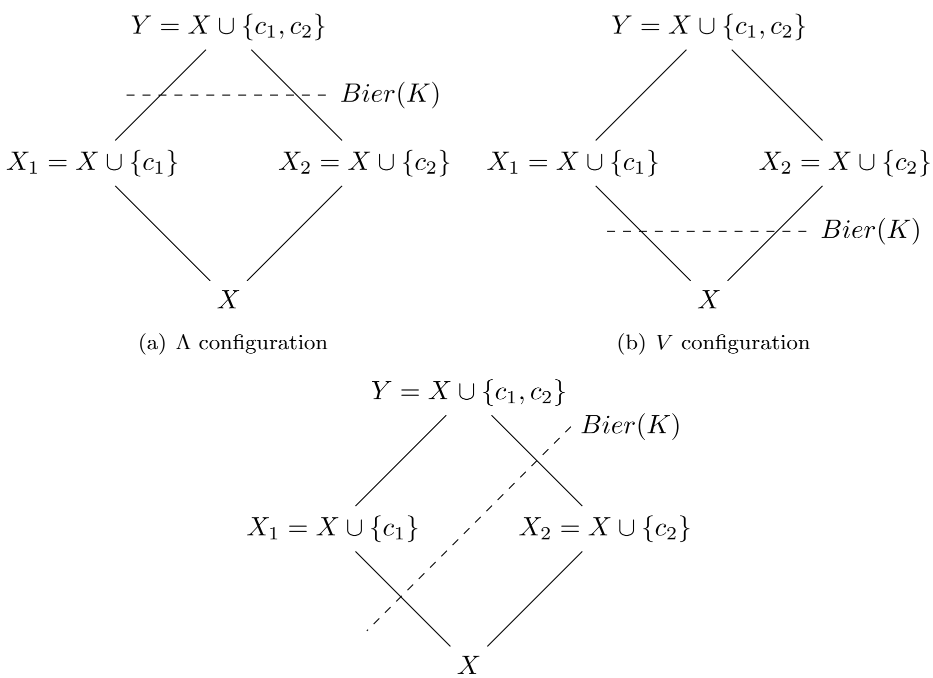
Proof. In the interval notation, the ridges in $Bier(K)$ correspond to intervals (X, Y) where $Y = X \cup \{c_1, c_2\}$ and $c_1 \neq c_2$. The Λ-configurations correspond to the case when both X_1 and X_2 are in K, the V-configurations correspond to the case when neither X_1 nor X_2 are in K, and the X-configurations arise if precisely one of these sets is in K. □

Definition 18. *Let $K \subsetneq 2^{[n]}$ be a simplicial complex and $Bier(K)$ the associated Bier sphere. A K*-submodular function *(K-wall crossing function) is a function $f : Vert(Bier(K)) \to \mathbb{R}$ such that*

$$f(c_1) + f(c_2) + \Sigma_{i \in X} f(i) > \Sigma_{j \notin Y} f(\bar{j}) \quad \text{for each } \Lambda\text{-configuration} \tag{20}$$

$$f(\bar{c}_1) + f(\bar{c}_2) + \Sigma_{j \notin X} f(\bar{j}) > \Sigma_{i \in X} f(i) \quad \text{for each } V\text{-configuration} \tag{21}$$

$$f(c_2) + f(\bar{c}_2) > 0 \quad \text{for each } X\text{-configuration.} \tag{22}$$



(a) Λ configuration

(b) V configuration

(c) X configuration

Figure 2: Configurations of maximal adjacent simplices in $Bier(K)$.

Theorem 19. *Let $\mathcal{F} = Fan(K)$ be the radial fan arising from the canonical star-shaped realization of the associated Bier sphere $Bier(K)$. (The fan $\mathcal{F}$ is by Theorem 2 a coarsening of the braid fan.) Then $\mathcal{F}$ is a normal fan of a convex polytope if and only if the simplicial complex K admits a K-submodular function. Moreover, there is a bijection between convex realizations of $Bier(K)$ with radial fan $\mathcal{F}$ and K-submodular functions f.*

Proof. We apply Proposition 15 to the fan $\mathcal{F} = Fan(K)$.

Let $\delta = (\delta_1, \dots, \delta_n)$ be a circuit in H_0 where $\delta_i = e_i - \frac{u}{n}$ $(u = e_1 + \cdots + e_n)$. Let $\bar{\delta} = (\bar{\delta}_1, \dots, \bar{\delta}_n)$ be the opposite circuit where $\bar{\delta}_i := -\delta_i$. The vertices of $Bier(K)$ are $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ and for the corresponding representatives on the one dimensional cones of the fan $\mathcal{F} = Fan(K)$ we choose $\{\delta_1, \dots, \delta_n, \bar{\delta}_1, \dots, \bar{\delta}_n\}$.

Our objective is to identify the corresponding "wall crossing relations" (19), in each of the three cases listed in Figure 2, and to read off the associated "wall crossing inequalities" (18).

In order to identify the wall crossing relations in the case of the Λ and V configurations we observe that, if $[n] = S \cup T$ and $S \cap T = \emptyset$ then, up to a linear factor, the only dependence in the set $\{\delta_i\}_{i\in S} \cup \{\bar{\delta}_j\}_{j\in T}$ is the relation

$$\sum_{i\in S} \delta_i = \sum_{j\in T} \bar{\delta}_j\,.$$

The first two inequalities in Definition 18 are an immediate consequence. To complete the proof it is sufficient to observe that, in the case of an X configuration (22), the only dependence in the set $\{\delta_i\}_{i\in X} \cup \{\bar{\delta}_j\}_{j\notin Y} \cup \{\delta_{c_2}, \bar{\delta}_{c_2}\}$ is, up to a non-zero factor, the relation $\delta_{c_2} + \bar{\delta}_{c_2} = 0$. □

As an illustration we use Theorem 19 to show that Bier spheres of threshold complexes are polytopal. This result was originally obtained in [**13**] (Theorem 2.2) by a different method.

Suppose that $L = (l_1, l_2, \ldots, l_n) \in \mathbb{R}^n_+$ is a strictly positive vector. The associated measure (weight distribution) μ_L on $[n]$ is defined by $\mu_L(I) = \sum_{i\in I}\ l_i$ (for each $I \subseteq [n]$).

Given a threshold $\nu > 0$, the associated threshold complex is $T_{\mu_L<\nu} := \{I \subseteq [n] \mid \mu_L(I) < \nu\}$. Without loss of generality we assume that $\mu_L([n]) = l_1 + \cdots + l_n = 1$. Moreover ([**13**]. Remark 2.1) we can always assume, without loss of generality, that $\mu_L(I) \neq \nu$ for each $I \subseteq [n]$, which implies that the Alexander dual of K is $K^\circ = T_{\mu_L\leq 1-\nu} = T_{\mu_L<1-\nu}$.

Corollary 20. ([**13**], Theorem 2.2) *$Bier(T_{\mu_L<\nu})$ is isomorphic to the boundary sphere of a convex polytope which can be realized as a polar dual of a generalized permutohedron.*

Proof. Following Theorem 19, it is sufficient to construct a K-submodular function $f : [n] \cup [\bar{n}] \to \mathbb{R}$ where $[n] \cup [\bar{n}] = Vert(Bier(K)) = \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. Let us show that the function defined by

$$f(i) = (1-\nu)l_i \qquad f(\bar{j}) = \nu l_j \qquad (i, j = 1, \ldots, n) \tag{23}$$

is indeed K-submodular for $K = T_{\mu_L<\nu}$. The inequalities (20) and (21), for the function f defined by (23), take (in the notation of Definition 18 and Figure 2) the following form

$$\nu\mu_L(Y) > (1-\nu)\mu_L(Y^c) \qquad (1-\nu)\mu_L(X) < \nu\mu_L(X^c)\,. \tag{24}$$

However, in a threshold complex, both inequalities (24) hold without any restrictions on a simplex $X \in K$ and a non-simplex $Y \notin K$. (For example the second inequality in (24) is a consequence of $\mu_L(X) < \nu$ and $\mu(X^c) > 1-\nu$.)

The convex polytope obtained by this construction is indeed the polar dual of a generalized permutohedron since the complete fan $\mathcal{F} = Fan(K)$ is a coarsening of the braid fan. □

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