Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **17** (2023), 120–137.
https://doi.org/10.2298/AADM211211006D

THE STRUCTURE OF THE 2-FACTOR TRANSFER DIGRAPH COMMON FOR RECTANGULAR, THICK CYLINDER AND MOEBIUS STRIP GRID GRAPHS

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Dedicated to the memory of a distinguished scientist and colleague,
Ratko Tošić (1942 - 2022)

In this paper, we prove that all but one of the components of the transfer digraph $\mathcal{D}^*_m$ needed for the enumeration of 2-factors in the rectangular, thick cylinder and Moebius strip grid graphs of the fixed width m ($m \in N$) are bipartite digraphs and that their orders could be expressed in term of binomial coefficients. In addition, we prove that the set of vertices of each component consists of all the binary m-words for which the difference of numbers of zeros in odd and even positions is constant.

1. INTRODUCTION

Robotic and biochips technology trends actualize the problem of generating and enumeration of Hamiltonian paths in grid graphs [**15, 16**]. The counting of Hamiltonian cycles on specific grid graphs was the subject of interest in [**1**]-[**6**], [**12**], [**13**], [**17**] and [**18**]. The transfer matrix approach has been proven to be the most suitable for this and similar problems [**9, 11, 14**]. Namely, the specificity of the considered graphs is reflected in possibility of grouping their vertices in columns that are suitable for coding as words of fixed length over some alphabet. Whether

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2020 Mathematics Subject Classification. 05C38, 05C50, 05A15, 05C30, 05C85.
Keywords and Phrases. 2-factor, Transfer matrix, Thick grid cylinder, Moebius strip.

the subgraph induced by the vertices from the same column is the path P_m or cycle C_m (m is the number of vertices in columns) we refer to these grid graphs as *linear* (the rectangular, thick cylinder and Moebius strip grid graphs) or *circular* ones (thin cylinder, torus and Klein bottle grid graphs). In this paper we deal with the former case. In the latter case the coding words are circular and the reader interested in that topic is referred to [**8**].

Definition 1. ***The Rectangular (grid) graph*** $RG_m(n)$, ***thin (grid) cylinder*** $TnC_m(n)$ *and* ***thick (grid) cylinder*** $TkC_m(n)$ *(*$m, n \in N$*) are* $P_m \times P_n$, $C_m \times P_n$ *and* $P_m \times C_n$*, respectively.*
The Moebius strip $MS_m(n)$ *is obtained from* $RG_m(n+1) = P_m \times P_{n+1}$ *by identification of corresponding vertices from the first and last column in the opposite direction and without duplicating edges. The value* m *is called the* ***width*** *of the grid graph.*

The thick grid cylinder $TkC_m(n)$ can be also obtained from $RG_m(n+1) = P_m \times P_{n+1}$ by identification of corresponding vertices from the first and last column (in the same direction) and without duplicating edges (see Figure 1).

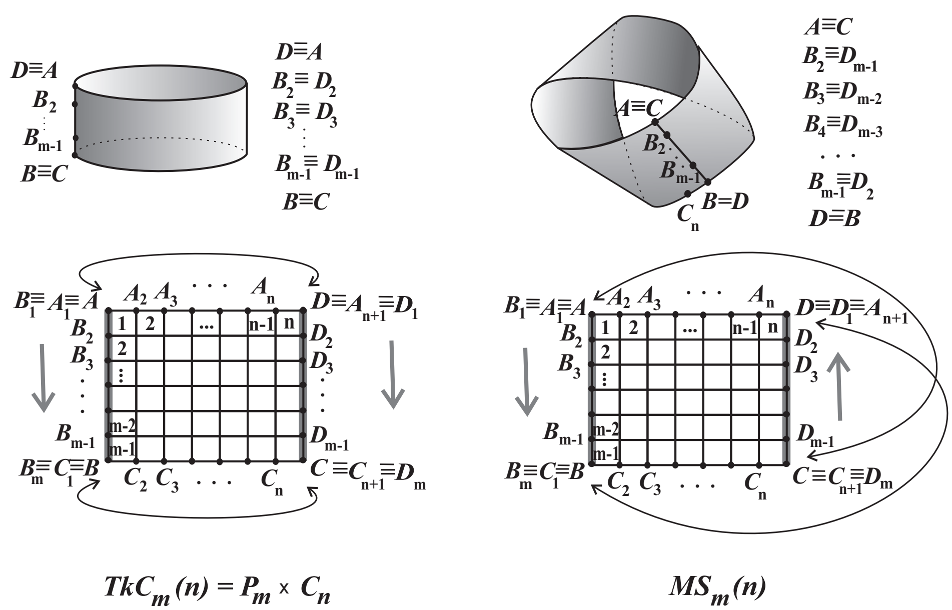

Figure 1: The thick grid cylinder $TkC_m(n)$ and the Moebius strip $MS_m(n)$

In recent papers [**1**]-[**4**] dealing with Hamiltonian cycles in the rectangular grid graph, thin and thick cylinder and their triangular variants some open questions have occurred. For example, the numbers of the contractible and non-contractible Hamiltonian cycles for thin cylinder graph are asymptotically equal (when $n \to \infty$) [**4**] and the same is valid for its triangular variant [**2**]. For the thick grid cylinder the contractible Hamiltonian cycles are more numerous than the non-contractible ones iff m is even [**3**]. The coefficient for dominant eigenvalue for non-contractible HCs is equal to 1 (computational data for $m \leq 10$) [**1**]. Also, positive dominant characteristic root for contractible HCs in a thick grid cylinder is equal to the same one associated with rectangular grid graph (computational data for $m \leq 10$) [**1, 5**].

Motivated to approach more closely to the answers on these questions we started with the investigation of 2-factors in these graphs as a generalization of Hamiltonian cycles [**7**]. Additionally, we expanded our research to the new class of grid graphs - Moebius strips $MS_m(n)$. We wondered if the same or similar properties related to HCs would remain valid for 2-factors or not, and wanted to see if some conclusions for 2-factors could help in proving the mentioned conjectures for HCs. For example, the property of the coefficient for dominant eigenvalue for non-contractible HCs appeared also in case of 2-factors for both $TkG_m(n)$ and $MS_m(n)$.

A spanning 2-regular subgraph of a graph is called a *2-factor*. Obviously, it is a union of disjoint cycles. In Figure 2, the boundary of the (gray) figure, known as *Adinkra*, consists of 7 cycles and represents a 2-factor of the rectangular grid graph $RG_{14}(14) = P_{14} \times P_{14}$. (In theoretical physics, adinkras are geometric objects that encode mathematical relationships between supersymmetric particles. The name "adinkra" is linked to West African symbols that represent wise sayings [**10**].)

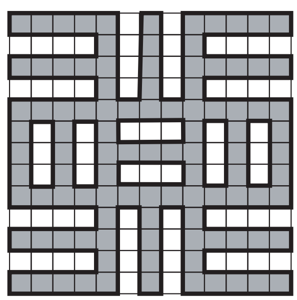

Figure 2: Adinkra "nea onnim no sua a, ohu" ("he who does not know can become knowledgeable through learning")

For a given 2-factor of a grid graph G there exist six possible situations in any vertex shown in Figure 3. Namely, for any vertex $v \in V(G)$ exactly two edges of the considered 2-factor (bold lines) are incident to v. The letters attached to these situations are called *code letters*

Definition 2. *For a given 2-factor of a linear grid graph G of width m and with $m \cdot n$ vertices ($m, n \in N$), the* ***code matrix*** $[\alpha_{i,j}]_{m\times n}$ *is a matrix of order $m \times n$ with entries from $\{a, b, c, d, e, f\}$ where $\alpha_{i,j}$ is the code letter for the i-th vertex in j-th column of G.*

The possibility that two code letters appear as neighbors in the same column (row) of the code matrix is shown in the auxiliary digraph $\mathcal{D}_{ud}$ ($\mathcal{D}_{lr}$) in Figure 3.

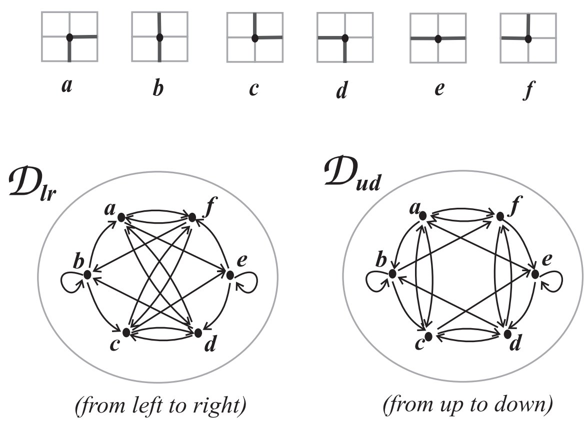

Figure 3: Six possible situations in any vertex for a given 2-factor (above); the digraphs $\mathcal{D}_{ud}$ and $\mathcal{D}_{lr}$ (below)

The code matrix $[\alpha_{i,j}]_{m\times n}$ for a given 2-factor of a linear grid graph G has the following properties [**7**]:

1. **Column conditions:** For every fixed j $(1 \le j \le n)$,

   (a) the ordered pairs $(\alpha_{i,j}, \alpha_{i+1,j})$, where $1 \le i \le m-1$, must be arcs in the digraph $\mathcal{D}_{ud}$.

   (b) $\alpha_{1,j} \in \{a, d, e\}$ and $\alpha_{m,j} \in \{c, e, f\}$.

2. **Adjacency of column condition:** For every fixed j, where $1 \le j \le n-1$, the ordered pairs $(\alpha_{i,j}, \alpha_{i,j+1})$, where $1 \le i \le m$, must be arcs in the digraph $\mathcal{D}_{lr}$.

3. **First and Last Column conditions:**

   (a) If $G = RG_m(n)$, then the alpha-word of the first column consists of the letters from the set $\{a, b, c\}$ and of the last column of the letters from the set $\{b, d, f\}$.

   (b) If $G = TkC_m(n)$, then the ordered pairs $(\alpha_{i,n}, \alpha_{i,1})$, where $1 \le i \le m$, must be arcs in the digraph $\mathcal{D}_{lr}$.

   (c) If $G = MS_m(n)$, then the ordered pairs $(\overline{\alpha}_{i,n}, \alpha_{m-i+1,1})$, $1 \le i \le m$, must be arcs in the digraph $\mathcal{D}_{lr}$,
   where $\overline{a} = c, \overline{b} = b, \overline{c} = a$, $\overline{d} = f, \overline{e} = e$ and $\overline{f} = d$ (the adequate label obtained by applying reflection symmetry with the horizontal axis as its line of symmetry).

This enables that the counting of such code matrices (in fact 2-factors) is reduced to the counting of directed walks in an auxiliary digraph $\mathcal{D}_m \overset{\text{def}}{=} (V(\mathcal{D}_m), E(\mathcal{D}_m))$, common for all linear graphs. The set of its vertices $V(\mathcal{D}_m)$ consists of all possible words $\alpha_{1,j}\alpha_{2,j}\ldots\alpha_{m,j}$ over alphabet $\{a,b,c,d,e,f\}$ (called *alpha-words*) which fulfill *Column conditions*. An arc $(v,u)\in E(\mathcal{D}_m)$ joins $v=\alpha_{1,j}\alpha_{2,j}\ldots\alpha_{m,j}$ to $u=\alpha_{1,j+1}\alpha_{2,j+1}\ldots\alpha_{m,j+1}$, i.e. $v\to u$ iff the *Adjacency of column condition* is satisfied for the ordered pair (v,u) (i.e. the vertex v can be the previous column for the vertex u in the code matrix $[\alpha_{i,j}]_{m\times n}$ for a 2-factor of G).

Definition 3. *The* ***outlet word*** *of a vertex* $\alpha\equiv\alpha_1\alpha_2\ldots\alpha_m\in V(\mathcal{D}_m)$ *is the binary word* $o(\alpha)\equiv o_1o_2\ldots o_m$*, where* $o_j\stackrel{\text{def}}{=}\begin{cases}0, & if\ \ \alpha_j\in\{b,d,f\}\\ 1, & if\ \ \alpha_j\in\{a,c,e\}\end{cases}$, $1\le j\le m$.
$\overline{\alpha}\stackrel{\text{def}}{=}\overline{\alpha}_m\overline{\alpha}_{m-1}\ldots\overline{\alpha}_1\in V(\mathcal{D}_m)$ *and* $\overline{o(\alpha)}\stackrel{\text{def}}{=}o(\overline{\alpha})$

Example 4. *The first and fifth column of the code matrix for the 2-factor in Figure 2 are the words* $(ac)^2ab^4c(ac)^2$ *and* $e(df)^2db^2f(df)^2e$*, respectively. Their outlet words are* $1^50^41^5$ *and* $10^{12}1$.

The digraph $\mathcal{D}^*_m\stackrel{\text{def}}{=}(V(\mathcal{D}^*_m),E(\mathcal{D}^*_m))$ is obtained by gluing all the vertices from $V(\mathcal{D}_m)$ having the same corresponding outlet word and replacing all the arcs from $E(\mathcal{D}_m)$ starting from these glued vertices and ending at same vertex with only one arc (see Figure 4). In [**7**] it is proved that every binary word from $\{0,1\}^m$ except the word $(01)^k0$ when $m=2k+1$ $(k\in N)$ belongs to $V(\mathcal{D}^*_m)$ and that the adjacency matrix $\mathcal{T}^*_m$ of the digraph $\mathcal{D}^*_m$ is a symmetric binary matrix, i.e. if $v\to w$, then $w\to v$, for all $v,w\in V(\mathcal{D}^*_m)$ (for this reason we occasionally use the label $v\leftrightarrow w$). Consequently, each component of $\mathcal{D}^*_m$ is a strongly connected digraph. By implementation of the algorithm for obtaining the digraphs $\mathcal{D}^*_m$ the data for $m\le 12$ gathered in [**7**] suggest that all the components of $\mathcal{D}^*_m$ except one are bipartite digraphs and that the order of each component could be expressed in term of binomial coefficients. In this paper, we prove that these assumptions are true. Moreover, for every component we give a characterisation of its set of vertices.

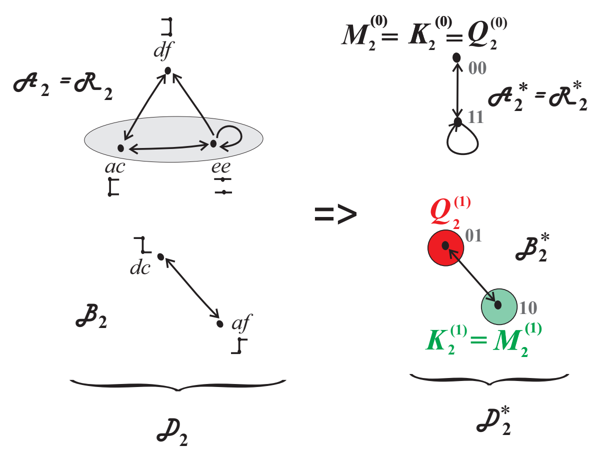

Figure 4: The digraphs $\mathcal{D}_2$ and $\mathcal{D}^*_2$

Theorem 5. *(Conjecture in [7]) For each $m \geq 2$, the digraph $\mathcal{D}^*_m$ has exactly $\left\lfloor \frac{m}{2} \right\rfloor + 1$ components, i.e. $\mathcal{D}^*_m = \mathcal{A}^*_m \cup (\bigcup_{s=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}^{*(s)}_m)$, where $\mid V(\mathcal{B}^{*(1)}_m) \mid \geq \mid V(\mathcal{B}^{*(2)}_m) \mid \geq \ldots \geq \mid V(\mathcal{B}^{*(\lfloor m/2 \rfloor)}_m) \mid$ and $\mathcal{A}^*_m$ is the one containing 1^m. All the components $\mathcal{B}^{*(s)}_m$ $(1 \leq s \leq \left\lfloor \frac{m}{2} \right\rfloor)$ are bipartite digraphs.*
If m is odd, then $\mid V(\mathcal{B}^{*(s)}_m) \mid = \binom{m+1}{(m+1)/2 - s}$ and $\mid V(\mathcal{A}^*_m) \mid = \binom{m}{(m-1)/2}$.
If m is even, then $\mid V(\mathcal{B}^{*(s)}_m) \mid = 2\binom{m}{m/2 - s}$ and $\mid V(\mathcal{A}^*_m) \mid = \binom{m}{m/2}$.
The vertices v and $\overline{v}$ belong to the same component. When the component is bipartite they are placed in the same class iff m is odd.

Example 6. *The digraph $\mathcal{D}^*_4$ depicted in Figure 5 has $\left\lfloor \frac{4}{2} \right\rfloor + 1 = 3$ components of which all but the one with the loop are bipartite digraphs. The cardinalities of its components are $\binom{4}{4/2} = 6$, $2\binom{4}{4/2-1} = 8$ and $2\binom{4}{4/2-2} = 2$. Note that the vertices v and $\overline{v}$ are placed in the different classes (of different colors).*

The transfer matrix $\mathcal{T}^*_m = [a_{ij}]$ which is used for the enumeration of 2-factors in the considered grid graphs is the adjacency (binary) matrix of $\mathcal{D}^*_m$.

Theorem 7. *([7]) If $f^G_m(n)$ denotes the number of 2-factors of the linear grid graph G of width m with $m \cdot n$ vertices, then*

$$f^G_m(n) = \begin{cases} a^{(n)}_{1,1}, & \text{if } G = RG, \\ tr((\mathcal{T}^*_m)^n) = \sum\limits_{v_i \in V(\mathcal{D}^*_m)} a^{(n)}_{i,i}, & \text{if } G = TkC, \\ tr(\mathcal{P}^*_m \cdot (\mathcal{T}^*_m)^n) = \sum\limits_{\substack{v_i, v_j \in V(\mathcal{D}^*_m) \\ \overline{v}_i = v_j}} a^{(n)}_{i,j}, & \text{if } G = MS, \end{cases}$$

where $v_1 \equiv 0^m$ (corresponding to the first row and first column of $\mathcal{T}^*_m$) and $\mathcal{P}^*_m$ is the permutation matrix of order $\mid V(\mathcal{D}^*_m) \mid$ which represents the product of all transpositions $(v, \overline{v})$ $(v, \overline{v} \in V(\mathcal{D}^*_m))$.

Example 8. *The 2-factor in Figure 2 corresponds to the closed directed walk of length 14 in $\mathcal{D}^*_{14}$ starting and finishing with 0^{14}, namely, $0^{14} \to 1^5 0^4 1^5 \to 1^6 0^2 1^6 \to 1^5 0^4 1^5 \to 1^6 0^2 1^6 \to 1 0^{12} 1 \to \ldots \to 1^6 0^2 1^6 \to 1^5 0^4 1^5 \to 0^{14}$.*

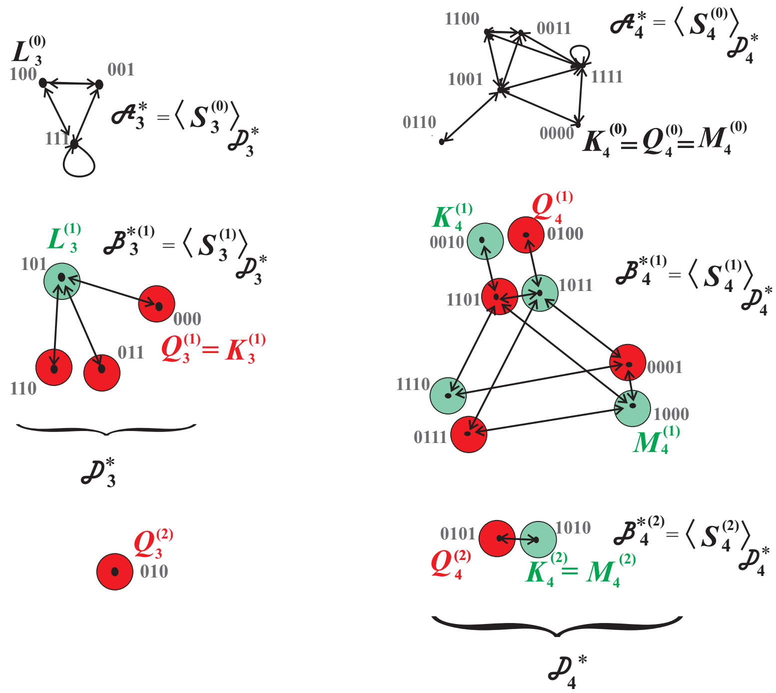

Figure 5: **The digraphs $\mathcal{D}_3^*$ and $\mathcal{D}_4^*$**

The main goal in this paper is the first proof of Theorem 5.
In Section 2, we introduce the sets of binary words of length m, denoted by $S_m^{(s)}$ $(0 \leq s \leq \lfloor m/2 \rfloor)$ (and their representatives) by means of counting the numbers of zeros in even and odd positions. We prove that two representatives of different sets $S_m^{(s_1)}$ and $S_m^{(s_2)}$, $s_1 \neq s_2$ can not be connected by a directed walk in $\mathcal{D}_m^*$.

In Section 3, we prove that the subdigraphs of $\mathcal{D}_m^*$ induced by $S_m^{(s)}$ for $0 \leq s \leq \lfloor m/2 \rfloor$ are strongly connected (hence the components of $\mathcal{D}_m^*$) and moreover for $s \geq 1$ they are bipartite digraphs. The cardinality of each class of these bipartite digraphs is determined. We prove that v and $\overline{v}$ belong to the same set $S_m^{(s)}$, for any $v \in V(\mathcal{D}_m^*)$. In this way, we complete the proof of Theorem 5.

2. PRELIMINARIES

Definition 9. *For a binary word x of length m ($m \in N$) we denote by $odd(x)$ ($even(x)$) the total number of 0's at odd (even) positions in x. The difference $odd(x) - even(x)$ is labeled as $Z(x)$.*

In order to prove Theorem 5, we introduce for each integer $m \geq 1$ the sets of binary words of length m: $S_m^{(0)}$, $S_m^{(1)} = R_m^{(1)} \cup G_m^{(1)}$, $S_m^{(2)} = R_m^{(2)} \cup G_m^{(2)}, \ldots, S_m^{(\lfloor m/2 \rfloor)} = R_m^{(\lfloor m/2 \rfloor)} \cup G_m^{(\lfloor m/2 \rfloor)}$ in the following way:

Definition 10. *The set $S_m^{(0)}$ $(m \in N)$ consists of all the binary m-words whose number of 0's at odd positions is equal to the number of 0's at even positions. For $1 \leq s \leq \lfloor m/2 \rfloor$, $S_m^{(s)} \stackrel{\text{def}}{=} R_m^{(s)} \cup G_m^{(s)}$ where the words in $R_m^{(s)}$ and $G_m^{(s)}$ are all the binary words x of the length m for which $Z(x) = s$ and $Z(x) = -s$, respectively. Additionally, if m is odd, then $R_m^{(\lceil m/2 \rceil)} \stackrel{\text{def}}{=} \{0(10)^{\lfloor m/2 \rfloor}\}$.*

Clearly, $x \in S_m^{(s)}$ iff $\mid Z(x) \mid = s$ $(0 \leq s \leq \lfloor m/2 \rfloor)$. In this way we define the collection of the sets $S_m^{(s)}, 0 \leq s \leq \lfloor m/2 \rfloor$ which represents for even $m \in N$ a partition of the set $\{0,1\}^m$, i.e. $\bigcup_{s=0}^{\lfloor m/2 \rfloor} S_m^{(s)} = \{0,1\}^m = V(\mathcal{D}_m^*)$. When m is odd, the singleton set $R_m^{(\lceil m/2 \rceil)}$ has the only m-word x which is not in set of vertices of $\mathcal{D}_m^*$ [**7**] and $Z(x) = \lfloor m/2 \rfloor + 1$. Then, we have $\bigcup_{s=0}^{\lfloor m/2 \rfloor} S_m^{(s)} = \{0,1\}^m \backslash R_m^{(\lceil m/2 \rceil)} = V(\mathcal{D}_m^*)$, too .

Example 11. *It is easy to check that*
$S_1^{(0)} = \{1\}$, $R_1^{(1)} = \{0\}$ *and* $S_1^{(0)} = \{1\} = V(\mathcal{D}_1^*)$

$S_2^{(0)} = \{00, 11\}$, $S_2^{(1)} = R_2^{(1)} \cup G_2^{(1)}$, $R_2^{(1)} = \{01\}$ *and* $G_2^{(1)} = \{10\}$ *and*
$S_2^{(0)} \cup S_2^{(1)} = \{0,1\}^2 = V(\mathcal{D}_2^*)$

$S_3^{(0)} = \{100, 111, 001\}$, $S_3^{(1)} = R_3^{(1)} \cup G_3^{(1)}$, $R_3^{(1)} = \{000, 011, 110\}$, $G_3^{(1)} = \{101\}$,
$R_3^{(2)} = \{010\}$ *and* $S_3^{(0)} \cup S_3^{(1)} = \{0,1\}^3 \backslash \{010\} = V(\mathcal{D}_3^*)$

Also, perceive that $R_{2k}^{(k)}$, $G_{2k}^{(k)}$, $G_{2k+1}^{(k)}$ and $R_{2k+1}^{(k+1)}$ are singleton sets, i.e. $R_{2k}^{(k)} = \{(01)^k\}$, $G_{2k}^{(k)} = \{(10)^k\}$, $G_{2k+1}^{(k)} = \{(10)^k 1\}$ and $R_{2k+1}^{(k+1)} = \{(01)^k 0\}$ $(k \in N)$. Additionally, the word 1^m belongs to $S_m^{(0)}$ for any $m \in N$.

Considering the entries of $S_m^{(s)}$ as the vertices of $\mathcal{D}_m^*$ we call the words from $R_m^{(s)}$ and $G_m^{(s)}$ *red* and *green vertices*, respectively. The only binary word of length m that does not belong to $\bigcup_{s=0}^{\lfloor m/2 \rfloor} S_m^{(s)} = V(\mathcal{D}_m^*)$ is $(01)^k 0 \in R_m^{(k+1)}$ where $m = 2k+1$. Still, we call it *red vertex* (by definition).

For an arbitrary set of binary words A, the label $1A$ (or $0A$) denotes the set of all words $1a$ $(0a)$, where $a \in A$. Note that the following equalities hold for any binary word x:

Proposition 12.

a) $Z(0x) = -Z(x) + 1$,
b) $Z(1x) = -Z(x)$,
c) $Z(00x) = Z(11x) = Z(x)$,
d) $Z(10x) = Z(x) - 1$ *and*
e) $Z(01x) = Z(x) + 1$.

Proof. Straightforward. □

Observe that a vertex from $S_m^{(0)}$ is obtained by adding the prefix 1 to a vertex from $S_{m-1}^{(0)}$ or the prefix 0 to a red vertex from $R_{m-1}^{(1)}$. A red vertex from $R_m^{(s)}$, $1 \le s \le \lfloor m/2 \rfloor$ is obtained by adding the prefix 1 to a green vertex from $G_{m-1}^{(s)}$ or the prefix 0 to a green vertex from $G_{m-1}^{(s-1)}$ (when $s \ge 2$) or to a vertex from $S_{m-1}^{(0)}$ (when $s = 1$). Similarly, a green vertex from $G_m^{(s)}$, $1 \le s \le \lfloor m/2 \rfloor$ is obtained by adding the prefix 1 to a red vertex from $R_{m-1}^{(s)}$ or the prefix 0 to a red vertex from $R_{m-1}^{(s+1)}$ when $s < \lfloor m/2 \rfloor$.

In what follows, we will prove that the subdigraphs of $\mathcal{D}_m^*$ induced by $S_m^{(s)}$ $(0 \le s \le \lfloor m/2 \rfloor)$, labeled as $\langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$, are $\mathcal{A}_m^*$ (the component which contains 1^m), $\mathcal{B}_m^{*(1)}$, $\mathcal{B}_m^{*(2)}, \ldots, \mathcal{B}_m^{*(\lfloor m/2 \rfloor)}$ (which satisfy $\mid V(\mathcal{B}_m^{*(1)}) \mid \ge \mid V(\mathcal{B}_m^{*(2)}) \mid \ge \ldots \ge \mid V(\mathcal{B}_m^{*(\lfloor m/2 \rfloor)}) \mid$), respectively, and that each $\mathcal{B}_m^{*(s)}$ is the bichromatic (hence bipartite) digraph $(R_m^{(s)}, G_m^{(s)})$ $(s \ge 1)$.

Definition 13. *We refer to the zero-word* $Q_{2k}^{(0)} \overset{\text{def}}{=} 0^{2k} \in S_{2k}^{(0)}$, *the words* $Q_{2k}^{(s)} \overset{\text{def}}{=} (01)^s 0^{2k-2s} \in R_{2k}^{(s)}$ $(1 \le s \le k)$ *and* $Q_{2k+1}^{(s)} \overset{\text{def}}{=} (01)^{s-1} 0^{2k-2s+3} \in R_{2k+1}^{(s)}$ $(1 \le s \le k+1)$ *as the* ***queens****. The words* $(10)^s 0^{2k-2s} \in S_{2k}^{(s)}$ $(0 \le s \le k)$ *are called the* ***maidens*** *and labeled as* $M_{2k}^{(s)}$. *Similarly, the word* $L_{2k+1}^{(k)} \overset{\text{def}}{=} (10)^k 1 \in S_{2k+1}^{(k)}$ *and the words* $L_{2k+1}^{(s)} \overset{\text{def}}{=} (10)^{s+1} 0^{2k-2s-1} \in S_{2k+1}^{(s)}$ $(0 \le s < k)$ *are called the* ***court ladies****.*

Note that $M_{2k}^{(0)} \equiv Q_{2k}^{(0)}$, $M_{2k}^{(s)} \in G_{2k}^{(s)}$ and $L_{2k+1}^{(s)} \in G_{2k+1}^{(s)}$ where $s \ge 1$. In this way, when $s \ge 1$ we have provided the representatives for the red and green sets: the queens $Q_m^{(s)}$ $(m \in N)$ for the first ones, and the maidens $M_{2k}^{(s)}$ $(k \in N)$ and the court ladies $L_{2k+1}^{(s)}$ $(k \in N)$ for the second ones. In these cases, we treat the queen $Q_m^{(s)}$ as the main representative for the entire set $S_m^{(s)}$. Find that the only set $S_m^{(s)}$ without a queen is $S_m^{(0)}$ when m is odd. In this case, the court lady $L_m^{(0)} \equiv 10^{m-1}$ becomes the main representative for $S_m^{(0)}$, while when m is even this role takes the queen $Q_m^{(0)} \equiv M_m^{(0)} = 0^m$.

If we add the prefix 0 to the maiden $M_{2k}^{(s)} \in S_{2k}^{(s)}$ $(0 \le s \le k)$, she becomes the queen in $R_{2k+1}^{(s+1)}$. If we add the prefix 1 to the queen $Q_{2k}^{(s)}$ $(0 \le s \le k)$, she becomes the court lady $L_{2k+1}^{(s)} \in S_{2k+1}^{(s)}$. Reversible "aging" process, i.e. "rejuvenation"

arises during the forming stage of the representatives for the red and green subsets of $S_{2k}^{(s)}$, where $s \geq 1$ ($k \in N$). Namely, by adding 1 as a prefix to the queen $Q_{2k-1}^{(s)} \in R_{2k-1}^{(s)}$, where $s \geq 1$, we obtain the maiden $M_{2k}^{(s)} \in G_{2k}^{(s)}$. Additionally, the queen $Q_{2k}^{(s)} \in R_{2k}^{(s)}$ ($1 \leq s \leq k$) is obtained by adding 0 as a prefix to the court lady $L_{2k-1}^{(s-1)} \in S_{2k-1}^{(s-1)}$.
This rule does not apply for the queen from $S_{2k}^{(0)}$. In fact, she arises from the queen from $R_{2k-1}^{(1)}$, i.e. $Q_{2k}^{(0)} = 0^{2k} = 0Q_{2k-1}^{(1)} \in S_{2k}^{(0)}$.

Definition 14. *The word $\overline{Q}_m^{(s)}$ ($0 \leq s \leq \lfloor m/2 \rfloor$) is called the* ***king*** *and labeled as $K_m^{(s)}$.*

Clearly, for $s = 0$ we have $K_{2k}^{(0)} \equiv Q_{2k}^{(0)} \equiv M_{2k}^{(0)} = 0^{2k}$ ($k \in N$). When $s \geq 1$, in case $m = 2k$ ($k \in N$), $K_{2k}^{(s)} = 0^{2k-2s}(10)^s \in G_{2k}^{(s)}$, while in case $m = 2k+1$ ($k \in N$), $K_{2k+1}^{(s)} \in R_{2k+1}^{(s)}$ because $K_{2k+1}^{(s)} = 0^{2k-2s+3}(10)^{s-1} \in R_{2k+1}^{(s)}$. (When the maiden is present in $S_m^{(s)}$ ($s \geq 1$), then the king takes "the same side" (color) as the maiden, but if the court lady is present in $S_m^{(s)}$, then he takes "the opposite side" (color) of the court lady.)

Lemma 15. *The court lady $L_{2k-1}^{(0)} \in S_{2k-1}^{(0)}$ ($k \in N$) and any queen $Q_{2k-1}^{(s)} \in S_{2k-1}^{(s)}$ ($1 \leq s \leq k-1$) can not be connected by a directed walk in $\mathcal{D}_{2k-1}^*$. The same is valid for any two queens $Q_m^{(s_1)}$ and $Q_m^{(s_2)}$ where $0 \leq s_1 < s_2 \leq \lfloor \frac{m}{2} \rfloor$ ($m \in N$).*

Proof. Recall that this statement is trivially valid for the main representatives of $S_m^{(s_1)}$ and $S_m^{(s_2)}$ ($0 \leq s_1 < s_2 \leq \lfloor \frac{m}{2} \rfloor$) when s_1 and s_2 (i.e. the numbers of their 1's) have opposite parity.

The remaining cases are discussed below. In all cases we give indirect proofs. Thus, we suppose the opposite, that in $\mathcal{D}_m^*$ there exists a directed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ of length $k \in N$, where v_0 and v_k are the main representatives of the two considered sets (no matter in which direction).

The corresponding part of the grid for the directed walk from v_1 to v_k has $m \cdot k$ vertices in the grid (see Figure 6). Note that this rectangular grid graph is bichromatic (we color its vertices in gray and black). In this grid the directed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ determines a spanning union of paths (open paths and cycles). In this union each cycle (if exists) has the same number of vertices of both colors. Further, when $m \cdot k$ is even, then the numbers of gray and black vertices of the grid $P_m \times P_k$ are the same. Otherwise, when $m \cdot k$ is odd, these numbers differ by 1, the all four corner vertices (with the degree 2) are of the same color - the one whose vertices are more numerous.

Case I: $s_1 = 0$, m and s_2 ($s_2 > 0$) are even.
The directed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ determines a spanning union of $\frac{s_2}{2}$ open paths and a few cycles or without them. The end vertices of these open paths (rounded in Figure 6 a) belong to the last column of the grid $P_m \times P_k$. They

are of the same color and related to 1's of the queen $v_k = Q_m^{(s_2)} = (01)^{s_2}0^{m-2s_2}$. Consequently, the numbers of gray and black vertices in the considered grid graph $P_m \times P_k$ differ by $\dfrac{s_2}{2}$. But, these numbers must be equal because $m \cdot k$ is even. Contradiction.

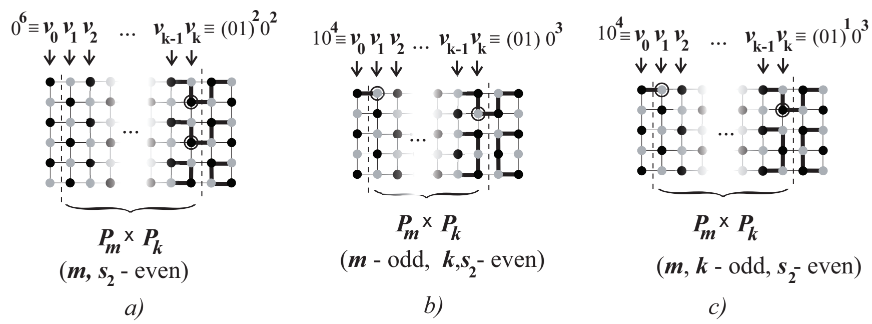

Figure 6: The main representatives of $S_m^{(0)}$ and $S_m^{(s_2)}$ $(1 \leq s_2 \leq \lfloor \frac{m}{2} \rfloor)$ can not be connected by a directed walk in $\mathcal{D}_m^*$.

Case II: $s_1 = 0$, m is odd and s_2 $(s_2 > 0)$ is even.

In the directed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ we have $v_0 = L_m^{(0)} = 10^{m-1}$ and $v_k = Q_m^{(s_2)} = (01)^{s_2-1}0^{m-2s_2+2}$.

If k is even (see Figure 6 b), the end vertices of all $\dfrac{s_2}{2}$ $(\dfrac{s_2}{2} \geq 1)$ open paths are of the same color which implies that in the grid graph $P_m \times P_k$ the number of vertices of this color is greater than the one of opposite color which is impossible because $m \cdot k$ is even.

If k is odd (see Figure 6 c), exactly one open path has end vertices of different colors. The end vertices of the remaining $\dfrac{s_2-2}{2}$ $(\dfrac{s_2-2}{2} \geq 0)$ open paths are all of the same color. Without loss of generality, let us say this color is black. We conclude that the number of black vertices is greater than or equal to the number of gray vertices. On the other hand, since $m \cdot k$ is odd, in the considered grid the number of gray vertices (among them are corner ones) must be greater by 1 than the number of black vertices. Contradiction.

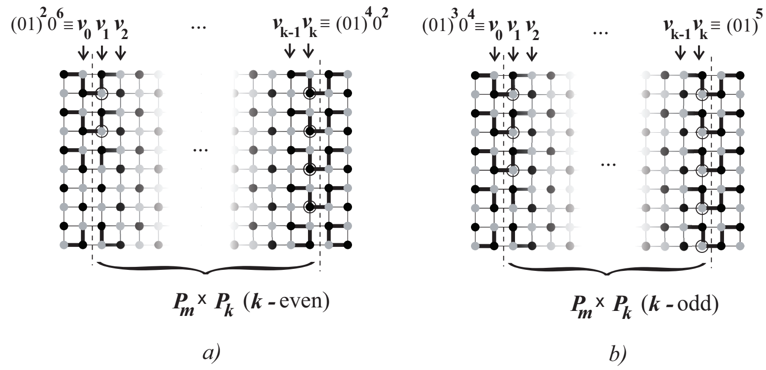

Figure 7: Two queens from $S_m^{(s_1)}$ and $S_m^{(s_2)}$ $(1 \leq s_1 \leq s_2 \leq \lfloor \frac{m}{2} \rfloor)$ can not be connected by a directed walk in $\mathcal{D}_m^*$.

Case III: $s_2 > s_1 > 0$ and k is even.
The considered grid graph has the same number of vertices of each color $(m \cdot k$ is even). However, its spanning graph has $\dfrac{s_1+s_2}{2}$ open paths, when m is even, and $\dfrac{s_1+s_2}{2} - 1$ open paths, when m is odd. These open paths cover $\dfrac{s_2-s_1}{2} \geq 1$ more vertices of one color than another (see Figure 7 a). Contradiction.

Case IV: $s_2 > s_1 > 0$ and k is odd.
If k is odd, then all open paths have end vertices of the same color, let us say gray (see Figure 7 b). Consequently, gray vertices are more numerous.
But, the upper corner vertices are black. This implies that the number of the black vertices is greater by 1 (when m is odd) or equal to the number of gray vertices (when m is even). Contradiction. □

We need a few assertions that can be easily obtained from the definition of 2-factor for the linear grid graph G.

Proposition 16. *For $v, w \in V(\mathcal{D}_{m_1}^*)$ and $x, y \in V(\mathcal{D}_{m_2}^*)$ $(m_1, m_2 \in N)$*

a) if $v \leftrightarrow w$ and $x \leftrightarrow y$, then $vx \leftrightarrow wy$ in $\mathcal{D}_{m_1+m_2}^*$,

b) if $vx \leftrightarrow wy$ in $\mathcal{D}_{m_1+m_2}^*$ and $v \leftrightarrow w$, then $x \leftrightarrow y$.

Proof. Straightforward. □

Proposition 17. *A direct predecessor and successor for a word from $V(\mathcal{D}_m^*)$ having*

a) a prefix 0 must have a prefix 1,

b) a prefix 01 must have a prefix 10.

Proof. Straightforward. □

Proposition 18. *If a word w from $V(\mathcal{D}_m^*)$ has a prefix* 00 *and $w \leftrightarrow v$ for some $v \in V(\mathcal{D}_m^*)$, then for the word u obtained from w by replacing that prefix with* 11 *we have $u \leftrightarrow v$, too.*

Proof. Observe a 2-factor of a linear grid graph G for which the words w and v are related to adjacent columns in G. The square of the grid corresponding to the prefix 00 of w has two vertical edges which belong to the 2-factor. If we replace these edges with other two (horizontal) of the square, the obtained subgraph of G is a 2-factor, too. □

3. PROOF OF THEOREM 5

Lemma 19. *For $m \geq 2$ and $0 \leq s \leq \lfloor m/2 \rfloor$, the subdigraphs of $\mathcal{D}_m^*$ induced by the sets $S_m^{(s)}$ are strongly connected. Additionally, for $s \geq 1$ they are bipartite digraphs, i.e $\langle S_m^{(s)} \rangle_{\mathcal{D}_m^*} = (R_m^{(s)}, G_m^{(s)})$. There is no edge which joins a vertex from $S_m^{(s)}$ to a vertex from $S_m^{(t)}$, where $0 \leq s < t \leq \lfloor m/2 \rfloor$, i.e. $\mathcal{D}_m^* = \bigcup_{s=0}^{\lfloor m/2 \rfloor} \langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$.*

Proof. The proof is by induction on m. The statement for $m \leq 3$ (the base cases) trivially holds (see Figure 4, Figure 5 and Example 11). Let us suppose that the statement is true for all digraphs $\mathcal{D}_w^*$, when $w < m$ and prove it for $m \geq 4$.

<u>$\langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$ $(0 \leq s \leq \lfloor m/2 \rfloor)$ is strongly connected.</u>

We introduce the following sets:

$\mathcal{O}_m^{(s)} \stackrel{\text{def}}{=} \{1v \mid v \in \{0,1\}^{m-1} \wedge \mid Z(v) \mid = s\}$,

$\mathcal{J}_m^{(s)} \stackrel{\text{def}}{=} \{00v \mid v \in \{0,1\}^{m-2} \wedge \mid Z(v) \mid = s\}$ and

$\mathcal{K}_m^{(s)} \stackrel{\text{def}}{=} \{01v \mid v \in \{0,1\}^{m-2} \wedge \mid Z(v) + 1 \mid = s\}$.

Obviously, $S_m^{(s)} = \mathcal{O}_m^{(s)} \cup \mathcal{J}_m^{(s)} \cup \mathcal{K}_m^{(s)}$.

Note that $\langle \mathcal{O}_m^{(s)} \rangle_{\mathcal{D}_m^*}$ is isomorphic to $\langle S_{m-1}^{(s)} \rangle_{\mathcal{D}_{m-1}^*}$. Since the latter digraph is strongly connected by inductive hypothesis (abbreviated I.H.), the same is valid for the former digraph, too. Let us prove that each vertex $x \in \mathcal{J}_m^{(s)} \cup \mathcal{K}_m^{(s)}$ is a direct predecessor for a vertex from $\mathcal{O}_m^{(s)}$.

If $x = 00v \in \mathcal{J}_m^{(s)}$ where $v \in S_{m-2}^{(s)}$ $(\mid Z(v) \mid = s)$, then by I.H. there exists $w \in S_{m-2}^{(s)}$ for which $v \leftrightarrow w$ in $\langle S_{m-2}^{(s)} \rangle_{\mathcal{D}_{m-2}^*}$ and $Z(w) = -Z(v)$, i.e. $\mid Z(w) \mid = \mid Z(v) \mid = s$. Since $00 \leftrightarrow 11$ in $\mathcal{D}_2^*$, Proposition 16a implies that $00v \leftrightarrow 11w$ where $11w \in \mathcal{O}_m^{(s)}$.

If $x = 01v$ where $v \in \{0,1\}^{m-2} \wedge Z(v) = -1 \pm s$, then by I.H. there exists $w \in \{0,1\}^{m-2}$ for which $v \leftrightarrow w$ (hence $Z(w) = -Z(v)$) in $\langle S_{m-2}^{(|Z(v)|)} \rangle_{\mathcal{D}_{m-2}^*}$. Since $01 \leftrightarrow 10$ in $\mathcal{D}_2^*$ and $Z(10w) = -1 + Z(w) = -1 - Z(v) = \mp s$, using Proposition 16a again, we conclude that $x = 01v \leftrightarrow 10w$ where $10w \in \mathcal{O}_m^{(s)}$.

<u>$\langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$ $(1 \leq s \leq \lfloor m/2 \rfloor)$ is the bipartite digraph $(R_m^{(s)}, G_m^{(s)})$.</u>

We need to prove that $R_m^{(s)}$ and $G_m^{(s)}$ are stable sets, i.e. $\langle R_m^{(s)}\rangle_{\mathcal{D}_m^*}$ and $\langle G_m^{(s)}\rangle_{\mathcal{D}_m^*}$ are edgeless digraphs. For this purpose we consider the set $\{x \in \{0,1\}^m \mid Z(x) = k\}$, where $k = s$ or $k = -s$. It is the union $\mathcal{O}_m^{(k)} \cup \mathcal{J}_m^{(k)} \cup \mathcal{K}_m^{(k)}$ where

$\mathcal{O}_m^{(k)} \stackrel{\text{def}}{=} \{1v \mid v \in \{0,1\}^{m-1} \wedge Z(v) = -k\}$,

$\mathcal{J}_m^{(k)} \stackrel{\text{def}}{=} \{00v \mid v \in \{0,1\}^{m-2} \wedge\ Z(v) = k\}$ and

$\mathcal{K}_m^{(k)} \stackrel{\text{def}}{=} \{01v \mid v \in \{0,1\}^{m-2} \wedge\ Z(v) = k-1\}$.

The digraph $\langle \mathcal{J}_m^{(k)} \cup \mathcal{K}_m^{(k)}\rangle_{\mathcal{D}_m^*}$ is an edgeless digraph because two words with a prefix 0 can not be neighbors in $\mathcal{D}_m^*$ (Proposition 17a). The digraph $\langle \mathcal{O}_m^{(k)}\rangle_{\mathcal{D}_m^*}$ is also an empty digraph because it is isomorphic to $\langle\{v \in \{0,1\}^{m-1} \mid Z(v) = -k\}\rangle_{\mathcal{D}_{m-1}^*}$, which is $\langle G_{m-1}^{(s)}\rangle_{\mathcal{D}_m^*}$ (for $k = s$) or $\langle R_{m-1}^{(s)}\rangle_{\mathcal{D}_m^*}$ (for $k = -s$) (I.H.).

It remains to prove that there are no arcs between $\langle \mathcal{O}_m^{(k)}\rangle_{\mathcal{D}_m^*}$ and $\langle \mathcal{J}_m^{(k)}\rangle_{\mathcal{D}_1^*}$, neither between $\langle \mathcal{O}_m^{(k)}\rangle_{\mathcal{D}_m^*}$ and $\langle \mathcal{K}_m^{(k)}\rangle_{\mathcal{D}_m^*}$.

Assume the opposite: $(\exists x \in \mathcal{J}_m^{(k)})\ (\exists y \in \mathcal{O}_m^{(k)})\ x \leftrightarrow y$ $\ (Case\ I)$ or

$(\exists x \in \mathcal{K}_m^{(k)})(\exists y \in \mathcal{O}_m^{(k)})\ x \leftrightarrow y$ $\ (Case\ II)$.

Case I: $x = 00v \in \mathcal{J}_m^{(k)}$, $y = 1w \in \mathcal{O}_m^{(k)}$ and $x \leftrightarrow y$.

Using Proposition 18 we conclude that $11v \leftrightarrow 1w$, i.e. $1v \leftrightarrow w$. $Z(v) = Z(00v) = k$ implies $Z(1v) = -k$. On the other hand $Z(w) = -k$ because of $Z(1w) = k$. Since $1v \leftrightarrow w$, we conclude that $\{v \mid v \in \{0,1\}^{m-1} \wedge Z(v) = -k\}$ is not a stable set. Contradiction with I.H.

Case II: $x = 01v \in \mathcal{K}_m^{(k)}$ and $y \in \mathcal{O}_m^{(k)} = \{10w \mid w \in \{0,1\}^{m-2} \wedge Z(w) = k+1\} \cup \{11w \mid w \in \{0,1\}^{m-2} \wedge Z(w) = k\}$ and $x \leftrightarrow y$.

Taking in mind Proposition 17b and Proposition 16b, we have $y = 10w$ and $v \leftrightarrow w$ where $Z(w) = k+1$ and $Z(v) = k-1$. Consequently, the digraphs inducted by $\{v \in \{0,1\}^{m-2} \mid Z(v) = k+1\}$ and $\{v \in \{0,1\}^{m-2} \mid Z(v) = k-1\}$ are not disjoint in $\mathcal{D}_{m-2}^*$. Contradiction with I.H.

There is no arc connecting a vertex in $S_m^{(s)}$ to a vertex in $S_m^{(t)}$, $0 \le s < t \le \lfloor m/2 \rfloor$.

Assuming the opposite, from already stated strong connectivity of the digraphs $\langle S_m^{(s)}\rangle_{\mathcal{D}_m^*}$ and $\langle S_m^{(t)}\rangle_{\mathcal{D}_m^*}$ we obtain that the main representatives of $S_m^{(s)}$ and $S_m^{(t)}$ are connected by a directed walk in $\mathcal{D}_m^*$ which is in contrary to Lemma 15. Consequently, we have $\mathcal{D}_m^* = \bigcup_{s=0}^{\lfloor m/2 \rfloor} \langle S_m^{(s)}\rangle_{\mathcal{D}_m^*}$.

This completes the verification of the main statement for $m \ge 4$. Hence, by induction the result is true for all integer $m \ge 2$, as required. □

Lemma 20. *If $v \in \mathcal{S}_m^{(s)}$ $(0 \le s \le \lfloor m/2 \rfloor, m \in N)$, then the vertex $\overline{v}$ belongs to the same set $S_m^{(s)}$. Moreover, if m is odd and $s > 0$, then the vertices v and $\overline{v}$ are in the same class (of the same color). If m is even and $s > 0$, then the vertices v and $\overline{v}$ are in the different classes (of different colors).*

Proof. Let $v \to v_1 \to v_2 \to \ldots \to v_t = Q_m^{(s)}$ be a directed walk of length t ($t \geq 0$) which connects an arbitrary vertex $v \in S_m^{(s)}$ where $s \geq 1$ to the queen in the same set. Then, the directed walk $\overline{v} \to \overline{v}_1 \to \overline{v}_2 \to \ldots \to \overline{v}_t = K_m^{(s)}$, as well as $\overline{v}_t = K_m^{(s)} \to \overline{v}_{t-1} \to \overline{v}_{t-2} \to \ldots \to \overline{v}_1 \to \overline{v}$, is of the same length t. When m is odd, the king $K_m^{(s)}$ is colored red, i.e. $K_m^{(s)} \in R_m^{(s)}$. Every directed walk between him and the queen $Q_m^{(s)}$ is of even length, so the same is valid for vertices v and $\overline{v}$. When m is even, the king $K_m^{(s)}$ and the queen $Q_m^{(s)}$ are of different colors (i.e. in different classes). Therefore, the same holds for v and $\overline{v}$.

For $v \in \mathcal{S}_m^{(0)}$, $\overline{v}$ must belong to the same set. Namely, if we assume the opposite, i.e. $\overline{v} \in \mathcal{S}_m^{(s)}$, where $s \neq 0$, then $\overline{\overline{v}} = v \in \mathcal{S}_m^{(s)}$, which is impossible. □

Lemma 21. *If* $1 \leq s \leq k$ $(k \in N)$, *then*

$$a)\quad | R_{2k}^{(s)} |=| G_{2k}^{(s)} |= \binom{2k}{k-s}, \quad | S_{2k}^{(s)} |= 2 \cdot \binom{2k}{k-s} \quad \textit{and} \quad | S_{2k}^{(0)} |= \binom{2k}{k}.$$

$$b)\quad | R_{2k+1}^{(s)} |= \binom{2k+1}{k-s+1}, \quad | G_{2k+1}^{(s)} |= \binom{2k+1}{k-s},$$
$$| S_{2k+1}^{(s)} |= \binom{2k+2}{k-s+1} \quad \textit{and} \quad | S_{2k+1}^{(0)} |= \binom{2k+1}{k}.$$

Proof. Let i be the number of 0's at odd position in a word $v \in V(\mathcal{D}_m^*)$, i.e. $i = odd(v)$. Using the definition of the sets $S_m^{(0)}$, $R_m^{(s)}$ and $G_m^{(s)}$ $(1 \leq s \leq k)$ and Vandermonde's convolution we analyze all the cases.

- $\underline{m = 2k, v \in S_m^{(0)}}$

$$| S_{2k}^{(0)} |= \sum_{i=0}^{k} \binom{k}{i}\binom{k}{i} = \sum_{i=0}^{k} \binom{k}{i}\binom{k}{k-i} = \binom{2k}{k}$$

- $\underline{m = 2k+1, v \in S_m^{(0)}}$

$$| S_{2k+1}^{(0)} |= \sum_{i=0}^{k} \binom{k+1}{i}\binom{k}{i} = \sum_{i=0}^{k} \binom{k+1}{i}\binom{k}{k-i} = \binom{2k+1}{k}$$

- $\underline{m = 2k, v \in S_m^{(s)}}$

$$| R_{2k}^{(s)} |= \sum_{i=0}^{k} \binom{k}{i}\binom{k}{i-s} = \sum_{i=0}^{k} \binom{k}{i}\binom{k}{k-i+s} = \binom{2k}{k+s} = \binom{2k}{k-s}$$

$$| G_{2k}^{(s)} |= \sum_{i=0}^{k} \binom{k}{i}\binom{k}{i+s} = \sum_{i=0}^{k} \binom{k}{i}\binom{k}{k-i-s} = \binom{2k}{k-s}$$

- $\underline{m = 2k+1, v \in S_m^{(s)}}$

$$| R_{2k+1}^{(s)} |= \sum_{i=0}^{k} \binom{k+1}{i}\binom{k}{i-s} = \sum_{i=0}^{k} \binom{k+1}{i}\binom{k}{k-i+s} = \binom{2k+1}{k+s} =$$

$$\binom{2k+1}{k-s+1}$$

$$\mid G_{2k+1}^{(s)} \mid = \sum_{i=0}^{k} \binom{k+1}{i} \binom{k}{i+s} = \sum_{i=0}^{k} \binom{k+1}{i} \binom{k}{k-i-s} = \binom{2k+1}{k-s}.$$

□

Now, from Lemma 19 we conclude that the components of the transfer digraph $\mathcal{D}_m^*$ are $\langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$ where $0 \leq s \leq \lfloor m/2 \rfloor$. Note that the digraph $\langle S_m^{(0)} \rangle_{\mathcal{D}_m^*}$ is the component which contains the vertex 1^m. It implies that $\langle S_m^{(0)} \rangle_{\mathcal{D}_m^*} = \mathcal{A}_m^*$. Having in mind that $\mathcal{D}_m^* = \mathcal{A}_m^* \cup (\bigcup_{s=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}_m^{*(s)}) = \bigcup_{s=0}^{\lfloor m/2 \rfloor} \langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$ and $\mid S_m^{(s_1)} \mid > \mid S_m^{(s_2)} \mid$ for $1 \leq s_1 < s_2 \leq \lfloor m/2 \rfloor$, we conclude that $\mathcal{B}_m^{*(s)} = \langle S_m^{(s)} \rangle_{\mathcal{D}_m^*}$ for all $s = 1, 2, \ldots, \lfloor m/2 \rfloor$. Consequently, all the components $\mathcal{B}_m^{*(s)}$ $(1 \leq s \leq \lfloor m/2 \rfloor)$ are bipartite digraphs. Lemma 20 and Lemma 21 further complete the proof of the Theorem 5.

Acknowledgements. The authors are indebted to the anonymous referees for their valuable suggestions and helpful comments which improved the clarity of the presentation. The authors would like to express their gratitude to Roddy Bogawa for his meticulous reading of the first draft of the manuscript and on many useful suggestions.

This work was supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia (Grants No. 451-03-9/2022-14/200125, 451-03-68/2022-14/200156) and the Project of the Department for fundamental disciplines in technology, Faculty of Technical Sciences, University of Novi Sad "Application of general disciplines in technical and IT sciences".

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(Received 11. 12. 2021.)
(Revised 07. 02. 2023.)

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