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# COMPLETE MONOTONICITY INVOLVING THE DIVIDED DIFFERENCE OF POLYGAMMA FUNCTIONS

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For $r, s \in \mathbb{R}$ and $\rho = \min\{r, s\}$, let

$$D[x+r, x+s; \psi_{n-1}] \equiv -\phi_n(x)$$

be the divided difference of the functions $\psi_{n-1} = (-1)^n \psi^{(n-1)}$ ($n \in \mathbb{N}$) on $(-\rho, \infty)$, where $\psi^{(n)}$ stands for the polygamma functions. In this paper, we present the necessary and sufficient conditions for the functions

$$\begin{aligned} x &\mapsto \prod_{i=1}^{k} \phi_{m_i}(x) - \lambda_k \prod_{i=1}^{k} \phi_{n_i}(x), \\ x &\mapsto \prod_{i=1}^{k} \phi_{n_i}(x) - \mu_k \phi_{s_{n_k}}(x) \end{aligned}$$

to be completely monotonic on $(-\rho, \infty)$, where $m_i, n_i \in \mathbb{N}$ for $i = 1, .., k$ with $k \geq 2$ and $s_{n_k} = \sum_{i=1}^{k} n_i$. These generalize known results and gives an answer to a problem.

## 1. INTRODUCTION

Recall that a function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and

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$n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (see [**8, 29**]). A obvious observation is that, if $f(x)$ and $g(x)$ are completely monotonic on $I$, then $pf(x) + qg(x)$ for $p, q > 0$ and $f(x)g(x)$ are also completely monotonic on $I$ (see [**18**, Theorem 1]).

The Euler's gamma and psi (digamma) functions are defined, for $x > 0$, by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. The derivatives $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ are called polygamma functions. The functions $\psi^{(n)}(x)$ for $n \in \mathbb{N}_0$ have the following integral and series representations [**1**, Sections 6.3, 6.4])

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \frac{1}{k(x+k)},$$

$$(-1)^{n-1} \psi^{(n)}(x) = \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt = \frac{n!}{x^{n+1}} + n! \sum_{k=1}^\infty \frac{1}{(x+k)^{n+1}},$$

where $\gamma = 0.57721...$ is Euler's constant.

Denote by $\psi_n = (-1)^{n-1} \psi^{(n)} = \left|\psi^{(n)}\right|$, $n \in \mathbb{N}_0$. Clearly, $\psi_n$ for $n \in \mathbb{N}$ is completely monotonic on $(0, \infty)$. More properties, including monotonicity, convexity, complete monotonicity and inequalities, of $\psi_n$ can be found in [**2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 19, 20, 24, 25, 30**], and recent papers [**11, 21, 22, 23, 26, 27, 28, 31, 32, 33, 34, 35**].

For $r, s \in \mathbb{R}$ and $n \in \mathbb{N}$, let $\phi_n^{[r,s]}(x)$ be defined on $(-\min\{r, s\}, \infty)$ by

$$\phi_n^{[r,s]}(x) = \begin{cases} (-1)^{n-1} \dfrac{\psi^{(n-1)}(x+r) - \psi^{(n-1)}(x+s)}{r-s} & \text{if } r \neq s, \\ (-1)^{n-1} \psi^{(n)}(x+s) = \psi_n(x+s) & \text{if } r = s, \end{cases} \tag{1}$$

or equivalently,

$$\phi_n^{[r,s]}(x) = \frac{\int_s^r \psi_n(x+t)\, dt}{r-s} \text{ if } r \neq s \text{ and } \phi_n^{[s,s]}(x) = \psi_n(x+s).$$

It was shown in [**33**] that, for $r \geq s$, the function $x \mapsto \phi_n^{[r,s]}(x)$ has the asymptotic relations

$$(x+s)^n \frac{\phi_n^{[r,s]}(x)}{(n-1)!} = 1 - \frac{n(r-s-1)}{2} \frac{1}{x+r} + O\left((x+r)^{-2}\right) \tag{2}$$

as $x \to \infty$, and

$$\begin{aligned} \lim_{x \to -s+} (x+s)^n \frac{\phi_n^{[r,s]}(x)}{(n-1)!} &= \frac{1}{|r-s|} && \text{if } r \neq s, \\ \lim_{x \to -s+} (x+s)^{n+1} \frac{\phi_n^{[r,s]}(x)}{n!} &= 1 && \text{if } r = s. \end{aligned} \tag{3}$$

Let $\boldsymbol{x}_{[k]} = (x_i)_{1\le i\le k} = (x_1, x_2, ..., x_k)$ and $\boldsymbol{y}_{[k]} = (y_i)_{1\le i\le k} = (y_1, y_2, ..., y_k) \in \mathbb{R}^k$. A $k$-tuple $\boldsymbol{x}_{[k]}$ is said to be strictly majorized by another $k$-tuple $\boldsymbol{y}_{[k]}$ (in symbol $\boldsymbol{x}_{[k]} \prec \boldsymbol{y}_{[k]}$) if $x_i \ge x_{i+1}$, $y_i \ge y_{i+1}$ for $1 \le i \le k-1$ with $\boldsymbol{x}_{[k]} \ne \boldsymbol{y}_{[k]}$,

$$\sum_{i=1}^{j} x_i \le \sum_{i=1}^{j} y_i \text{ for } 1 \le j \le k-1 \text{ and } \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$$

(see [**17**, p. 8, Definition A.1]). In what follows, we will use another equivalent definition of $\boldsymbol{x}_{[k]} \prec \boldsymbol{y}_{[k]}$, which can be stated as follows:

**Definition 1.** *Let $\boldsymbol{x}_{[k]} = (x_i)_{1\le i\le k} = (x_1, x_2, ..., x_k)$ and $\boldsymbol{y}_{[k]} = (y_i)_{1\le i\le k} = (y_1, y_2, ..., y_k) \in \mathbb{R}^k$. The $k$-tuple $\boldsymbol{x}_{[k]}$ is said to be strictly majorized by another $k$-tuple $\boldsymbol{y}_{[k]}$, denoted by $\boldsymbol{x}_{[k]} \prec \boldsymbol{y}_{[k]}$ or $\boldsymbol{y}_{[k]} \succ \boldsymbol{x}_{[k]}$, if $\boldsymbol{x}_{[k]} \ne \boldsymbol{y}_{[k]}$,*

$$\begin{aligned} x_1 &\le x_2 \le \cdots \le x_k, \quad y_1 \le y_2 \le \cdots \le y_k, \\ \sum_{i=1}^{j} x_i &\ge \sum_{i=1}^{j} y_i \text{ for } j = 1, ..., k-1 \text{ and } \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i. \end{aligned}$$

For the sake of statements and proofs in the sequel, we also need several assumptions and notations:

(i) $\boldsymbol{n}^*_{[k]} = (n^*_i)_{1\le i\le k}$, $\boldsymbol{n}'_{[k]} = (n'_i)_{1\le i\le k}$ and $(\boldsymbol{n}+\boldsymbol{1})_{[k]} = (n_i + 1)_{1\le i\le k}$;

(ii) $\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ and $\beta_{\boldsymbol{n}_{[k]}}$ are defined by

$$\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}} = \prod_{i=1}^{k} \frac{\Gamma(m_i)}{\Gamma(n_i)} \text{ and } \beta_{\boldsymbol{n}_{[k]}} = \frac{\prod_{i=1}^{k} \Gamma(n_i)}{\Gamma(s_{n_k})}, \tag{4}$$

where $s_{n_k} = \sum_{i=1}^{k} n_i$.

Let $F^{[r,s]}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x; \lambda_k)$ and $G^{[r,s]}_{\boldsymbol{n}_{[k]}}(x; \mu_k)$ be defined on $(-\min\{r,s\}, \infty)$ by

$$F^{[r,s]}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x; \lambda_k) = \prod_{i=1}^{k} \phi^{[r,s]}_{m_i}(x) - \lambda_k \prod_{i=1}^{k} \phi^{[r,s]}_{n_i}(x), \tag{5}$$

$$G^{[r,s]}_{\boldsymbol{n}_{[k]}}(x; \mu_k) = \prod_{i=1}^{k} \phi^{[r,s]}_{n_i}(x) - \mu_k \phi^{[r,s]}_{s_{n_k}}(x), \tag{6}$$

where $\lambda_k, \mu_k \in \mathbb{R}$, $s_{n_k} = \sum_{i=1}^{k} n_i$, $k \ge 2$. In particular, due to $\phi^{[0,0]}_n(x) = \psi_n(x)$, we have

$$\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x; \lambda_k) = F^{[0,0]}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x; \lambda_k) = \prod_{i=1}^{k} \psi_{m_i}(x) - \lambda_k \prod_{i=1}^{k} \psi_{n_i}(x), \tag{7}$$

$$\mathcal{G}_{\boldsymbol{n}_{[k]}}(x; \mu_k) = G^{[0,0]}_{\boldsymbol{n}_{[k]}}(x; \mu_k) = \prod_{i=1}^{k} \psi_{n_i}(x) - \mu_k \psi_{s_{n_k}}(x). \tag{8}$$

In 2017, Yang [**33**] obtained the complete monotonicity of the function

$$x \mapsto \phi_{n+1}^{[r,s]}(x)^2 - \lambda\phi_n^{[r,s]}(x)\,\phi_{n+2}^{[r,s]}(x)$$

on $(-\min\{r,s\},\infty)$, which extended Alzer and Wells's result in [**5**, Corollary 2.3]. In 2019, Qi and Agarwal [**23**] proposed an problem on the complete monotonicity of the function

$$x \mapsto \phi_{n+1}^{[r,s]}(x)^2 - \lambda\phi_{n-k+1}^{[r,s]}(x)\,\phi_{n+k+1}^{[r,s]}(x)$$

on $(-\min\{r,s\},\infty)$, where $k,n \in \mathbb{N}$. Later, Gao [**11**] considered the complete monotonicity of the functions $F_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}^{[c,0]}(x;\lambda_2)$ and $G_{\boldsymbol{n}_{[2]}}^{[c,0]}(x;\mu_2)$ on $(0,\infty)$, where $m_1,m_2,n_1,n_2 \in \mathbb{N}_0$ satisfying

$$n_2 > m_2 \geq m_1 > n_1 \quad \text{and} \quad n_1+n_2 = m_1+m_2, \tag{9}$$

and $c>0$. Precisely, Gao's result can be stated as two theorems (see also [**21, 22**]).

**Theorem 2.** Let $m_1,m_2,n_1,n_2 \in \mathbb{N}$ satisfying (9) and $\lambda_2 \in \mathbb{R}$, $c>0$. Then the following statements are valid:

(i) For $0<c<1$, $F_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}^{[c,0]}(x;\lambda_2)$ is completely monotonic on $(0,\infty)$ if and only if $\lambda_2 \leq \alpha_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}$, where $\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ is given in (4).

(ii) For $c>1$, $-F_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}^{[c,0]}(x;\lambda_2)$ is completely monotonic on $(0,\infty)$ if and only if $\lambda_2 \geq \alpha_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}$.

(iii) For $c>0$, $-F_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}^{[c,0]}\left(x;\alpha_{(\boldsymbol{m+1})_{[2]},(\boldsymbol{n+1})_{[2]}}\right)$ is completely monotonic on $(0,\infty)$.

**Theorem 3.** Let $n_1,n_2 \in \mathbb{N}$ and $\mu_2 \in \mathbb{R}$, $c>0$. Then the following statements are valid:

(i) For $0<c<1$, $G_{\boldsymbol{n}_{[2]}}^{[c,0]}(x;\mu_2)$ and $-G_{\boldsymbol{n}_{[2]}}^{[c,0]}(x;\mu_2)$ are completely monotonic on $(0,\infty)$ if and only if $\mu_2 \leq \beta_{\boldsymbol{n}_{[2]}}$ and $\mu_2 \geq c^{-1}\beta_{\boldsymbol{n}_{[2]}}$, respectively, where $\beta_{\boldsymbol{n}_{[2]}}$ is given in (4).

(ii) For $c>1$, $G_{\boldsymbol{n}_{[2]}}^{[c,0]}(x;\mu_2)$ and $-G_{\boldsymbol{n}_{[2]}}^{[c,0]}(x;\mu_2)$ are completely monotonic on $(0,\infty)$ if and only if $\mu_2 \leq c^{-1}\beta_{\boldsymbol{n}_{[2]}}$ and $\mu_2 \geq \beta_{\boldsymbol{n}_{[2]}}$, respectively.

**Remark 4.** *(i) Clearly, the conditions (9) imply $(m_1,m_2) \prec (n_1,n_2)$.*

*(ii) Let $c=|r-s|$ and $\rho=\min\{r,s\}$. We see clearly that*

$$F_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}^{[|r-s|,0]}(x+\rho;\lambda_2) = F_{\boldsymbol{m}_{[2]},\boldsymbol{n}_{[2]}}^{[r,s]}(x;\lambda_2) \quad \textit{and} \quad G_{\boldsymbol{n}_{[2]}}^{[|r-s|,0]}(x+\rho;\mu_2) = G_{\boldsymbol{n}_{[2]}}^{[r,s]}(x;\mu_2).$$

*Then replacing (9), $c$, $x$ and $(0,\infty)$ by $(m_1,m_2) \prec (n_1,n_2)$, $|r-s|$, $x+\rho$ and $(-\rho,\infty)$, respectively, Theorems 2 and 3 are still true.*

*(iii) For convenience, in what follows we always denote $\phi_n$, $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ and $G_{\boldsymbol{n}_{[k]}}$ for $\phi_n^{[p,q]}$, $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}^{[r,s]}$ and $G_{\boldsymbol{n}_{[k]}}^{[r,s]}$, respectively, unless special explanation.*

Recently, Qi [**22**, Remark 19] proposed a problem on discussing necessary and sufficient conditions for the functions $\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ and $-\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ to be respectively completely monotonic on $(0,\infty)$, where $\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]} \in \mathbb{N}_0^k$.

Motivated by Theorems 2, 3 and Remark 4, as well Qi's problem mentioned above, it is the aim of this paper to give generalizations of Theorems 2 and 3. Our first result is the following theorem.

**Theorem 5.** *Let* $\boldsymbol{m}_{[k]} = (m_1, ..., m_k)$ *and* $\boldsymbol{n}_{[k]} = (n_1, ..., n_k) \in \mathbb{N}^k$ *for* $k \geq 2$ *satisfy* $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}_{[k]}$, *let* $\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ *be given by (4), and let the function* $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ *be defined on* $(-\rho,\infty)$ *by (5), where* $\rho = \min\{r,s\}$. *Then the following statements hold:*

*(i) If* $0 < |r-s| < 1$, *then the function* $x \mapsto F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ *is completely monotonic on* $(-\rho,\infty)$ *if and only if* $\lambda_k \leq \alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$.

*(ii) If* $|r-s| > 1$, *then the function* $x \mapsto -F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ *is completely monotonic on* $(-\rho,\infty)$ *if and only if* $\lambda_k \geq \alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$.

*(iii) For* $|r-s| > 0$, *the function* $x \mapsto -F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\alpha_{(\boldsymbol{m}+\mathbf{1})_{[k]},(\boldsymbol{n}+\mathbf{1})_{[k]}}\right)$ *is completely monotonic on* $(-\rho,\infty)$.

Let $r \to s = 0$ in Theorem 5, we can prove the following corollary.

**Corollary 6.** *Let* $\boldsymbol{m}_{[k]} = (m_1, ..., m_k)$ *and* $\boldsymbol{n}_{[k]} = (n_1, ..., n_k) \in \mathbb{N}^k$ *for* $k \geq 2$ *satisfy* $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}_{[k]}$, *let* $\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ *be given by (4), and let the function* $\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ *be defined on* $(0,\infty)$ *by (7). Then the following statements are valid:*

*(i) The function* $x \mapsto \mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ *is completely monotonic on* $(0,\infty)$ *if and only if* $\lambda_k \leq \alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$.

*(ii) The function* $x \mapsto -\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ *is completely monotonic on* $(0,\infty)$ *if and only if* $\lambda_k \geq \alpha_{(\boldsymbol{m}+\mathbf{1})_{[k]},(\boldsymbol{n}+\mathbf{1})_{[k]}}$.

Our second result is stated as follows.

**Theorem 7.** *Let* $n_i \in \mathbb{N}$, $i = 1, 2, ..., k$ *with* $k \geq 2$, *and* $s_{n_k} = \sum_{i=1}^k n_i$, *let* $\beta_{\boldsymbol{n}_{[k]}}$ *be given by (4), and let the function* $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *be defined by (6). Then the following statements hold:*

*(i) For* $0 < |r-s| < 1$, $x \mapsto G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *and* $x \mapsto -G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *are completely monotonic on* $(-\rho,\infty)$ *if and only if* $\mu_k \leq \beta_{\boldsymbol{n}_{[k]}}$ *and* $\mu_k \geq |r-s|^{1-k}\beta_{\boldsymbol{n}_{[k]}}$, *respectively.*

*(ii) For* $|r-s| > 1$, *the functions* $x \mapsto G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *and* $x \mapsto -G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *are completely monotonic on* $(-\rho,\infty)$ *if and only if* $\mu_k \leq |r-s|^{1-k}\beta_{\boldsymbol{n}_{[k]}}$ *and* $\mu_k \geq \beta_{\boldsymbol{n}_{[k]}}$, *respectively.*

Let $r \to s = 0$ in Theorem 7, the following proposition is immediate.

**Corollary 8.** *Let* $n_i \in \mathbb{N}$, $i = 1, 2, ..., k$ *with* $k \geq 2$, *and* $s_{n_k} = \sum_{i=1}^k n_i$, *and let the function* $\mathcal{G}_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *be given by (8). Then the function* $x \mapsto \mathcal{G}_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ *is completely monotonic on* $(0,\infty)$ *if and only if* $\mu_k \leq \beta_{\boldsymbol{n}_{[k]}}$.

**Remark 9.** *Clearly, Corollaries 6 and 8 solve partly Qi's problem in [**22**, Remark 19].*

The proofs of main results are given in Section 3. In Section 4, we list several consequences of main results.

## 2. LEMMAS

To prove main results, we need the following lemmas.

**Lemma 10.** *Let $m_i, n_i \in \mathbb{N}$ for $i = 1, ..., k$ with $k \geq 2$ satisfy $\sum_{i=1}^{k} m_i = \sum_{i=1}^{k} n_i$, and let $\phi_n(x)$ be defined on $(-\rho, \infty)$ by (1), where $\rho = \min\{r, s\}$. Then*

$$\lim_{x \to -\rho^+} \prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} = \begin{cases} \prod_{i=1}^{k} \frac{\Gamma(m_i)}{\Gamma(n_i)} = \alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}} & \text{if } |r - s| > 0, \\ \prod_{i=1}^{k} \frac{\Gamma(m_i + 1)}{\Gamma(n_i + 1)} = \alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]}, (\boldsymbol{n}+\boldsymbol{1})_{[k]}} & \text{if } r = s, \end{cases} \tag{10}$$

$$\lim_{x \to \infty} \prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} = \prod_{i=1}^{k} \frac{\Gamma(m_i)}{\Gamma(n_i)} = \alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}}. \tag{11}$$

*Proof.* Assume that $\rho = \min\{r, s\} = s$. When $r > s$, using the first limit relation of (3) we have

$$\lim_{x \to -s^+} \prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} = \lim_{x \to -s^+} \prod_{i=1}^{k} \frac{(m_i - 1)!\,(x + s)^{-m_i} / (r - s)}{(n_i - 1)!\,(x + s)^{-n_i} / (r - s)} = \prod_{i=1}^{k} \frac{(m_i - 1)!}{(n_i - 1)!}.$$

When $r = s$, using the second limit relation of (3) we have

$$\lim_{x \to -s^+} \prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} = \lim_{x \to -s^+} \prod_{i=1}^{k} \frac{m_i!\,(x + s)^{-m_i - 1}}{n_i!\,(x + s)^{-n_i - 1}} = \prod_{i=1}^{k} \frac{m_i!}{n_i!}.$$

By the asymptotic formula (2) it is deduced that

$$\prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} \sim \prod_{i=1}^{k} \frac{(m_i - 1)!\,(x + s)^{-m_i}}{(n_i - 1)!\,(x + s)^{-n_i}} \to \prod_{i=1}^{k} \frac{(m_i - 1)!}{(n_i - 1)!}$$

as $x \to \infty$, which completes the proof. □

**Remark 11.** *Assume that $r > s$. Then from (10) and (11) we have*

$$\lim_{x \to -s^+} \prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} = \lim_{x \to \infty} \prod_{i=1}^{k} \frac{\phi_{m_i}(x)}{\phi_{n_i}(x)} = \alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}}.$$

*Therefore, for $r > s$, the function $\prod_{i=1}^{k} (\phi_{m_i}/\phi_{n_i})$ is not monotonic on $(-s, \infty)$.*

**Lemma 12.** *For $k, \ell \in \mathbb{N}$ with $\ell \leq k-1$, let $\boldsymbol{m}_{[k+1]} = (m_i)_{1 \leq i \leq k+1}$ and $\boldsymbol{n}_{[k+1]} = (n_i)_{1 \leq i \leq k+1} \in \mathbb{N}^{k+1}$ satisfy $\boldsymbol{m}_{[k+1]} \prec \boldsymbol{n}_{[k+1]}$, and let $n_i^* = n_i$ for $1 \leq i \leq k-\ell$, $n_{k-\ell+\mathbf{1}}^* = n_{k-\ell+1} + n_{k-\ell+2} - m_{k+1}$, $n_i^* = n_{i+1}$ for $k-\ell+2 \leq i \leq k$ and $\ell \geq 2$. Suppose that*

$$n_1 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} < n_{k-\ell+2} \leq n_{k-\ell+3} \leq \cdots \leq n_{k+1}. \tag{12}$$

*Then $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}_{[k]}^*$ if $n_{k-\ell+\mathbf{1}} \leq m_{k+1}$. Moreover, we have*

$$\frac{\alpha_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}}{\alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}^*}} = \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}, \tag{13}$$

$$\frac{\alpha_{(\boldsymbol{m}+\mathbf{1})_{[k+1]}, (\boldsymbol{n}+\mathbf{1})_{[k+1]}}}{\alpha_{(\boldsymbol{m}+\mathbf{1})_{[k]}, (\boldsymbol{n}+\mathbf{1})_{[k]}^*}} = \alpha_{(\boldsymbol{m}'+\mathbf{1})_{[2]}, (\boldsymbol{n}'+\mathbf{1})_{[2]}}, \tag{14}$$

*where $\boldsymbol{m}'_{[2]} = (m'_1, m'_2) = \left(n_{k-\ell+1}^*, m_{k+1}\right)$ and $\boldsymbol{n}'_{[2]} = (n'_1, n'_2) = (n_{k-\ell+1}, n_{k-\ell+2})$.*

*Proof.* (i) Due to $\boldsymbol{m}_{[k+1]} \prec \boldsymbol{n}_{[k+1]}$ and $n_{k-\ell+\mathbf{1}} \leq m_{k+1}$, it is easy to check that

$$\begin{aligned} n_{k-\ell+2}^* - n_{k-\ell+\mathbf{1}}^* &= n_{k-\ell+3} - (n_{k-\ell+\mathbf{1}} + n_{k-\ell+\mathbf{2}} - m_{k+1}) \geq 0, \\ n_{k-\ell+\mathbf{1}}^* - n_{k-\ell}^* &= (n_{k-\ell+\mathbf{1}} + n_{k-\ell+\mathbf{2}} - m_{k+1}) - n_{k-\ell} > 0. \end{aligned}$$

From the above two inequalities we get that

$$\begin{aligned} n_k^* &= n_{k+1} \geq n_k = n_{k-1}^* \geq \cdots \geq n_{k-\ell+3} = n_{k-\ell+2}^* \geq n_{k-\ell+\mathbf{1}}^* > n_{k-\ell}^* \\ &= n_{k-\ell} \geq n_{k-\ell-1} = n_{k-\ell-1}^* \geq \cdots \geq n_1^*. \end{aligned}$$

For $1 \leq j \leq k-\ell$, it is clear that

$$\sum_{i=1}^{j} m_i \geq \sum_{i=1}^{j} n_i = \sum_{i=1}^{j} n_i^*.$$

For $k-\ell+1 \leq j \leq k$, we have

$$\begin{aligned} \sum_{i=1}^{j} m_i - \sum_{i=1}^{j} n_i^* &= \sum_{i=1}^{j} m_i - \sum_{i=1}^{k-\ell} n_i^* - n_{k-\ell+\mathbf{1}}^* - \sum_{i=k-\ell+2}^{j} n_i^* \\ &= \sum_{i=1}^{j} m_i - \sum_{i=1}^{k-\ell} n_i - \sum_{i=k-\ell+3}^{j+1} n_i - (n_{k-\ell+\mathbf{1}} + n_{k-\ell+\mathbf{2}} - m_{k+1}) \\ &= \sum_{i=1}^{j+1} m_i - \sum_{i=1}^{j+1} n_i + (m_{k+1} - m_{j+1}) \geq 0, \end{aligned}$$

where the equality holds when $j = k$. These show that $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}_{[k]}^*$.

(ii) A direct computation yields

$$\frac{\alpha_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}}{\alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}^*}} = \frac{\prod_{i=1}^{k+1} \left(\Gamma(m_i) / \Gamma(n_i)\right)}{\prod_{i=1}^{k} \left(\Gamma(m_i) / \Gamma(n_i^*)\right)} = \frac{\Gamma\left(n_{k-\ell+1}^*\right) \Gamma(m_{k+1})}{\Gamma(n_{k-\ell+1}) \Gamma(n_{k-\ell+2})} = \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}},$$

and

$$\begin{aligned}\frac{\alpha_{(\boldsymbol{m}+\mathbf{1})_{[k+1]},(\boldsymbol{n}+\mathbf{1})_{[k+1]}}}{\alpha_{(\boldsymbol{m}+\mathbf{1})_{[k]},(\boldsymbol{n}+\mathbf{1})^{*}_{[k]}}} &= \frac{\prod_{i=1}^{k+1}\left(\Gamma\left(m_i+1\right)/\Gamma\left(n_i+1\right)\right)}{\prod_{i=1}^{k}\left(\Gamma\left(m_i+1\right)/\Gamma\left(n_i^{*}+1\right)\right)} \\ &= \frac{\Gamma\left(m_{k+1}+1\right)\Gamma\left(n^{*}_{k-\ell+1}+1\right)}{\Gamma\left(n_{k-\ell+1}+1\right)\Gamma\left(n_{k-\ell+2}+1\right)} = \alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}},\end{aligned}$$

thereby completing the proof. □

**Lemma 13.** *Let $n_1, n_2, ..., n_k \in \mathbb{N}$ with $k \geq 2$ and let $\phi_n(x)$ be defined on $(-\rho, \infty)$ by (1), where $\rho = \min\{r, s\}$. Then we have*

$$\begin{aligned}\lim_{x\to-\rho+}\frac{\prod_{i=1}^{k}\phi_{n_i}(x)}{\phi_{s_{n_k}}(x)} &= \begin{cases}\dfrac{\prod_{i=1}^{k}\Gamma\left(n_i\right)}{\left|r-s\right|^{k-1}\Gamma\left(s_{n_k}\right)} = \dfrac{\beta_{\boldsymbol{n}_{[k]}}}{\left|r-s\right|^{k-1}} & \text{if } r\neq s, \\ \infty & \text{if } r=s,\end{cases} \\ \lim_{x\to\infty}\frac{\prod_{i=1}^{k}\phi_{n_i}(x)}{\phi_{s_{n_k}}(x)} &= \frac{\prod_{i=1}^{k}\Gamma\left(n_i\right)}{\Gamma\left(s_{n_k}\right)} = \beta_{\boldsymbol{n}_{[k]}},\end{aligned}$$

*where $s_{n_k} = \sum_{i=1}^{k} n_i$.*

*Proof.* Assume that $\rho = \min\{r, s\} = s$. When $r > s$, using the first limit relation of (3) we have

$$\begin{aligned}\lim_{x\to-s^{+}}\frac{\prod_{i=1}^{k}\phi_{n_i}(x)}{\phi_{s_{n_k}}(x)} &= \lim_{x\to-s^{+}}\frac{\prod_{i=1}^{k}\left[\left(n_i-1\right)!\left(x+s\right)^{-n_i}/\left(r-s\right)\right]}{\left(s_{n_k}-1\right)!\left(x+s\right)^{-s_{n_k}}/\left(r-s\right)} \\ &= \frac{\prod_{i=1}^{k}\Gamma\left(n_i\right)}{\left|r-s\right|^{k-1}\Gamma\left(s_{n_k}\right)}.\end{aligned}$$

When $r = s$, using the second limit relation of (3) we have

$$\begin{aligned}\lim_{x\to-s^{+}}\frac{\prod_{i=1}^{k}\phi_{n_i}(x)}{\phi_{s_{n_k}}(x)} &= \lim_{x\to-s^{+}}\frac{\prod_{i=1}^{k}n_i!\left(x+s\right)^{-n_i-1}}{s_{n_k}!\left(x+s\right)^{-s_{n_k}-1}} \\ &= \frac{\prod_{i=1}^{k}n_i!}{s_{n_k}!}\lim_{x\to-s^{+}}\frac{1}{\left(x+s\right)^{k}} = \infty.\end{aligned}$$

By the asymptotic formula (2) it is deduced that

$$\frac{\prod_{i=1}^{k}\phi_{n_i}(x)}{\phi_{s_{n_k}}(x)} \sim \frac{\prod_{i=1}^{k}\left(n_i-1\right)!\left(x+s\right)^{-n_i}}{\left(s_{n_k}-1\right)!\left(x+s\right)^{-s_{n_k}}} \to \frac{\prod_{i=1}^{k}\left(n_i-1\right)!}{\left(s_{n_k}-1\right)!}$$

as $x \to \infty$, which completes the proof. □

## 3. PROOFS OF MAIN RESULTS

We are now in a position to prove our main results.

### 3.1 Proofs of Theorem 5

(i) For $0<|r-s|<1$, the necessary condition for $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ to be completely monotonic on $(-\rho,\infty)$ follows from the limit relation

$$\lambda_k \le \lim_{x\to\infty}\frac{\prod_{i=1}^{k}\phi_{m_i}(x)}{\prod_{i=1}^{k}\phi_{n_i}(x)} = \alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}} := \lambda_k^{[0]},$$

where the limit relation holds due to (11). To prove that $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ is completely monotonic on $(-\rho,\infty)$ if $\lambda_k\le\lambda_k^{[0]}$, it suffices to prove $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$. By virtu of Theorem 2 (i) and Remark 4 (ii) we see that $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ for $k=2$. Suppose that $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ for certain $k\ge2$. If we prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$, then by induction, $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ for all $k\ge2$.

Since $\boldsymbol{m}_{[k+1]}\prec\boldsymbol{n}_{[k+1]}$, we have

$$n_1\le m_1\le m_2\le\cdots\le m_k\le m_{k+1}\le n_{k+1}. \tag{15}$$

We now prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ stepwise. To this end, we first write

$$\begin{aligned}
&F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)=F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\alpha_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\right)\\
&=\left[\prod_{i=1}^{k+1}\phi_{m_i}(x)-\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\phi_{m_{k+1}}(x)\prod_{i=1}^{k}\phi_{n_i^*}(x)\right]\\
&+\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\left[\phi_{m_{k+1}}(x)\prod_{i=1}^{k}\phi_{n_i^*}(x)-\frac{\alpha_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}}{\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}}\prod_{i=1}^{k+1}\phi_{n_i}(x)\right]\\
&:=S_1(x)+\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\times S_2(x).
\end{aligned}\tag{16}$$

It is easy to check that

$$\begin{aligned}
S_1(x) &= \phi_{m_{k+1}}(x)\left[\prod_{i=1}^{k}\phi_{m_i}(x)-\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\prod_{i=1}^{k}\phi_{n_i^*}(x)\right]\\
&= \phi_{m_{k+1}}(x)F_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\left(x;\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\right).
\end{aligned}$$

If $\boldsymbol{m}_{[k]}\prec\boldsymbol{n}^*_{[k]}$, then by the induction assumption $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\left(x;\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}^*_{[k]}}\right)$ is completely monotonic on $(-\rho,\infty)$. Since $\phi_{m_{k+1}}(x)$ is completely monotonic on $(-\rho,\infty)$, we see that $S_1(x)$ is completely monotonic on $(-\rho,\infty)$ if $\boldsymbol{m}_{[k]}\prec\boldsymbol{n}^*_{[k]}$.

**Step 1**: Let $n_i^* = n_i$ for $i = 1, ..., k-1$ and $n_k^* = n_k + n_{k+1} - m_{k+1}$. Taking $\ell = 1$ in Lemma 12 gives that $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}_{[k]}^*$ and (13) holds. Therefore, $S_1(x)$ in (16) is completely monotonic on $(-\rho, \infty)$. While $S_2(x)$ in (16) can be written as

$$\begin{aligned} S_2(x) &= \left[\prod_{i=1}^{k-1} \phi_{n_i}(x)\right] \left[\phi_{n_k^*}(x)\, \phi_{m_{k+1}}(x) - \frac{\alpha_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}}{\alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}^*}} \phi_{n_k}(x)\, \phi_{n_k+1}(x)\right] \\ &= \left[\prod_{i=1}^{k-1} \phi_{n_i}(x)\right] F_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\left(x; \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\right), \end{aligned}$$

where $\boldsymbol{m}'_{[2]} = (m'_1, m'_2) = (n_k^*, m_{k+1})$, $\boldsymbol{n}'_{[2]} = (n'_1, n'_2) = (n_k, n_{k+1})$.

**Case 1.1**: $n_k \leq m_{k+1}$. Since $n_k^* = n_k + n_{k+1} - m_{k+1} \geq n_k, m_{k+1} \geq n_k$ and $n_k^* + m_{k+1} = n_k + n_{k+1}$, we have $(n_k^*, m_{k+1}) \prec (n_k, n_{k+1})$ if $n_k^* \leq m_{k+1}$ and $(m_{k+1}, n_k^*) \prec (n_k, n_{k+1})$ if $n_k^* > m_{k+1}$. Using Theorem 2 (i) and Remark 4 (ii) we see that $F_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\left(x; \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\right)$ is completely monotonic on $(-\rho, \infty)$. Moreover, the functions $\phi_{n_i}(x)$ for $i = 1, ..., k-1$ are clearly completely monotonic on $(-\rho, \infty)$. Then $S_2(x)$ is also completely monotonic on $(-\rho, \infty)$. It follows from the relation (16) that $F_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}\left(x; \lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho, \infty)$.

**Case 1.2**: $n_k > m_{k+1}$. This together with the relation (15) yields

$$n_1 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} < n_k \leq n_{k+1}. \tag{17}$$

To prove that $F_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}\left(x; \lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho, \infty)$ in this case, it suffices to prove that $F_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}\left(x; \lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho, \infty)$ under the relation (17), which is given by Step 2.

**Step 2**: Let $n_i^* = n_i$ for $i = 1, ..., k-2$, $n_{k-1}^* = n_{k-1} + n_k - m_{k+1}$ and $n_k^* = n_{k+1}$.

**Case 2.1**: $n_{k-1} \leq m_{k+1}$. Taking $\ell = 2$ in Lemma 12 gives that $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}_{[k]}^*$ and (13) holds. Hence, by the induction assumption $S_1(x)$ in the relation (16) is completely monotonic on $(-\rho, \infty)$. While $S_2(x)$ in (16) can be written as

$$\begin{aligned} S_2(x) &= \frac{\prod_{i=1}^{k+1} \phi_{n_i}(x)}{\phi_{n_{k-1}}(x)\, \phi_{n_k}(x)} \left[\phi_{n_{k-1}^*}(x)\, \phi_{m_{k+1}}(x) - \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}} \phi_{n_{k-1}}(x)\, \phi_{n_k}(x)\right] \\ &= \left[\phi_{n_{k+1}}(x) \prod_{i=1}^{k-2} \phi_{n_i}(x)\right] F_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\left(x; \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\right), \end{aligned}$$

where $\boldsymbol{m}'_{[2]} = (m'_1, m'_2) = \left(n_{k-1}^*, m_{k+1}\right)$, $\boldsymbol{n}'_{[2]} = (n'_1, n'_2) = (n_{k-1}, n_k)$.

Since $n_{k-1}^* = n_{k-1} + n_k - m_{k+1} > n_{k-1}$, $m_{k+1} \geq n_{k-1}$ and $n_{k-1}^* + m_{k+1} = n_{k-1} + n_k$, we have $\left(n_{k-1}^*, m_{k+1}\right) \prec (n_{k-1}, n_k)$ if $n_{k-1}^* \leq m_{k+1}$ and $\left(m_{k+1}, n_{k-1}^*\right) \prec (n_{k-1}, n_k)$ if $n_{k-1}^* > m_{k+1}$, which implies that $F_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\left(x; \alpha_{\boldsymbol{m}'_{[2]}, \boldsymbol{n}'_{[2]}}\right)$ is completely monotonic on $(-\rho, \infty)$. Moreover, the functions $\phi_{n_i}(x)$ for $i = 1, ..., k+1$ are clearly completely monotonic on $(-\rho,\infty)$, so is $S_2(x)$. Hence $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$.

**Case 2.2**: $n_{k-1}>m_{k+1}$. This together with the relation (17) yields

$$n_1\leq m_1\leq m_2\leq\cdots\leq m_k\leq m_{k+1}<n_{k-1}\leq n_k\leq n_{k+1}. \tag{18}$$

To prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ in this case, it suffices to prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ under the relation (18), which is given by Step 3.

**Step 3**: Let $n_i^*=n_i$ for $i=1,...,k-3$, $n_{k-2}^*=n_{k-2}+n_{k-1}-m_{k+1}$, $n_{k-1}^*=n_k$, and $n_k^*=n_{k+1}$.

**Case 3.1**: $n_{k-2}\leq m_{k+1}$. Taking $\ell=3$ in Lemma 12 gives that $\boldsymbol{m}_{[k]}\prec\boldsymbol{n}_{[k]}^*$ and (13) holds. By the induction assumption $S_1(x)$ in the relation (16) is completely monotonic on $(-\rho,\infty)$. While $S_2(x)$ in (16) can be written as

$$\begin{aligned} S_2(x) &= \frac{\prod_{i=1}^{k+1}\phi_{n_i}(x)}{\phi_{n_{k-2}}(x)\phi_{n_{k-1}}(x)}\left[\phi_{n_{k-2}^*}(x)\phi_{m_{k+1}}(x)-\alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\phi_{n_{k-2}}(x)\phi_{n_{k-1}}(x)\right] \\ &= \left[\phi_{n_k}(x)\phi_{n_{k+1}}(x)\prod_{i=1}^{k-3}\phi_{n_i}(x)\right]F_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\left(x;\alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\right), \end{aligned}$$

where $\boldsymbol{m}'_{[2]}=(m'_1,m'_2)=\left(n_{k-2}^*,m_{k+1}\right)$, $\boldsymbol{n}'_{[2]}=(n'_1,n'_2)=(n_{k-2},n_{k-1})$.

Since $n_{k-2}^*=n_{k-2}+n_{k-1}-m_{k+1}>n_{k-2}$, $m_{k+1}\geq n_{k-2}$ and $n_{k-2}^*+m_{k+1}=n_{k-2}+n_{k-1}$, we have $\left(n_{k-2}^*,m_{k+1}\right)\prec(n_{k-2},n_{k-1})$ if $n_{k-2}^*\leq m_{k+1}$ and $\left(m_{k+1},n_{k-2}^*\right)\prec(n_{k-2},n_{k-1})$ if $n_{k-2}^*>m_{k+1}$, which implies that the function $F_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\left(x;\alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\right)$ is completely monotonic on $(-\rho,\infty)$. This together with the facts that the functions $\phi_{n_i}(x)$ for $i=1,...,k+1$ are clearly completely monotonic on $(-\rho,\infty)$ means that $S_2(x)$ is completely monotonic on $(-\rho,\infty)$, and so is $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$.

**Case 3.2**: $n_{k-2}>m_{k+1}$. This together with the relation (18) yields

$$n_1\leq m_1\leq m_2\leq\cdots\leq m_k\leq m_{k+1}<n_{k-2}\leq n_{k-1}\leq n_k\leq n_{k+1}. \tag{19}$$

To prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ in this case, it suffices to prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ under the relation (19), which is given by Step 4.

Repeating such step $\ell-1$ times, it remains to prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ under the relation (12).

**Step $\ell$**: Let $n_i^*=n_i$ for $1\leq i\leq k-\ell$, $n_{k-\ell+\mathbf{1}}^*=n_{k-\ell+\mathbf{1}}+n_{k-\ell+\mathbf{2}}-m_{k+1}$, $n_i^*=n_{i+1}$ for $k-\ell+2\leq i\leq k$.

**Case** $\ell.1$: $n_{k-\ell+\mathbf{1}} \leq m_{k+1}$. Lemma 12 indicates that $\boldsymbol{m}_{[k]} \prec \boldsymbol{n}^*_{[k]}$ and (13) holds. By the induction assumption $S_1(x)$ in the relation (16) is completely monotonic on $(-\rho, \infty)$. While $S_2(x)$ in (16) can be written as

$$\begin{aligned} S_2(x) &= \frac{\prod_{i=1}^{k+1} \phi_{n_i}(x)}{\phi_{n_{k-\ell+1}}(x)\,\phi_{n_{k-\ell+2}}(x)} \\ &\quad \times \left[\phi_{n^*_{k-\ell+\mathbf{1}}}(x)\,\phi_{m_{k+1}}(x) - \alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\phi_{n_{k-\ell+1}}(x)\,\phi_{n_{k-\ell+2}}(x)\right] \\ &= \left[\prod_{i=1}^{k-\ell} \phi_{n_i}(x) \prod_{i=k-\ell+3}^{k+1} \phi_{n_i}(x)\right] F_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\left(x; \alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\right), \end{aligned}$$

where $\boldsymbol{m}'_{[2]} = (m'_1, m'_2) = \left(n^*_{k-\ell+1}, m_{k+1}\right)$, $\boldsymbol{n}'_{[2]} = (n'_1, n'_2) = (n_{k-\ell+1}, n_{k-\ell+2})$. Since $n^*_{k-\ell+\mathbf{1}} = n_{k-\ell+\mathbf{1}} + n_{k-\ell+\mathbf{2}} - m_{k+1} > n_{k-\ell+\mathbf{1}}$ and $m_{k+1} \geq n_{k-\ell+\mathbf{1}}$, we have $\left(n^*_{k-\ell+1}, m_{k+1}\right) \prec (n_{k-\ell+1}, n_{k-\ell+2})$ if $n^*_{k-\ell+1} \leq m_{k+1}$ and $\left(m_{k+1}, n^*_{k-\ell+1}\right) \prec (n_{k-\ell+1}, n_{k-\ell+2})$ if $n^*_{k-\ell+1} > m_{k+1}$, which implies that $F_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\left(x; \alpha_{\boldsymbol{m}'_{[2]},\boldsymbol{n}'_{[2]}}\right)$ is completely monotonic on $(-\rho, \infty)$. In view of that the functions $\phi_{n_i}(x)$ for $i = 1, ..., k+1$ are clearly completely monotonic on $(-\rho, \infty)$, so is $S_2(x)$, and so is $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x; \lambda^{[0]}_{k+1}\right)$.

**Case** $\ell.2$: $n_{k-\ell+\mathbf{1}} > m_{k+1}$. This together with the relation (12) yields

$$n_1 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} < n_{k-\ell+1} \leq n_{k-\ell+2} \leq \cdots \leq n_{k+1}. \tag{20}$$

To prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x; \lambda^{[0]}_{k+1}\right)$ is completely monotonic on $(-\rho, \infty)$ in this case, it suffices to prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x; \lambda^{[0]}_{k+1}\right)$ is completely monotonic on $(-\rho, \infty)$ under the relation (20), which is given by Step $\ell + 1$.

Repeating such step $k-1$ times, it remains to prove that $F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x; \lambda^{[0]}_{k+1}\right)$ is completely monotonic on $(-\rho, \infty)$ under the relation

$$n_1 \leq m_1 \leq \cdots \leq m_k \leq m_{k+1} < n_2 \leq n_3 \leq \cdots \leq n_k \leq n_{k+1}. \tag{21}$$

**Step** $k$**.** We write

$$\begin{aligned} &F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x; \lambda^{[0]}_{k+1}\right) = F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x; \alpha_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\right) \\ &= \left[\prod_{i=1}^{k+1} \phi_{m_i}(x) - \alpha_{\boldsymbol{m}^*_{[k]},\boldsymbol{n}^*_{[k]}}\phi_{m_1}(x) \prod_{i=1}^{k} \phi_{n^*_i}(x)\right] \\ &+\alpha_{\boldsymbol{m}^*_{[k]},\boldsymbol{n}^*_{[k]}}\left[\phi_{m_1}(x) \prod_{i=1}^{k} \phi_{n^*_i}(x) - \frac{\alpha_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}}{\alpha_{\boldsymbol{m}^*_{[k]},\boldsymbol{n}^*_{[k]}}} \prod_{i=1}^{k+1} \phi_{n_i}(x)\right] \\ &:= I_1(x) + \alpha_{\boldsymbol{m}^*_{[k]},\boldsymbol{n}^*_{[k]}} I_2(x), \end{aligned}$$

where $\boldsymbol{m}_k^* = (m_1^*, ..., m_k^*) = (m_2, m_3, ..., m_{k+1})$, $\boldsymbol{m}_2' = (m_1', m_2') = (m_1, n_1^*)$, $\boldsymbol{n}_2' = (n_1', n_2') = (n_1, n_2)$, $\boldsymbol{n}_k^* = (n_1^*, ..., n_k^*) = (n_1 + n_2 - m_1, n_3, ..., n_{k+1})$.

An easy verification shows that

$$\boldsymbol{m}_k^* = (m_2, m_3, ..., m_{k+1}) \prec (n_1 + n_2 - m_1, n_3, ..., n_{k+1}) = \boldsymbol{n}_k^*.$$

By the induction assumption $F_{\boldsymbol{m}_{[k]}^*, \boldsymbol{n}_{[k]}^*}\left(x; \alpha_{\boldsymbol{m}_{[k]}^*, \boldsymbol{n}_{[k]}^*}\right)$ is completely monotonic on $(-\rho, \infty)$.

Since $m_1 \geq n_1$ and $n_1^* = n_1 + n_2 - m_1 \geq n_1$, we have $(m_1, n_1^*) \prec (n_1, n_2)$ if $m_1 \leq n_1^*$ and $(n_1^*, m_1) \prec (n_1, n_2)$ if $m_1 > n_1^*$. Moreover, it is readily seen that

$$\frac{\alpha_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}}{\alpha_{\boldsymbol{m}_{[k]}^*, \boldsymbol{n}_{[k]}^*}} = \frac{\prod_{i=1}^{k+1} \left(\Gamma(m_i) / \Gamma(n_i)\right)}{\prod_{i=1}^{k} \left(\Gamma(m_i^*) / \Gamma(n_i^*)\right)} = \frac{\Gamma(n_1^*)\Gamma(m_1)}{\Gamma(n_1)\Gamma(n_2)} = \alpha_{\boldsymbol{m}_{[2]}', \boldsymbol{n}_{[2]}'}.$$

It follows that that $F_{\boldsymbol{m}_{[2]}', \boldsymbol{n}_{[2]}'}\left(x; \alpha_{\boldsymbol{m}_{[2]}', \boldsymbol{n}_{[2]}'}\right)$ is completely monotonic on $(-\rho, \infty)$.

Note that $I_1(x)$ and $I_2(x)$ can be written as

$$\begin{aligned} I_1(x) &= \phi_{m_1}(x) \left[\prod_{i=1}^{k} \phi_{m_i^*}(x) - \alpha_{\boldsymbol{m}_{[k]}^*, \boldsymbol{n}_{[k]}^*} \prod_{i=1}^{k} \phi_{n_i^*}(x)\right] \\ &= \phi_{m_1}(x) F_{\boldsymbol{m}_{[k]}^*, \boldsymbol{n}_{[k]}^*}\left(x; \alpha_{\boldsymbol{m}_{[k]}^*, \boldsymbol{n}_{[k]}^*}\right), \end{aligned}$$

$$\begin{aligned} I_2(x) &= \left[\prod_{i=3}^{k+1} \phi_{n_i}(x)\right] \left[\phi_{n_1^*}(x)\phi_{m_1}(x) - \alpha_{\boldsymbol{m}_{[2]}', \boldsymbol{n}_{[2]}'} \phi_{n_1}(x)\phi_{n_2}(x)\right] \\ &= \left[\prod_{i=3}^{k+1} \phi_{n_i}(x)\right] F_{\boldsymbol{m}_{[2]}', \boldsymbol{n}_{[2]}'}\left(x; \alpha_{\boldsymbol{m}_{[2]}', \boldsymbol{n}_{[2]}'}\right). \end{aligned}$$

By the induction assumption $I_1(x)$ is completely monotonic on $(-\rho, \infty)$.

Since $\phi_{m_1}(x)$ and $\phi_{n_i}(x)$ for $1 \leq i \leq k+1$ are completely monotonic on $(-\rho, \infty)$, we deduce that $F_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}\left(x; \lambda_{k+1}^{[0]}\right)$ represented by (16) is completely monotonic on $(-\rho, \infty)$.

Taking into account the above $k$ times steps, we find $F_{\boldsymbol{m}_{[k+1]}, \boldsymbol{n}_{[k+1]}}\left(x; \lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho, \infty)$. By induction $F_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}}\left(x; \lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho, \infty)$ for all $k \geq 2$, thereby completing the proof of the first statement.

(ii) For $|r - s| > 1$, the necessary condition for $-F_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}}(x; \lambda_k)$ is completely monotonic on $(-\rho, \infty)$ follows from the limit relation

$$\lambda_k \geq \lim_{x \to -\rho^+} \frac{\prod_{i=1}^{k} \phi_{m_i}(x)}{\prod_{i=1}^{k} \phi_{n_i}(x)} = \alpha_{\boldsymbol{m}_{[k]}, \boldsymbol{n}_{[k]}} = \lambda_k^{[0]},$$

where the limit relation holds due to (10).

Similarly, to prove that $-F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ is completely monotonic on $(-\rho,\infty)$ if $\lambda_k \geq \lambda_k^{[0]}$, it suffices to prove $-F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$. By Theorem 2 (ii) and Remark 4 (ii) we see that $-F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ for $k=2$. Suppose that $-F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ for certain $k\geq 2$. Clearly, multiplying by "$-1$" on each side in those equations containing letters "$F$", "$S$" and "$I$" in the proof of the first assertion, we find that $-F_{\boldsymbol{m}_{[k+1]},\boldsymbol{n}_{[k+1]}}\left(x;\lambda_{k+1}^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$. By induction, $-F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\lambda_k^{[0]}\right)$ is completely monotonic on $(-\rho,\infty)$ for all $k\geq 2$.

(iii) Using the same technic as the proof of part (ii) and noting that the identity (14), the third assertion follows. This completes the proof.

## 3.2 Proof of Corollary 6

(i) Taking $r\to s=0$ in Theorem 5 (i) gives the first statement.

(ii) If $-\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ is completely monotonic on $(0,\infty)$, then

$$-\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k) = -\prod_{i=1}^{k}\psi_{m_i}(x) + \lambda_k\prod_{i=1}^{k}\psi_{n_i}(x) \geq 0$$

for all $x>0$, and then,

$$\lambda_k \geq \lim_{x\to 0+}\frac{\prod_{i=1}^{k}\psi_{m_i}(x)}{\prod_{i=1}^{k}\psi_{n_i}(x)} = \prod_{i=1}^{k}\frac{m_i!}{n_i!} = \alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]},(\boldsymbol{n}+\boldsymbol{1})_{[k]}},$$

where the equality holds due to the limit relation (10) for $r\to s=0$, which proves the necessity.

Suppose that $\lambda_k \geq \alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]},(\boldsymbol{n}+\boldsymbol{1})_{[k]}}$. To prove the sufficiency, we note that

$$\begin{aligned}-\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k) &= -\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]},(\boldsymbol{n}+\boldsymbol{1})_{[k]}}\right)\\ &\quad+\left(\lambda_k-\alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]},(\boldsymbol{n}+\boldsymbol{1})_{[k]}}\right)\prod_{i=1}^{k}\psi_{n_i}(x).\end{aligned}$$

Taking $r\to s=0$ in Theorem 5 (iii), we find that $-\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}\left(x;\alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]},(\boldsymbol{n}+\boldsymbol{1})_{[k]}}\right)$ is completely monotonic on $(0,\infty)$; the functions $\psi_{n_i}(x)$ for $i=1,...,k$ are completely monotonic on $(0,\infty)$. Consequently, $-\mathcal{F}_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ is completely monotonic on $(0,\infty)$ if $\lambda_k\geq\alpha_{(\boldsymbol{m}+\boldsymbol{1})_{[k]},(\boldsymbol{n}+\boldsymbol{1})_{[k]}}$, which proves the sufficiency, and the proof is done.

### 3.3 Proof of Theorem 7

We only prove that the functions $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ is completely monotonic on $(-\rho,\infty)$ if and only if $\mu_k \le \beta_{\boldsymbol{n}_{[k]}}$ in the case of $0<|r-s|<1$, other statements in the cases of $0<|r-s|<(>)\,1$ can be proven in the same way.

For $0<|r-s|<1$, if the functions $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ is completely monotonic on $(-\rho,\infty)$ then $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)\ge 0$ for all $x>-\rho$, which implies

$$\mu_k \le \lim_{x\to\infty} \frac{\prod_{i=1}^k \phi_{n_i}(x)}{\phi_{s_{n_k}}(x)} = \frac{\prod_{i=1}^k \Gamma(n_i)}{\Gamma(s_{n_k})} = \beta_{\boldsymbol{n}_{[k]}}.$$

Suppose that $\mu_k \le \beta_{\boldsymbol{n}_{[k]}}$. Since

$$G_{\boldsymbol{n}_{[k]}}(x;\mu_k) = \prod_{i=1}^k \phi_{n_i}(x) - \beta_{\boldsymbol{n}_{[k]}}\phi_{s_{n_k}}(x) + \left(\beta_{\boldsymbol{n}_{[k]}}-\mu_k\right)\phi_{s_{n_k}}(x),$$

and $\phi_{s_{n_k}}(x)$ is completely monotonic on $(-\rho,\infty)$, to prove that $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ is completely monotonic on $(-\rho,\infty)$, it suffices to prove that $G_{\boldsymbol{n}_{[k]}}\left(x;\beta_{\boldsymbol{n}_{[k]}}\right)$ is completely monotonic on $(-\rho,\infty)$, which can be proven by induction. By Theorem 3 and Remark 4 (i), we see that $G_{\boldsymbol{n}_{[k]}}\left(x;\beta_{\boldsymbol{n}_{[k]}}\right)$ is completely monotonic on $(-\rho,\infty)$ for $k=2$. Assume that $G_{\boldsymbol{n}_{[k]}}\left(x;\beta_{\boldsymbol{n}_{[k]}}\right)$ is completely monotonic on $(-\rho,\infty)$ for certain $k\ge 2$. Note that

$$\begin{aligned} G_{\boldsymbol{n}_{[k+1]}}\left(x;\beta_{\boldsymbol{n}_{[k+1]}}\right) &= \left[\prod_{i=1}^{k+1}\phi_{n_i}(x) - \beta_{\boldsymbol{n}_{[k]}}\phi_{s_{n_k}}(x)\,\phi_{n_{k+1}}(x)\right] \\ &\quad + \left[\beta_{\boldsymbol{n}_{[k]}}\phi_{s_{n_k}}(x)\,\phi_{n_{k+1}}(x) - \beta_{\boldsymbol{n}_{[k+1]}}\phi_{s_{n_{k+1}}}(x)\right] \\ : &= P_1(x) + P_2(x), \end{aligned}$$

where $P_1$ and $P_2$ stand for the first and second functions in the above two square brackets, respectively. Since $\phi_{n_{k+1}}(x)$ is completely monotonic on $(-\rho,\infty)$ and $G_{\boldsymbol{n}_{[k]}}\left(x;\beta_{\boldsymbol{n}_{[k]}}\right)$ is so by the induction assumption, the function

$$P_1(x) = \phi_{n_{k+1}}(x)\,G_{\boldsymbol{n}_{[k]}}\left(x;\beta_{\boldsymbol{n}_{[k]}}\right)$$

is also completely monotonic on $(-\rho,\infty)$. While $P_2(x)$ can be written as

$$P_2(x) = \beta_{\boldsymbol{n}_{[k]}}\left[\phi_{s_{n_k}}(x)\,\phi_{n_{k+1}}(x) - \frac{\beta_{\boldsymbol{n}_{[k+1]}}}{\beta_{\boldsymbol{n}_{[k]}}}\phi_{s_{n_{k+1}}}(x)\right],$$

where

$$\begin{aligned} s_{n_k} + n_{k+1} &= \sum_{i=1}^k n_i + n_{k+1} = s_{n_{k+1}}, \\ \frac{\beta_{\boldsymbol{n}_{[k+1]}}}{\beta_{\boldsymbol{n}_{[k]}}} &= \frac{\prod_{i=1}^{k+1}(n_i-1)!}{\left(s_{n_{k+1}}-1\right)!} \Bigg/ \frac{\prod_{i=1}^{k}(n_i-1)!}{(s_{n_k}-1)!} = \frac{(s_{n_k}-1)!\,(n_{k+1}-1)!}{\left(s_{n_{k+1}}-1\right)!}. \end{aligned}$$

Using Theorem 3 and Remark 4 (i) again we deduce that $P_2(x)$ is completely monotonic on $(-\rho,\infty)$. Therefore, $G_{\boldsymbol{n}_{[k+1]}}\left(x;\beta_{\boldsymbol{n}_{[k+1]}}\right)$ is completely monotonic on $(-\rho,\infty)$, and by induction, it is completely monotonic on $(-\rho,\infty)$ for all $k\geq 2$. This by induction completes the proof.

## 4. COROLLARIES

In this section, we give some consequences of Theorems 5 and 7. Taking $\boldsymbol{m}_{[n]}=(m+n-1,...,m+n-1)$ and $\boldsymbol{n}_{[n]}=(n,...,n,mn)$ implies that $\boldsymbol{m}_{[n]}\prec\boldsymbol{n}_{[n]}$. By Theorem 5 we have

**Corollary 14.** *Let $m,n\in\mathbb{N}$, let the function $F_{m,n,\lambda_{m,n}}=\phi_{m+n-1}^{n}-\lambda_{m,n}\phi_n^{n-1}\phi_{mn}$ be defined on $(-\rho,\infty)$, where $\phi_n$ is defined on $(-\rho,\infty)$ by (1) and $\rho=\min\{r,s\}$. Then the following statements hold:*

*(i) If $0<|r-s|<1$, then the function $F_{m,n,\lambda_{m,n}}$ is completely monotonic on $(-\rho,\infty)$ if and only if*

$$\lambda_{m,n}\leq\frac{\Gamma^n(m+n-1)}{\Gamma^{n-1}(n)\,\Gamma(mn)}=c_{m,n}.$$

*In particular, letting $r=s\to 0$, the function $\psi_{m+n-1}^{n}-c_{m,n}\psi_n^{n-1}\psi_{mn}$ is completely monotonic on $(0,\infty)$.*

*(ii) If $|r-s|>1$, then the function $F_{m,n,\lambda_{m,n}}$ is completely monotonic on $(-\rho,\infty)$ if and only if $\lambda_{m,n}\geq c_{m,n}$.*

*(iii) If $|r-s|>0$, then the function $-F_{m,n,c_{m+1,n+1}}$ is completely monotonic on $(-\rho,\infty)$. In particular, letting $r=s\to 0$, the function $-\psi_{m+n-1}^{n}+c_{m+1,n+1}\psi_n^{n-1}\psi_{mn}$ is completely monotonic on $(0,\infty)$.*

Taking $n_i=n$ for $i=1,...,k$ in Theorem 7 we have

**Corollary 15.** *Let $n,k\in\mathbb{N}$ with $k\geq 2$ and let $\phi_n$ be defined on $(-\rho,\infty)$ by (1), where $\rho=\min\{r,s\}$. The following statements hold:*

*(i) For $0<|r-s|<1$, the functions $\phi_n^k-\mu_k\phi_{kn}$ and its negativity are completely monotonic on $(-\rho,\infty)$ if and only if*

$$\mu_k\leq\frac{\Gamma^k(n)}{\Gamma(kn)}\quad\text{and}\quad\mu_k\geq|r-s|^{1-k}\,\frac{\Gamma^k(n)}{\Gamma(kn)},$$

*respectively. In particular, the function $\psi_n^k-\mu_k\psi_{kn}$ is completely monotonic on $(0,\infty)$ if and only if $\mu_k\leq\Gamma^k(n)/\Gamma(kn)$.*

*(ii) For $|r-s|>1$, the function $\phi_n^k-\mu_k\phi_{kn}$ and its negativity are completely monotonic on $(-\rho,\infty)$ if and only if*

$$\mu_k\leq|r-s|^{1-k}\,\frac{\Gamma^k(n)}{\Gamma(kn)}\quad\text{and}\quad\mu_k\geq\frac{\Gamma^k(n)}{\Gamma(kn)},$$

*respectively.*

**Remark 16.** *Taking $k = 2$ in the above proposition, we see that the function $x \mapsto \psi_n^2(x) - \mu_2\psi_{2n}(x)$ is completely monotonic on $(0,\infty)$ if and only if $\mu_2 \le \Gamma^2(n)/\Gamma(2n)$, which was proven in [**21**, Theorem 3.2] by Qi.*

**Corollary 17.** *Let $m, n \in \mathbb{N}$ and let $\phi_n$ be defined on $(-\rho,\infty)$ by (1), where $\rho = \min\{r, s\}$. The following statements are valid:*

*(i) For $0 < |r-s| < 1$, the functions*

$$\begin{aligned} f^{[1]}_{m,n,\lambda_{31}} &= \phi_n\phi_{m+1}\phi_{m+n} - \lambda_{31}\phi_n\phi_m\phi_{m+n+1}, \\ f^{[2]}_{m,n,\lambda_{32}} &= \phi_{n+1}\phi_m\phi_{m+n} - \lambda_{32}\phi_n\phi_m\phi_{m+n+1} \end{aligned}$$

*are both completely monotonic on $(-\rho,\infty)$ if and only if $\lambda_{31} \le m/(m+n)$ and $\lambda_{32} \le n/(m+n)$.*

*(ii) For $|r-s| > 1$, the functions $-f^{[1]}_{m,n,\lambda_{31}}$ and $-f^{[2]}_{m,n,\lambda_{32}}$ are both completely monotonic on $(-\rho,\infty)$ if and only if $\lambda_{31} \ge m/(m+n)$ and $\lambda_{32} \ge n/(m+n)$.*

*(iii) For $|r-s| > 0$, the functions $-f^{[1]}_{m,n,\lambda_{310}}$ and $-f^{[2]}_{m,n,\lambda_{320}}$ are completely monotonic on $(-\rho,\infty)$, where*

$$\lambda_{310} = \frac{m+1}{m+n+1} \quad \textit{and} \quad \lambda_{320} = \frac{n+1}{m+n+1}.$$

*Proof.* It is easy to check that

$$(m+1, m+n) \prec (m, m+n+1) \text{ and } (n+1, m+n) \prec (n, m+n+1).$$

(i) For $0 < |r-s| < 1$, the complete monotonicity of $\phi_n$ and $\phi_m$, and Theorem 5 (i) imply that the functions

$$\begin{aligned} f^{[1]}_{m,n,\lambda_{31}} &= \phi_n\left(\phi_{m+1}\phi_{m+n} - \lambda_{31}\phi_m\phi_{m+n+1}\right) \\ f^{[2]}_{m,n,\lambda_{32}} &= \phi_m\left(\phi_{n+1}\phi_{m+n} - \lambda_{32}\phi_n\phi_{m+n+1}\right) \end{aligned}$$

are both completely monotonic on $(-\rho,\infty)$ if and only if

$$\begin{aligned} \lambda_{31} &\le \frac{m!\,(m+n-1)!}{(m-1)!\,(m+n)!} = \frac{m}{m+n}, \\ \lambda_{32} &\le \frac{n!\,(m+n-1)!}{(n-1)!\,(m+n)!} = \frac{n}{m+n}. \end{aligned}$$

(ii) Similarly, for $|r-s| > 1$, by Theorem 5 (ii) the functions $-f^{[1]}_{m,n,\lambda_{31}}$ and $-f^{[2]}_{m,n,\lambda_{32}}$ are both completely monotonic on $(-\rho,\infty)$ if and only if $\lambda_{31} \ge m/(m+n)$ and $\lambda_{32} \ge n/(m+n)$.

(iii) A direct computation yields

$$\begin{aligned} \lambda_{310} &= \frac{(m+1)!\,(m+n)!}{m!\,(m+n+1)!} = \frac{m+1}{m+n+1}, \\ \lambda_{320} &= \frac{(n+1)!\,(m+n)!}{n!\,(m+n+1)!} = \frac{n+1}{m+n+1}. \end{aligned}$$

For $|r-s|>0$, by Theorem 5 (iii) the functions $-f^{[1]}_{m,n,\lambda_{310}}$ and $-f^{[2]}_{m,n,\lambda_{320}}$ are both completely monotonic on $(-\rho,\infty)$. This completes the proof. $\square$

Note that

$$\begin{aligned} & f^{[1]}_{m,n,\lambda_{31}}+f^{[2]}_{m,n,\lambda_{32}} \\ = \quad & \phi_n\phi_{m+1}\phi_{m+n}+\phi_{n+1}\phi_m\phi_{m+n}-(\lambda_{31}+\lambda_{32})\,\phi_n\phi_m\phi_{m+n+1} \\ = \quad & -\phi_{m+n}\left(\phi_n\phi_m\right)'-(\lambda_{31}+\lambda_{32})\,\phi_n\phi_m\phi_{m+n+1}. \end{aligned}$$

From Corollary 17 we get immediately the following corollary.

**Corollary 18.** *Let $m,n\in\mathbb{N}$ and let $\phi_n$ be defined on $(-\rho,\infty)$ by (1), where $\rho=\min\{r,s\}$. The following statements hold:*

*(i) For $0<|r-s|<1$, the function*

$$f_{m,n,\lambda_3}=-\phi_{m+n}\left(\phi_n\phi_m\right)'-\lambda_3\phi_{m+n+1}\phi_n\phi_m$$

*is completely monotonic on $(-\rho,\infty)$ if and only if $\lambda_3\le 1$. In particular, the function $-\psi_{m+n}\left(\psi_n\psi_m\right)'-\psi_{m+n+1}\psi_n\psi_m$ is completely monotonic on $(0,\infty)$.*

*(ii) For $|r-s|>1$, the function $f_{m,n,\lambda_3}$ is completely monotonic on $(-\rho,\infty)$ if and only if $\lambda_3\ge 1$.*

*(iii) For $|r-s|>0$, the functions $-f_{m,n,\lambda_{30}}$ is completely monotonic on $(0,\infty)$, where $\lambda_{30}=(m+n+2)\,/\,(m+n+1)$. In particular, the function*

$$\psi_{m+n}\left(\psi_n\psi_m\right)'+\frac{m+n+2}{m+n+1}\psi_{m+n+1}\psi_n\psi_m$$

*is completely monotonic on $(0,\infty)$.*

**Remark 19.** *Qi [**22**, Remark 17] guessed that, for $m,n\in\mathbb{N}$ the function*

$$\psi^{(m+n)}\left(\psi^{(m)}\psi^{(n)}\right)'-\psi^{(m+n+1)}\psi^{(m)}\psi^{(n)}$$

*should be completely monotonic on $(0,\infty)$. Since $\psi_n=(-1)^{n-1}\psi^{(n)}$, we have*

$$-\psi_{m+n}\left(\psi_n\psi_m\right)'-\psi_n\psi_m\psi_{m+n+1}=-\psi^{(m+n)}\left(\psi^{(m)}\psi^{(n)}\right)'-\psi^{(m+n+1)}\psi^{(m)}\psi^{(n)},$$

*which is completely monotonic on $(0,\infty)$ by Corollary 18 (i). Similarly, by Corollary 18 (iii), the function*

$$\begin{aligned} & \psi_{m+n}\left(\psi_n\psi_m\right)'+\frac{m+n+2}{m+n+1}\psi_{m+n+1}\psi_n\psi_m \\ = \quad & \psi^{(m+n)}\left(\psi^{(m)}\psi^{(n)}\right)'+\frac{m+n+2}{m+n+1}\psi^{(m+n+1)}\psi^{(m)}\psi^{(n)} \end{aligned}$$

*is completely monotonic on $(0,\infty)$.*

## 5. CONCLUSIONS

In this paper, we found the necessary and sufficient conditions for the functions $F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ and $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$, defined by (5) and (6), respectively, to be completely monotonic on $(-\min\{r,s\},\infty)$, which generalize Gao's results in [**11**]. In particular, the functions $\pm F_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}(x;\lambda_k)$ are completely monotonic in $x$ on $(0,\infty)$ if and only if $\lambda_k \leq \alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ $(\lambda_k \geq \alpha_{(\boldsymbol{m}+\mathbf{1})_{[k]},(\boldsymbol{n}+\mathbf{1})_{[k]}})$, and the function $G_{\boldsymbol{n}_{[k]}}(x;\mu_k)$ is completely monotonic in $x$ on $(0,\infty)$ if and only if $\mu_k \leq \beta_{\boldsymbol{n}_{[k]}}$, where $\alpha_{\boldsymbol{m}_{[k]},\boldsymbol{n}_{[k]}}$ and $\beta_{\boldsymbol{n}_{[k]}}$ are given in (4). This offers an answer to Qi's problem in [**22**].

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