Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **17** (2023), 159–178.
https://doi.org/10.2298/AADM220414024H

ASYMPTOTIC EXPANSIONS FOR THE WALLIS SEQUENCE AND SOME NEW MATHEMATICAL CONSTANTS ASSOCIATED WITH THE GLAISHER-KINKELIN AND CHOI-SRIVASTAVA CONSTANTS

Xue-Feng Han, Chao-Ping Chen and H. M. Srivastava*

The celebrated Wallis sequence W_n, which is defined by $W_n := \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$, is known to have the limit $\frac{\pi}{2}$ as $n \to \infty$. Without using the Bernoulli numbers B_n, the authors present several asymptotic expansions and a recurrence relation for determining the coefficients of each asymptotic expansion related to the Wallis sequence W_n and the newly-introduced constants D and E, which are analogous to the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C.

1. INTRODUCTION, MOTIVATION AND PRELIMINARIES

The famous Wallis sequence W_n, defined by

$$W_n := \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \qquad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}),$$

*Corresponding author. Chao-Ping Chen
2020 Mathematics Subject Classification. 11M06, 41A60, 40A05
Keywords and Phrases. Wallis' sequence, Asymptotic expansion, Glaisher-Kinkelin constant, Generalized Glaisher-Kinkelin constant, Barnes G-function, Choi-Srivastava constants, Stirling's formula, Riemann zeta function, Euler-Maclaurin summation formula.

has the limit value

$$W_\infty := \lim_{n\to\infty} W_n = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \frac{\pi}{2},$$

which was established by Wallis in 1655 (see also [**12**, p. 68]). Several elementary proofs of this well-known result can be found in, for example, [**3, 48, 69**].

An interesting geometric construction that produces the above limit value can be found in the work of Myerson [**56**]. Many formulas exist for the representation of π, and a collection of these formulas is listed in [**63, 64**]. For a history of π, see [**2, 10, 12, 33**].

Some inequalities and asymptotic formulas associated with the Wallis sequence W_n can be found in [**13, 21, 32, 35, 40, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 58**]. For example, Deng *et al.* [**32**] proved, for all $n \in \mathbb{N}$, that

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\alpha}\right) < W_n \leqq \frac{\pi}{2}\left(1-\frac{1}{4n+\beta}\right)$$

with the best possible constants α and β given by

$$\alpha = \frac{5}{2} \quad \text{and} \quad \beta = \frac{32-9\pi}{3\pi-8} = 2.614909986\cdots,$$

respectively.

Chen and Paris [**21**] showed that the following asymptotic expansion holds true for the Wallis sequence W_n:

$$\begin{aligned} W_n &\sim \frac{\pi}{2}\exp\left(\sum_{j=1}^{\infty}\frac{\nu_j}{n^j}\right) \\ &= \frac{\pi}{2}\exp\left(-\frac{1}{4n}+\frac{1}{8n^2}-\frac{5}{96n^3}+\frac{1}{64n^4}-\frac{1}{320n^5}+\frac{1}{384n^6}-\frac{25}{7168n^7}+\cdots\right), \end{aligned} \tag{1}$$

with the coefficients ν_j given by

$$\nu_j = \frac{(-1)^{j+1}\Big((4-2^{1-j})B_{j+1}-(j+1)\cdot 2^{-j}\Big)}{j(j+1)} \qquad (j \geqq 1), \tag{2}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ denote the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z|<2\pi).$$

By applying Lemma 3 in [**19**], Chen and Paris [**21**] deduced the following asymptotic expansion from (1):

$$W_n \sim \frac{\pi}{2}\sum_{j=0}^{\infty}\frac{\mu_j}{n^j}, \tag{3}$$

where the coefficients μ_j are given by the recurrence relation:

$$\mu_0 = 1, \quad \text{and} \quad \mu_j = \frac{1}{j}\sum_{k=1}^{j} k\, \nu_k\, \mu_{j-k} \qquad (j \geqq 1)$$

and the coefficients ν_j are given in (2). This produces the expansion in the inverse powers of n given by

$$W_n \sim \frac{\pi}{2}\left(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \frac{625}{65536n^6} - \frac{1843}{262144n^7} + \cdots\right) \qquad (n \to \infty).$$

The main object of the present paper is to first provide a recurrence relation for determining the coefficients of n^{-j} in the expansion (1) without the help of the Bernoulli numbers B_n. We also derive a recurrence relation for determining the coefficients of n^{-j} in the expansion (3) without using the coefficients ν_j.

The double gamma function Γ_2 and the multiple gamma functions Γ_n were introduced and investigated by Barnes in a series of papers [**4, 5, 6, 7**]. In fact, Barnes applied these functions in the theories of elliptic functions and theta functions. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [**70**, p. 264] and also by Gradshteyn and Ryzhik [**38**, p. 661, Entry 6.441 (4); p. 937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of the determinants of the Laplacians on the n-dimensional unit sphere S^n (see, for example, [**26, 43, 57, 61, 67, 68**]). The theory of the double gamma function Γ_2 has indeed found interesting applications in many other recent investigations (see, for details, [**65, 66**]).

Barnes [**4**] defined the double gamma function (or the Barnes G-function) $\Gamma_2 = 1/G$, which satisfies each of the following properties:

(i) $G(z+1) = \Gamma(z)G(z)$ for all complex z;

(ii) $G(1) = 1$;

(iii) As $n \to \infty$,

$$\ln G(z+n+2) = \frac{n+1+z}{2}\ln(2\pi) + \left(\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right)\ln n - \frac{3n^2}{4} - n - nz - \ln A + \frac{1}{12} + O(n^{-1}),$$

where Γ is the familiar (Euler's) gamma function and A is called the Glaisher-Kinkelin constant defined by

$$\ln A = \lim_{n\to\infty}\left\{\ln\left(\prod_{k=1}^{n} k^k\right) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln n + \frac{n^2}{4}\right\}, \tag{4}$$

the numerical value of A being $1.28242713\cdots$.

The Glaisher-Kinkelin constant A can be expressed as follows (see [**36**]):

$$A=\lim_{n\to\infty} n^{-n^2/2-n/2-1/12}e^{n^2/4}\prod_{k=1}^{n}k^k,$$

$$\frac{e^{1/12}}{A}=\lim_{n\to\infty}\frac{G(n+1)}{n^{n^2/2-1/12}(2\pi)^{n/2}e^{-3n^2/4}},$$

and (see [**25**, p. 129, Eq. (3.22)])

$$A=e^{\frac{1}{12}-\zeta'(-1)}=(2\pi)^{1/12}\left[e^{\gamma\pi^2/6-\zeta'(2)}\right]^{1/(2\pi^2)}, \tag{5}$$

where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [**30**]).

The Glaisher-Kinkelin constant A has drawn attention in many works (see, for example, [**16, 20, 25, 29, 30**]; see also [**4**]). One section in the book by Finch [**37**, pp. 135–138] was devoted to introduce this Glaisher-Kinkelin constant A. Moreover, the Glaisher-Kinkelin constant A plays an important role in the study of the Barnes G-function (see, for details, [**66**, Section 1.4]).

The following integral representation for the remainder $R_N(z)$ of the explicit expression for the Barnes G-function was established by Ferreira and López [**36**, Theorem 1].

Theorem 1. *An integral representation for the remainder $R_N(z)$ in the following explicit expression for the Barnes G-function*:

$$\ln G(z+1)=\frac{1}{4}z^2+z\ln\Gamma(z+1)-\left(\frac{1}{2}\,z^2+\frac{1}{2}\,z+\frac{1}{12}\right)\ln z-\ln A$$
$$+\sum_{k=1}^{N-1}\frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}}+R_N(z)\quad (N\in\mathbb{N};\ |\mathrm{Arg}(z)|<\pi)$$

where B_{2k+2} are the Bernoulli numbers, is given for $\Re(z)>0$ by

$$R_N(z)=\int_0^\infty\left(\frac{t}{e^t-1}-\sum_{k=0}^{2N}\frac{B_k}{k!}\,t^k\right)\,\frac{e^{-zt}}{t^3}\,\mathrm{d}t\qquad \big(\Re(z)>0\big).$$

Estimates for $|R_N(z)|$ were also found by Ferreira and López [**36**], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in the sectors of the complex plane cut along the negative real axis. Pedersen [**59**, Theorem 1.1] proved that, for any $N\geqq 1$, the function $x\mapsto(-1)^N R_N(x)$ is completely monotonic on $(0,\infty)$. Other asymptotic relations (avoiding the $\ln\Gamma$ term) was given by Ruijsenaars [**62**] and investigated by Pedersen [**60**], Koumandos [**41**], and by Koumandos and Pedersen [**42**]. Some upper and lower bounds for the double gamma function were derived in terms of the gamma, psi and polygamma functions (see [**8, 9, 14, 22**]). Chen [**15**] and Mortici [**54**] established several inequalities and asymptotic expansions for $\ln A$ in (4). Chen and Lin [**20**] and Chen [**16**] presented a class of asymptotic expansions related to the Glaisher-Kinkelin constant A and the Barnes G-function. Recently, Chen and Srivastava [**23**] presented a number of potentially useful properties of the Barnes G-function. The properties considered here include, for example, its integral representation, complete monotonicity, and continued-fraction approximation. We also derive continued-fraction approximations of the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C, which are analogous to the Glaisher-Kinkelin constant A and are given by (see [**29**, p.102] and [**30**])

$$\ln B = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \right\}$$

and

$$\ln C = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \right\}$$

for which the approximate numerical values are given by

$$B = 1.03091675\cdots \quad \text{and} \quad C = 0.97955746\cdots .$$

As $x \to \infty$, the Stirling formula for the Barnes G-function can be found as follows (see [**65**, p. 26]):

$$\ln G(x+1) = \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12} \right) \ln x + O(x^{-1}). \tag{6}$$

Chen [**17**] applied the formula (6) to produce the following complete asymptotic expansion:

$$\begin{aligned} \ln G(x+1) \sim \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12} \right) \ln x \\ - \frac{1}{240x^2} + \frac{1}{1008x^4} - \frac{1}{1440x^6} + \frac{1}{1056x^8} - \frac{691}{327600x^{10}} + \cdots \end{aligned} \tag{7}$$

and derived a recurrence relation for determining the coefficients of $1/x^j$ $(j \geqq 2)$ occurring in the expansion (7).

Just as the expression of the Glaisher-Kinkelin constant A in (5), the Choi-Srivastava constants B and C are also known to be expressible in terms of special values of the derivative of the Riemann zeta function $\zeta(s)$ as follows (see [**30**] and [**31**, Eq. (1.9)]):

$$\ln B = -\zeta'(-2) \qquad \text{and} \qquad \ln C = -\frac{11}{720} - \zeta'(-3). \tag{8}$$

Chen [**15**] established the asymptotic expansions related to the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C. Mortici [**54**] also dealt with the same problem. Cheng and Chen [**24**] and Chen and Choi [**18**] established some novel asymptotic expansions of the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C. Also, by using the Bernoulli numbers B_n, Chen [**15**] established the asymptotic expansions related to the constants A, B and C.

Recently, Chen [**17**] derived a recurrence relation for determining the coefficients of each asymptotic expansion related to the constants A, B and C, without using the Bernoulli numbers B_n. More precisely, Chen [**17**] proved the following results:

$$\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4} \sim \ln A + \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \cdots,$$

$$\sum_{k=1}^{n} k^2\ln k - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12} \sim \ln B - \frac{1}{360n} + \frac{1}{7560n^3} - \frac{1}{25200n^5} + \frac{1}{33264n^7} - \frac{691}{16216200n^9} + \cdots$$

and

$$\sum_{k=1}^{n} k^3\ln k - \left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\ln n + \frac{n^4}{16} - \frac{n^2}{12} \sim \ln C - \frac{1}{5040n^2} + \frac{1}{33600n^4} - \frac{1}{66528n^6} + \frac{691}{43243200n^8} - \frac{1}{34320n^{10}} + \cdots,$$

as $n \to \infty$ in each case.

In our present investigation, we introduce two new mathematical constants D and E, which are analogous to the constants A, B and C, defined by

$$\ln D = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^4\ln k - \left(\frac{1}{5}\,n^5+\frac{1}{2}\,n^4+\frac{1}{3}\,n^3-\frac{1}{30}\,n\right)\ln n + \frac{1}{25}\,n^5 - \frac{1}{12}\,n^3 + \frac{13}{360}\,n\right\} \tag{9}$$

and

$$\ln E = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^5\ln k - \left(\frac{1}{6}\,n^6+\frac{1}{2}\,n^5+\frac{5}{12}\,n^4-\frac{1}{12}\,n^2+\frac{1}{252}\right)\ln n + \frac{1}{36}\,n^6 - \frac{1}{12}\,n^4 + \frac{47}{720}\,n^2\right\}, \tag{10}$$

respectively, approximate numerical values of D and E being given by

$$D = 0.99204797\cdots \quad \text{and} \quad E = 1.00968038\cdots .$$

We also derive a recurrence relation for determining the coefficients of each of the asymptotic expansions, which are related to the constants D and E, without using the Bernoulli numbers B_n.

Remark. We thank a referee for drawing our attention toward the related developments in [**1, 11, 27**]. The mathematical constants D and E, which we have introduced and studied in this work, were studied in a generalized form which includes the Glaisher-Kinkelin constant A as well as the Choi-srivastava constants B and C (see, for details, [**1, 11**]; see also [**25**, p. 131, Eq. (4.10)] for the corrected form of [**1**, p. 198, Eq. (20)] as well as [**27**]).

2. ASYMPTOTIC EXPANSIONS FOR THE WALLIS SEQUENCE

Theorem 2 below provides a recurrence relation for determining the coefficients of n^{-j} in the expansion (1), without the help of the Bernoulli numbers B_n.

Theorem 2. *As* $n \to \infty$, *the following asymptotic expansion holds true*:

$$W_n \sim \frac{\pi}{2} \exp\left(\sum_{k=1}^{\infty} \frac{\nu_k}{n^k}\right),$$

where the coefficients ν_j *are given by the recurrence relation given by*

(11)
$$\nu_1 = -\frac{1}{4} \quad \textit{and} \quad \nu_k = \frac{(-1)^k}{k}\left\{\frac{3^{k+1} - 2^{k+2} + 1}{2^{k+1}(k+1)} - \sum_{j=1}^{k-1} \nu_j (-1)^j \binom{k}{k-j+1}\right\}$$

for $k \geqq 2$.

Proof. Upon setting

$$P_n = \ln\left(\frac{2}{\pi} W_n\right) \quad \text{and} \quad Q_n = \sum_{k=1}^{\infty} \frac{\nu_k}{n^k},$$

we can let $P_n \sim Q_n$ and

$$\Delta P_n := P_{n+1} - P_n \sim Q_{n+1} - Q_n =: \Delta Q_n$$

as $n \to \infty$, where ν_k are real numbers to be determined.

By making use of the fact that

$$W_n = \frac{\pi}{2} \cdot \frac{1}{n+\frac{1}{2}} \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \right)^2 = \frac{\pi}{2} \cdot \frac{\Gamma(n+1)^2}{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})},$$

we obtain

$$\begin{aligned} \Delta P_n &= 2\ln\left(1+\frac{1}{n}\right) - \ln\left(1+\frac{1}{2n}\right) - \ln\left(1+\frac{3}{2n}\right) \\ &= \sum_{k=2}^{\infty} \frac{(-1)^k(3^k - 2^{k+1}+1)}{2^k k}\, n^{-k}. \end{aligned} \tag{12}$$

Also, by direct computation, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\nu_k}{(n+1)^k} &= \sum_{k=1}^{\infty} \frac{\nu_k}{n^k}\left(1+\frac{1}{n}\right)^{-k} = \sum_{k=1}^{\infty} \frac{\nu_k}{n^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{n^j} \\ &= \sum_{k=1}^{\infty} \frac{\nu_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{n^j} \\ &= \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} \nu_j (-1)^{k-j} \binom{k-1}{k-j} \right\} n^{-k}. \end{aligned} \tag{13}$$

We thus find that

$$\Delta Q_n = \sum_{k=2}^{\infty} \left\{ \sum_{j=1}^{k} \nu_j (-1)^{k-j} \binom{k-1}{k-j} - \nu_k \right\} n^{-k}. \tag{14}$$

Now, upon equating the coefficients of n^{-k} on the right-hand sides of (12) and (14), we have

$$\begin{aligned} \frac{(-1)^k(3^k - 2^{k+1}+1)}{2^k k} &= \sum_{j=1}^{k} \nu_j (-1)^{k-j} \binom{k-1}{k-j} - \nu_k \\ &= \sum_{j=1}^{k-1} \nu_j (-1)^{k-j} \binom{k-1}{k-j} \end{aligned}$$

and

$$\begin{aligned} \frac{3^k - 2^{k+1}+1}{2^k k} &= \sum_{j=1}^{k-1} \nu_j (-1)^{j} \binom{k-1}{k-j} \\ &= \sum_{j=1}^{k-2} \nu_j (-1)^{j} \binom{k-1}{k-j} + (-1)^{k-1}(k-1)\nu_{k-1} \qquad (k \geqq 2), \end{aligned}$$

where (and elsewhere in this paper) an empty sum is understood to be zero.

For $k = 2$, we obtain $\nu_1 = -\frac{1}{4}$. Also, for $k \geqq 3$, we have

$$\nu_{k-1} = \frac{(-1)^{k-1}}{k-1} \left\{ \frac{3^k - 2^{k+1} + 1}{2^k k} - \sum_{j=1}^{k-2} \nu_j (-1)^j \binom{k-1}{k-j} \right\},$$

which can be written precisely as (11). The proof of Theorem 2 is thus completed. □

Our next result (Theorem 3) provides a recurrence relation for determining the coefficients of n^{-j} in the expansion (3) without the coefficients ν_j.

Theorem 3. *As $n \to \infty$, the following asymptotic expansion holds true*:

$$W_n \sim \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\mu_k}{n^k},$$

where the coefficients μ_k are given by the recurrence relation:

(15)

$$\mu_0 = 1, \quad \mu_1 = -\frac{1}{4}, \quad \textit{and} \quad \mu_k = -\frac{1}{k} \left\{ \frac{2k^2 - 2k + 1}{4} \mu_{k-1} + \sum_{j=0}^{k-2} \mu_j \binom{k}{k-j+1} \right\}$$

for $k \geqq 2$.

Proof. We first set

$$U_n = \frac{2}{\pi} W_n \quad \text{and} \quad V_n = \sum_{k=0}^{\infty} \frac{\mu_k}{n^k},$$

where $\mu_0 = 1$. We can then let $U_n \sim V_n$ and

$$\frac{U_n}{U_{n-1}} \sim \frac{V_n}{V_{n-1}} \qquad (n \to \infty),$$

where μ_k are real numbers to be determined as follows:

$$\frac{4n^2}{4n^2 - 1} \sim \frac{\sum_{k=0}^{\infty} \frac{\mu_k}{n^k}}{\sum_{k=0}^{\infty} \frac{\mu_k}{(n-1)^k}},$$

which yields

$$\sum_{k=0}^{\infty} \frac{\mu_k}{(n-1)^k} \sim \left(1 - \frac{1}{4n^2}\right) \sum_{k=0}^{\infty} \frac{\mu_k}{n^k} = \mu_0 + \frac{\mu_1}{n} + \sum_{k=2}^{\infty} \left(\mu_k - \frac{\mu_{k-2}}{4}\right) \frac{1}{n^k}. \tag{16}$$

Now, by direct computation, we get

$$\sum_{k=0}^{\infty} \frac{\mu_k}{(n-1)^k} = \sum_{k=0}^{\infty} \frac{\mu_k}{n^k}\left(1-\frac{1}{n}\right)^{-k} = \sum_{k=0}^{\infty} \frac{\mu_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{-k}{j} \frac{1}{n^j}$$

$$= \sum_{k=0}^{\infty} \frac{\mu_k}{n^k} \sum_{j=0}^{\infty} \binom{k+j-1}{j} \frac{1}{n^j} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \mu_j \binom{k-1}{k-j} \frac{1}{n^k}$$

(17)
$$= \mu_0 + \frac{\mu_1}{n} + \sum_{k=2}^{\infty}\sum_{j=0}^{k} \mu_j \binom{k-1}{k-j} \frac{1}{n^k}.$$

Thus, upon equating the coefficients of n^{-k} on the right-hand sides of (16) and (17), we find that

$$\mu_k - \frac{\mu_{k-2}}{4} = \sum_{j=0}^{k} \mu_j \binom{k-1}{k-j} \qquad (k \geqq 2)$$

and

$$-\frac{\mu_{k-2}}{4} = \sum_{j=0}^{k-1} \mu_j \binom{k-1}{k-j} = \sum_{j=0}^{k-2} \mu_j \binom{k-1}{k-j} + (k-1)\mu_{k-1}$$

$$= \sum_{j=0}^{k-3} \mu_j \binom{k-1}{k-j} + \frac{(k-1)(k-2)}{2}\mu_{k-2} + (k-1)\mu_{k-1}.$$

For $k=2$, we obtain $\mu_1 = -\frac{1}{4}$. Also, for $k \geqq 3$. we have

$$\mu_{k-1} = -\frac{1}{k-1}\left\{\left(\frac{1}{4}+\frac{(k-1)(k-2)}{2}\right)\mu_{k-2} + \sum_{j=0}^{k-3} \mu_j \binom{k-1}{k-j}\right\},$$

which can be written precisely as (15). This completes our proof of Theorem 3. □

3. ASYMPTOTIC EXPANSIONS RELATED TO THE CONSTANTS D AND E

In this section, we first recall the Euler-Maclaurin summation formula as follows (see, for example, [**39**, p. 318]; see also [**34**]):

(18)
$$\sum_{k=1}^{n} f(k) \sim C_0 + \int_a^n f(x)\,\mathrm{d}x + \frac{1}{2}\,f(n) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}\,f^{(2r-1)}(n),$$

where C_0 is an arbitrary constant to be determined in each special case and the B_{2k} are the Bernoulli numbers. Indeed, if we set

$$f(x) = x^4 \ln x \qquad \text{and} \qquad f(x) = x^5 \ln x$$

in (18) with $a = 1$, we are led to (9) and (10), respectively.

As in the cases of $\ln A$ in (5), $\ln B$ and $\ln C$ in (8), we can also express $\ln D$ and $\ln E$ as special cases of $\zeta'(s)$. In this connection, by using the Euler-Maclaurin summation formula (18), we can obtain a number of analytical representations of $\zeta(s)$, such as the known result recorded by Hardy [**39**, p. 333]:

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}\, n^{-s} \right\} \qquad \big(\Re(z) > -1\big),$$

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}\, n^{-s} + \frac{1}{12}\, sn^{-s-1} \right\} \qquad \big(\Re(z) > -3\big),$$

$$\begin{aligned} \zeta(s) = \lim_{n\to\infty} \Bigg\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}\, n^{-s} + \frac{1}{12}\, sn^{-s-1} \\ - \frac{1}{720}\, s(s+1)(s+2)n^{-s-3} \Bigg\} \qquad \big(\Re(z) > -5\big) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \zeta(s) = \lim_{n\to\infty} \Bigg\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}\, n^{-s} + \frac{1}{12}\, sn^{-s-1} - \frac{1}{720}\, s(s+1)(s+2)n^{-s-3} \\ + \frac{1}{30240}\, s(s+1)(s+2)(s+3)(s+4)n^{-s-5} \Bigg\} \qquad \big(\Re(z) > -7\big). \end{aligned} \tag{20}$$

By first differentiating both sides of (19) with respect to s and then setting $s = -4$, we obtain

$$\begin{aligned} -\zeta'(-4) = \lim_{n\to\infty} \Bigg\{ \sum_{k=1}^{n} k^4 \ln k - \left(\frac{1}{5}\, n^5 + \frac{1}{2}\, n^4 + \frac{1}{3}\, n^3 - \frac{1}{30}\, n \right) \ln n \\ + \frac{1}{25}\, n^5 - \frac{1}{12}\, n^3 + \frac{13}{360}\, n \Bigg\}, \end{aligned}$$

which, when compared with (9), yields

$$\ln D = -\zeta'(-4).$$

Also, by first differentiating both sides of (20) with respect to s and then setting $s = -5$, we obtain

$$\frac{137}{15120} - \zeta'(-5) = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^5 \ln k - \left(\frac{1}{6}\,n^6 + \frac{1}{2}\,n^5 + \frac{5}{12}\,n^4 - \frac{1}{12}\,n^2 + \frac{1}{252}\right)\ln n + \frac{1}{36}\,n^6 - \frac{1}{12}\,n^4 + \frac{47}{720}\,n^2\right\},$$

which, when compared with (10), yields

$$\ln E = \frac{137}{15120} - \zeta'(-5).$$

Without using the Bernoulli numbers, Theorems 4 and 5 provide a recurrence relation for determining the coefficients of each asymptotic expansion related to the above-defined constants D and E in (9) and (10), respectively.

Theorem 4. *As $n \to \infty$, the following asymptotic expansion holds true*:

$$\sum_{k=1}^{n} k^4 \ln k - \left(\frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n\right)\ln n + \frac{1}{25}\,n^5 - \frac{1}{12}\,n^3 + \frac{13}{360}\,n \sim \ln D + \sum_{k=1}^{\infty}\frac{d_k}{n^k},$$

where the coefficients d_k are given by the recurrence relation:

(21)
$$d_k = \frac{(-1)^{k+1}(k+14)}{30(k+2)(k+4)(k+5)(k+6)} - \frac{1}{k}\sum_{j=1}^{k-1} d_j(-1)^{k-j}\binom{k}{k-j+1} \qquad (k \geqq 2)$$

together with $d_1 = \frac{1}{1260}$. In terms of the constant D defined by (9), *it is asserted that*

$$\sum_{k=1}^{n} k^4 \ln k - \left(\frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n\right)\ln n + \frac{1}{25}\,n^5 - \frac{1}{12}\,n^3 + \frac{13}{360}\,n \sim \ln D + \frac{1}{1260n} - \frac{1}{25200n^3} + \frac{1}{83160n^5} - \frac{691}{75675600n^7} + \cdots .$$

Proof. If we set

$$X_n = \sum_{k=1}^{n} k^4 \ln k - \left(\frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n\right)\ln n + \frac{1}{25}\,n^5 - \frac{1}{12}\,n^3 + \frac{13}{360}\,n - \ln D$$

and

$$Y_n = \sum_{k=1}^{\infty} \frac{d_k}{n^k},$$

then we can let $X_n \sim Y_n$ and

$$\Delta X_n := X_{n+1} - X_n \sim Y_{n+1} - Y_n =: \Delta Y_n \qquad (n \to \infty),$$

where d_k are real numbers to be determined. Indeed, after some elementary transformations, we find that

$$\begin{aligned} \Delta X_n &= \left(-\frac{1}{5}\,n^5 - \frac{1}{2}\,n^4 - \frac{1}{3}\,n^3 + \frac{1}{30}\,n\right) \ln\left(1 + \frac{1}{n}\right) \\ &\qquad + \frac{2}{5}\,n^3 - \frac{1}{20}\,n + \frac{1}{5}\,n^4 + \frac{3}{20}\,n^2 - \frac{13}{1800} \\ &= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}(k-1)(k+13)}{30(k+1)(k+3)(k+4)(k+5)}\,\frac{1}{n^k}. \end{aligned} \tag{22}$$

Thus, by using (13), we obtain

$$\Delta V_n = \sum_{k=2}^{\infty} \left\{ \sum_{j=1}^{k} d_j (-1)^{k-j} \binom{k-1}{k-j} - d_k \right\} \frac{1}{n^k}. \tag{23}$$

Upon equating the coefficients of n^{-k} on the right-hand sides of (22) and (23), we get

$$\begin{aligned} \frac{(-1)^{k-1}(k-1)(k+13)}{30(k+1)(k+3)(k+4)(k+5)} &= \sum_{j=1}^{k} d_j (-1)^{k-j} \binom{k-1}{k-j} - d_k \\ &= \sum_{j=1}^{k-1} d_j (-1)^{k-j} \binom{k-1}{k-j} \\ &= \sum_{j=1}^{k-2} d_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1) d_{k-1} \qquad (k \geqq 2), \end{aligned}$$

which, for $k = 2$, yields $d_1 = \frac{1}{1260}$. Moreover, for $k \geqq 3$, we have

$$d_{k-1} = \frac{(-1)^k (k+13)}{30(k+1)(k+3)(k+4)(k+5)} + \frac{1}{k-1} \sum_{j=1}^{k-2} d_j (-1)^{k-j} \binom{k-1}{k-j},$$

which can be written as (21). The proof of Theorem 4 is now complete. □

Theorem 5. *As $n \to \infty$, the following asymptotic expansion holds true*:

$$\sum_{k=1}^{n} k^5 \ln k - \left(\frac{1}{6}\, n^6 + \frac{1}{2}\, n^5 + \frac{5}{12}\, n^4 - \frac{1}{12}\, n^2 + \frac{1}{252}\right) \ln n + \frac{1}{36}\, n^6 - \frac{1}{12}\, n^4 + \frac{47}{720}\, n^2 \sim \ln E + \sum_{k=1}^{\infty} \frac{e_k}{n^k},$$

where the coefficients e_k are given by the recurrence relation:

(24)

$$e_k = \frac{(-1)^k (k-1)(141 + 22k + k^2)}{252(k+1)(k+3)(k+5)(k+6)(k+7)} - \frac{1}{k} \sum_{j=1}^{k-1} e_j (-1)^{k-j} \binom{k}{k-j+1} \qquad (k \geqq 2)$$

together with $e_1 = 0$. Furthermore, in terms of the constant E defined by (10), *it is asserted that*

$$\sum_{k=1}^{n} k^5 \ln k - \left(\frac{1}{6}\, n^6 + \frac{1}{2}\, n^5 + \frac{5}{12}\, n^4 - \frac{1}{12}\, n^2 + \frac{1}{252}\right) \ln n + \frac{1}{36}\, n^6 - \frac{1}{12}\, n^4 + \frac{47}{720}\, n^2 \sim \ln E + \frac{1}{10080 n^2} - \frac{1}{66528 n^4} + \frac{691}{90810720 n^6} - \frac{1}{123552 n^8} + \cdots .$$

Proof. Upon setting

$$I_n = \sum_{k=1}^{n} k^5 \ln k - \left(\frac{1}{6}\, n^6 + \frac{1}{2}\, n^5 + \frac{5}{12}\, n^4 - \frac{1}{12}\, n^2 + \frac{1}{252}\right) \ln n + \frac{1}{36}\, n^6 - \frac{1}{12}\, n^4 + \frac{47}{720}\, n^2 - \ln E$$

and

$$J_n = \sum_{k=1}^{\infty} \frac{e_k}{n^k},$$

we can let $I_n \sim J_n$ and

$$\Delta I_n := I_{n+1} - I_n \sim J_{n+1} - J_n =: \Delta J_n \qquad (n \to \infty),$$

where e_k are real numbers to be determined. In fact, after some elementary transformations, we obtain

$$\Delta I_n = \left(-\frac{1}{6}\, n^6 - \frac{1}{2}\, n^5 - \frac{5}{12}\, n^4 + \frac{1}{12}\, n^2 - \frac{1}{252}\right) \ln\left(1 + \frac{1}{n}\right) + \frac{1}{6}\, n^5 + \frac{5}{12}\, n^4 + \frac{2}{9}\, n^3 - \frac{1}{12}\, n^2 - \frac{13}{360}\, n + \frac{7}{720}$$

(25) $$= \sum_{k=3}^{\infty} \frac{(-1)^k (k-1)(k-2)(k^2 + 20k + 120)}{252 k (k+2)(k+4)(k+5)(k+6)}\, \frac{1}{n^k}$$

and

$$\Delta J_n = -\frac{e_1}{n^2} + \sum_{k=3}^{\infty} \left\{ \sum_{j=1}^{k} e_j (-1)^{k-j} \binom{k-1}{k-j} - e_k \right\} \frac{1}{n^k}. \tag{26}$$

Now, by equating the coefficients of n^{-k} on the right-hand sides of (25) and (26), we find that $e_1 = 0$ and that

$$\begin{aligned} \frac{(-1)^k (k-1)(k-2)(k^2+20k+120)}{252k(k+2)(k+4)(k+5)(k+6)} &= \sum_{j=1}^{k} e_j (-1)^{k-j} \binom{k-1}{k-j} - e_k \\ &= \sum_{j=1}^{k-1} e_j (-1)^{k-j} \binom{k-1}{k-j} \\ &= \sum_{j=1}^{k-2} e_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1) e_{k-1} \end{aligned}$$

and

$$e_{k-1} = \frac{1}{k-1} \sum_{j=1}^{k-2} e_j (-1)^{k-j} \binom{k-1}{k-j} - \frac{(-1)^k \ (k-2)(k^2+20k+120)}{252k(k+2)(k+4)(k+5)(k+6)} \qquad (k \geqq 3),$$

which can be written precisely as (24). This evidently completes our proof of Theorem 5. □

4. AN OPEN PROBLEM

As the Euler-Mascheroni constant γ is involved with the classical gamma function Γ, the constants A, B and C have appeared naturally in the theory of the multiple gamma functions Γ_n (see, for example, [**66**, Section 1.4]) and play their respective roles as described in ([**65**, p. 39, p. 247], [**28**, p. 523, Eq. (2.50)] and [**25**]).

$$\int_0^{\frac{1}{2}} \ln \Gamma(t+1) \, \mathrm{d}t = -\frac{1}{2} - \frac{7}{24} \ln 2 + \frac{1}{4} \ln \pi + \frac{3}{2} \ln A, \tag{27}$$

$$\int_0^{\frac{1}{2}} \ln G(t+1) \, \mathrm{d}t = \frac{1}{24}(\ln 2 + 1) + \frac{1}{16} \ln \pi - \frac{1}{4} \ln A - \frac{7}{4} \ln B \tag{28}$$

and

(29)

$$\int_0^{\frac{3}{2}} \ln \Gamma_3(t+2) \, \mathrm{d}t = -\frac{259}{768} - \frac{29}{1920} \ln 2 + \frac{9}{16} \ln \pi - \frac{15}{16} \ln A - \frac{5}{4} \ln B + \frac{15}{16} \ln C,$$

where Γ_3 is the triple gamma function (see [**66**, p. 58]).

In view of (27), (28) and (29), we propose the following open problem.

Open Problem. Let α and β be two given positive numbers. Determine the constants $a_j \equiv a_j(\alpha, \beta)$ and $b_j \equiv b_j(\alpha, \beta)$ such that

$$\int_0^\alpha \ln \Gamma_4(t+\beta)\, \mathrm{d}t = a_1 + a_2 \ln 2 + a_3 \ln \pi + a_4 \ln A + a_5 \ln B + a_6 \ln C + a_7 \ln D$$

and

$$\int_0^\alpha \ln \Gamma_5(t+\beta)\, \mathrm{d}t = b_1 + b_2 \ln 2 + b_3 \ln \pi + b_4 \ln A + b_5 \ln B + b_6 \ln C + b_7 \ln D + b_8 \ln E.$$

5. CONCLUDING REMARKS AND OBSERVATIONS

In our present investigation, without using the Bernoulli numbers B_n, we have established several asymptotic expansions and a recurrence relation for determining the coefficients of each asymptotic expansion associated with the Wallis sequence W_n, defined by

$$W_n := \prod_{k=1}^n \frac{4k^2}{4k^2-1},$$

and the constants D and E, which are analogous to the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C. We have also pointed out the relevant connections of the formulas and results, which we have considered in this article, with various known or new results.

Acknowledgements. The authors thank the referee for helpful comments. This work was supported by the Key Science Research Project in the Universities of Henan Province (20B110007) and the Fundamental Research Funds for the Universities of the Henan Province (NSFRF210446).

REFERENCES

1. V.S. Adamchik: *Ploygamma functions of negative order.* J. Comput. Appl. Math. 100 (1998), 191–199.
2. R.P. Agarwal, H. Agarwal, S.K. Sen: *Birth, growth and computation of π to ten trillion digits.* Adv. Differ. Equ. **2013** (2013), Article 100.

3. T. Amdeberhan, O. Espinosa, V.H. Moll, A. Straub: *Wallis-Ramanujan-Schur-Feynman*. Amer. Math. Monthly **117** (2010), 618–632.
4. E.W. Barnes: *The theory of the G-function*. Quart. J. Math. **31** (1899), 264–314.
5. E.W. Barnes: *Genesis of the double gamma function*. Proc. London Math. Soc. **31** (1900), 358–381.
6. E.W. Barnes: *The theory of the double gamma function*. Philos. Trans. Roy. Soc. London Ser. A **196** (1901), 265–388.
7. E.W. Barnes: *On the theory of the multiple gamma functions*. Trans. Cambridge Philos. Soc. **19** (1904), 374–439.
8. N. Batir: *Inequalities for the double gamma function*. J. Math. Anal. Appl. **351** (2009), 182–185.
9. N. Batir, M. Cancan: *A double inequality for the double gamma function*. Internat. J. Math. Anal. (Ruse) **2** (2008), 329–335.
10. V. Beckmann: *A History of Pi*. St. Martin's Press, New York, 1971.
11. L. Bendersky, Sur la fonction Gamma généralisée, Acta Math. 61 (1933), 263–322.
12. L. Berggren, J. Borwein, P. Borwein (eds.): *Pi: A Source Book*. Second ed., Springer, New York, 2000.
13. T. Burić, N. Elezović, R. Šimić: *Asymptotic expansions of the multiple quotients of gamma functions with applications*. Math. Inequal. Appl. **16** (2013), 1159–1170.
14. C.-P. Chen: *Inequalities associated with Barnes G-function*, Exposition. Math. **29** (2011), 119–125.
15. C.-P. Chen: *Glaisher-Kinkelin constant*. Integral Transforms Spec. Funct. **23** (2012), 785–792.
16. C.-P. Chen: *Asymptotic expansions for Barnes G-function*. J. Number Theory **135** (2014), 36–42.
17. C.-P. Chen: *Complete asymptotic expansions for certain mathematical constants and special functions*. Appl. Anal. Discrete Math. **12** (2018), 493–507.
18. C.-P. Chen, J. Choi: *Unified treatment of several asymptotic expansions concerning some mathematical constants*. Appl. Math. Comput. **305** (2017), 348–363.
19. C.-P. Chen, N. Elezović, L. Vukšić: *Asymptotic formulae associated with the Wallis power function and digamma function*. J. Class. Anal. **2** (2013), 151–166.
20. C.-P. Chen, L. Lin: *Asymptotic expansions related to Glaisher-Kinkelin constant based on the Bell polynomials*. J. Number Theory **133** (2013), 2699–2705.
21. C.-P. Chen, R.B. Paris: *On the asymptotic expansions of products related to the Wallis, Weierstrass, and Wilf formulas*. Appl. Math. Comput. **293** (2017), 30–39.
22. C.-P. Chen, H.M. Srivastava: *Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function*. Integral Transforms Spec. Funct. **22** (2011), 1–15.
23. C.-P. Chen, H.M. Srivastava: *Some new properties of the Barnes G-function and related results*. Appl. Anal. Discrete Math. **15** (2021), 129–150.
24. J.-X. Cheng, C.-P. Chen: *Asymptotic expansions of the Glaisher-Kinkelin and Choi-Srivastava constants*. J. Number Theory, **144** (2014), 105–110.

25. J. Choi: *Some mathematical constants.* Appl. Math. Comput. **187** (2007), 122–140.

26. J. Choi: *Determinant of Laplacian on S^3.* Math. Japon. **40** (1994), 155–166.

27. J. Choi: *A set of mathematical constants arising naturally in the theory of the multiple Gamma functions.* Abstr. Appl. Anal. **2012** (2012), Art. ID 121795, 11 pp.

28. J. Choi, Y.J. Cho, H.M. Srivastava: *Series involving the zeta function and multiple Gamma functions.* Appl. Math. Comput. **159** (2004), 509–537.

29. J. Choi, H.M. Srivastava: *Certain classes of series involving the zeta function.* J. Math. Anal. Appl. **231** (1999), 91–117.

30. J. Choi, H.M. Srivastava: *Certain classes of series associated with the zeta function and multiple Gamma functions.* J. Comput. Appl. Math. **118** (2000), 87–109.

31. J. Choi, H.M. Srivastava: *Asymptotic formulas for the triple gamma function Γ_3 by means of its integral representation.* Appl. Math. Comput. **218** (2011), 2631–2640.

32. J.-E. Deng, T. Ban, C.-P. Chen: *Sharp inequalities and asymptotic expansion associated with the Wallis sequence.* J. Inequal. Appl. **2015** (2015), Art. ID 186, 11 pp.

33. W. Dunham: *Journey Through Genius*: *The Great Theorems of Mathematics.* Penguin, New York, 1990.

34. J. Edwards: *A Treatise on the Integral Calculus with Applications*: *Examples and Problems.* Vols. 1 and 2, Chelsea Publishing Company, New York, 1954.

35. N. Elezović, L. Lin, L. Vukšić: *Inequalities and asymptotic expansions for the Wallis sequence and the sum of the Wallis ratio.* J. Math. Inequal. **7** (2013), 679–695.

36. C. Ferreira, J.L. López: *An asymptotic expansion of the double gamma function.* J. Approx. Theory **111** (2001), 298–314.

37. S.R. Finch: *Mathematical Constants.* Cambridge University Press, Cambridge, London and New York, 2003.

38. I.S. Gradshteyn, I.M. Ryzhik: *Tables of Integrals*, *Series*, *and Products* (Corrected and Enlarged Edition prepared by A. Jeffrey). Academic Press, New York, 1980.

39. G.H. Hardy: *Divergent Series.* Clarendon (Oxford University) Press, Oxford, London and New York, 1949.

40. M.D. Hirschhorn: *Comments on the paper "Wallis sequence estimated through the Euler-Maclaurin formula*: *Even from the Wallis product π could be computed fairly accurately" by V. Lampret.* Austral. Math. Soc. Gaz. **32** (2005), 194–194.

41. S. Koumandos: *On Ruijsenaars' asymptotic expansion of the logarithm of the double gamma function.* J. Math. Anal. Appl. **341** (2008), 1125–1132.

42. S. Koumandos, H.L. Pedersen: *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function.* J. Math. Anal. Appl. **355** (2009), 33–40.

43. H. Kumagai: *The determinant of the Laplacian on the n-sphere.* Acta Arith. **91** (1999), 199–208.

44. V. Lampret: *An asymptotic approximation of Wallis' sequence.* Central European J. Math. **10** (2012), 775–787.

45. V. Lampret: *Wallis sequence estimated through the Euler-Maclaurin formula: Even from the Wallis product π could be computed fairly accurately.* Austral. Math. Soc. Gaz. **31** (2004), 328–339.

46. L. Lin: *Further refinements of Gurland's formula for π.* J. Inequal. Appl. **2013** (2013), Art. ID 48, 11 pp..

47. L. Lin, J.-E. Deng, C.-P. Chen: *Inequalities and asymptotic expansions associated with the Wallis sequence.* J. Inequal. Appl. **2014** (2014), Art. ID 251, 14 pp..

48. S.J. Miller: *A probabilistic proof of Wallis's formula for π.* Amer. Math. Monthly **115** (2008), 740–745.

49. C. Mortici: *Estimating π from the Wallis sequence.* Math. Commun. **17** (2012), 489–495.

50. C. Mortici: *Completely monotone functions and the Wallis ratio.* Appl. Math. Lett. **25** (2012), 717–722.

51. C. Mortici: *Sharp inequalities and complete monotonicity for the Wallis ratio.* Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 929–936 (2010).

52. C. Mortici: *A new method for establishing and proving new bounds for the Wallis ratio.* Math. Inequal. Appl. **13** (2010), 803–815.

53. C. Mortici: *New approximation formulas for evaluating the ratio of gamma functions.* Math. Comput. Model. **52** (2010), 425–433.

54. C. Mortici: *Approximating the constants of Glaisher-Kinkelin type.* J. Number Theory **133** (2013), 2465–2469.

55. C. Mortici, V.G. Cristea, D. Lu: *Completely monotonic functions and inequalities associated to some ratio of gamma function.* Appl. Math. Comput. **240** (2014), 168–174.

56. G. Myerson: *The limiting shape of a sequence of rectangles.* Amer. Math. Monthly **99** (1992), 279–280

57. B. Osgood, R. Phillips, P. Sarnak: *Extremals of determinants of Laplacians.* J. Funct. Anal. **80** (1988), 148–211.

58. E. Păltănea: *On the rate of convergence of Wallis' sequence.* Austral. Math. Soc. Gaz. **34** (2007), 34–38.

59. H.L. Pedersen: *On the remainder in an asymptotic expansion of the double gamma function.* Mediterr. J. Math. **2** (2005), 171–178.

60. H.L. Pedersen: *The remainder in Ruijsenaars' asymptotic expansion of Barnes double gamma function.* Mediterr. J. Math. **4** (2007), 419–433.

61. J.R. Quine, J. Choi: *Zeta regularized products and functional determinants on spheres.* Rocky Mountain J. Math. **26** (1996), 719–729.

62. S.N.M. Ruijsenaars: *On Barnes' multiple zeta and gamma functions.* Adv. Math. **156** (2000), 107–132.

63. A. Sofo: *Some representations of π.* Austral. Math. Soc. Gaz. **31** (2004), 184–189.

64. A. Sofo: *π and some other constants.* J. Inequal. Pure Appl. Math. **6** (5) (2005), Art. ID 138, 14 pp. (electronic).

65. H.M. Srivastava, J. Choi: *Series Associated with the Zeta and Related Functions.* Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

66. H.M. Srivastava, J. Choi: *Zeta and q-Zeta Functions and Associated Series and Integrals.* Elsevier Science Publishers, Amsterdam, London and New York, 2012.

67. I. Vardi: *Determinants of Laplacians and multiple gamma functions.* SIAM J. Math. Anal. **19** (1988), 493–507.

68. A. Voros: *Special functions, spectral functions and the Selberg Zeta function.* Commun. Math. Phys. **110** (1987), 439–465.

69. J. Wästlund: *An elementary proof of the Wallis product formula for Pi.* Amer. Math. Monthly **114** (2007), 914–917.

70. E.T. Whittaker, G.N. Watson: *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *With an Account of the Principal Transcendental Functions.* Fourth ed. (Reprinted), Cambridge University Press, Cambridge, London and New York, 1963.

(Received 14. 04. 2022.)
(Revised 25. 08. 2022.)

Xue-Feng Han
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province,
People's Republic of China.
E-Mail: *hanxuefeng8110@sohu.com*

Chao-Ping Chen
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province,
People's Republic of China.
E-Mail: *chenchaoping@sohu.com*

H. M. Srivastava
Department of Mathematics and Statistics,
University of Victoria,
Victoria, British Columbia V8W 3R4, Canada.
Department of Medical Research,
China Medical University Hospital,
China Medical University,
Taichung 40402, Taiwan, Republic of China.
Department of Mathematics and Informatics,
Azerbaijan University,
71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan.
Section of Mathematics, International Telematic
University Uninettuno, I-00186 Rome, Italy.
E-Mail: *harimsri@math.uvic.ca*