

## HÜGELSCHÄFFER EGG CURVE AND SURFACE

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In this paper we consider Hügelschäffer cubic curves which are generated using appropriate geometric constructions. The main result of this work is the mode of explicitly calculating the area of the egg-shaped part of the cubic curve using elliptic integrals. In this paper, we also analyze the Hügelschäffer surface of cubic curves for which we provide new forms of formulae for the volume and surface area of the egg-shaped part. Curves and surfaces of ovoid shape have wide applicability in aero-engineering and construction, and are also of biologic importance. With respect to this, in the final section, we consider some examples of the real applicability of this Hügelschäffer model.

### 1. INTRODUCTION

The starting point is the geometric construction of planar egg-shaped curves introduced by the German engineer FRITZ HÜGELSCHÄFFER in 1944, [8], [43], [44].

*Hügelschäffer curve definition.* Let there exist two circles, in an Euclidean plane:

$$\mathcal{C}_1 : X^2 + Y^2 = a^2$$

and

$$\mathcal{C}_2 : (X + w)^2 + Y^2 = b^2,$$

where  $a, b, w \geq 0$ . For the point  $P_1 = (X_1, Y_1) \in \mathcal{C}_1$  we define the point  $P_2 = (X_2, Y_2) \in \mathcal{C}_2$  by  $P_2 = O_2 P_1 \cap \mathcal{C}_2$ , where  $O_2 = (-w, 0)$  is the center of circle  $\mathcal{C}_2$ , see

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Figure 1. The locus of points  $Q = (x, y)$  where  $x = X_1$  and  $y = Y_2$ , satisfies the cubic algebraic equation

$$(1) \quad \mathcal{F}: \quad 2wxy^2 + b^2x^2 + (a^2 + w^2)y^2 - a^2b^2 = 0,$$

as shown in [38], see also [10]. The Hügelschäffer curve is defined by the locus of points  $Q = (x, y)$ . Let us emphasize that the previous definition of the cubic algebraic equation is consistent with Newton's hyperbolism [33]; for details about hyperbolism see considerations in [9], [38] and [39].

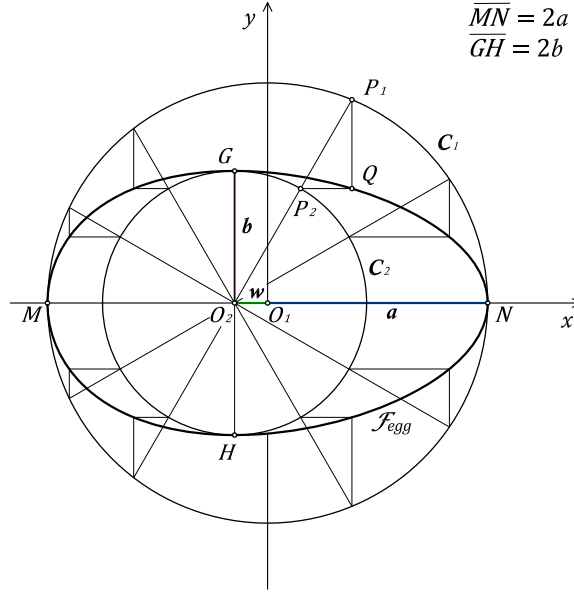


Figure 1: Hügelschäffer's construction of an egg curve

Let us assume that  $a, b, w \geq 0$ . We consider the cases where some of the parameters  $a, b, w$  equal zero. If  $b = 0$ , the algebraic curve  $\mathcal{F}$  degenerates into an union of lines. The cases where  $a = 0$  and/or  $w = 0$  are of special interest. If  $a = 0$  and  $w \neq 0$  the algebraic curve  $\mathcal{F}$  becomes a mixed cubic [52], [38], [11]. This type of curve was studied by G. de Longchamps in 1890, who named it. If  $a \neq 0$  and  $w = 0$  the algebraic curve  $\mathcal{F}$  becomes an ellipse. If  $a = w = 0$  the algebraic curve  $\mathcal{F}$  degenerates into an union of lines. In all other cases ( $b \neq 0 \wedge w \neq 0 \wedge a \neq w$ ) the algebraic curve is non-degenerated algebraic curve of the third order and in this paper we consider only such curves. Let us note that if  $w = a$  the algebraic curve  $\mathcal{F}$  degenerates into an union of a line and a parabola, [38].

Furthermore, let  $w > 0$ . Solving the algebraic equation (1) for  $y$ , two functions are obtained:

$$(2) \quad y = f_{1,2}(x) = \pm b \sqrt{\frac{a^2 - x^2}{2wx + a^2 + w^2}} = \pm H \sqrt{\frac{(\alpha - x)(x - \beta)}{x - \gamma}}: (-\infty, \gamma) \cup [-a, a] \rightarrow \mathbb{R}$$

where

$$(3) \quad \alpha = a, \beta = -a, \gamma = -\frac{a^2 + w^2}{2w}, H = \frac{b}{\sqrt{2w}}.$$

as considered in [38]. It is easy to check that  $\gamma < -a \Leftrightarrow (a - w)^2 > 0$ .

Functions  $f_{1,2}$  over  $[-a, a]$  determine the egg-shaped part of the cubic curve which we denote with  $\mathcal{F}_{egg}$ . Functions  $f_{1,2}$  over  $(-\infty, \gamma)$  determine the hyperbolic part of the cubic curve which we denote with  $\mathcal{F}_{hyp}$ , see Figure 2. The geometric construction of the curve  $\mathcal{F}_{hyp}$ , i.e. the extended hyperbolic part of the cubic curve, using two hyperbolae and Newton's hyperbolism was first given in the magister's thesis [38] by M. Petrović. It is noted that  $\mathcal{F} = \mathcal{F}_{egg} \cup \mathcal{F}_{hyp}$ .

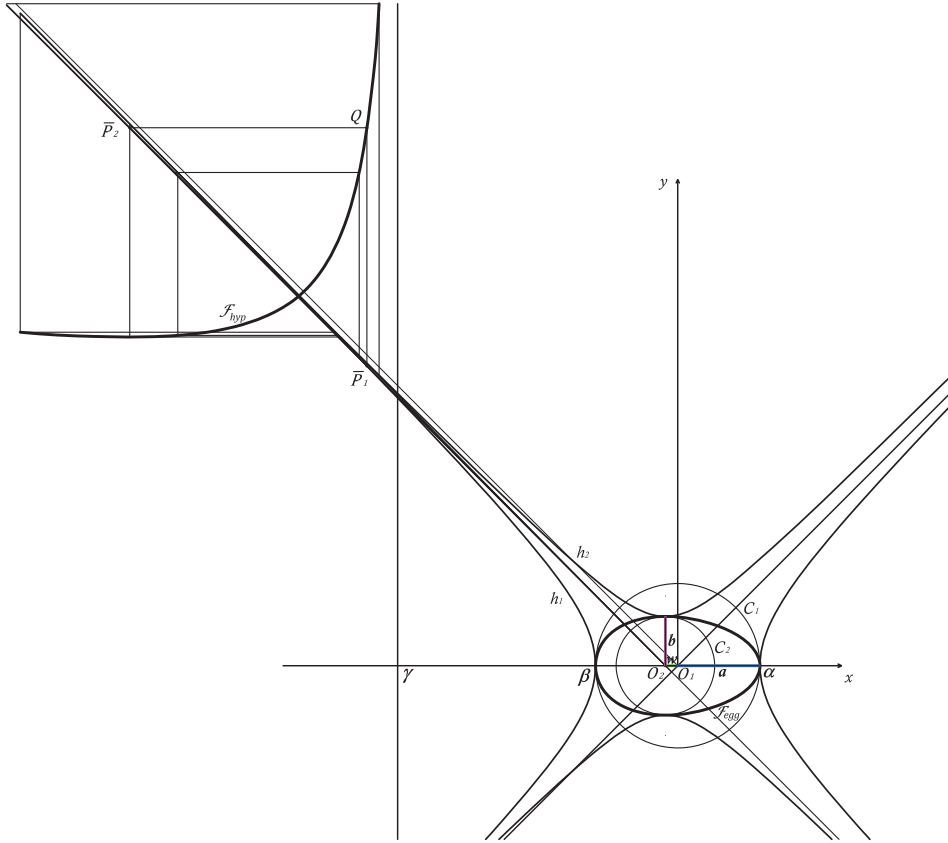


Figure 2: Hügelschäffer's construction of the egg-shaped part and the Petrović construction of the hyperbolic part of a cubic curve

Various analytical properties of the functions  $f_{1,2}$  given by (2) are considered in [38], [39]. It is simple to show the following result which is used in the next section. It holds that:

**Theorem 1.** Let  $u$  be the abscissa of the maximum of function  $y = f_1(x)$  over  $[-a, a]$ . Then:

$$(4) \quad \mathbf{u} = \begin{cases} -w & : w < a, \\ -a & : w = a, \\ -a^2/w & : w > a. \end{cases}$$

**Note 1.** The case where  $w = a$  leads to the degeneration of the cubic curve  $\mathcal{F}$ , so we do not take this case of the previous theorem into further consideration.

## 2. MAIN RESULTS

### 2.1. Hügelschäffer's egg-shaped curve

The starting point is the area of the surface bounded by the cubic curve  $\mathcal{F}_{egg}$  which is given by the integral:

$$\mathcal{A}_{egg} = 2 \int_{\beta}^{\alpha} f_1(x) dx.$$

The value  $\mathcal{A}_{egg}$  is obtained using the well-known integrals [12]:

*The elliptic integral of the first kind*

$$F(\theta, p) = \int_0^{\sin(\theta)} \frac{dt}{\sqrt{(1-t^2)(1-p^2t^2)}}, \quad 0 \leq p^2 \leq 1 \quad \wedge \quad 0 \leq \sin(\theta) \leq 1,$$

and *the elliptic integral of the second kind*

$$E(\theta, p) = \int_0^{\sin(\theta)} \frac{\sqrt{1-p^2t^2}}{\sqrt{1-t^2}} dt, \quad 0 \leq p^2 \leq 1 \quad \wedge \quad 0 \leq \sin(\theta) \leq 1.$$

Furthermore, let there be functions

$$\varphi(x) = \sqrt{\frac{(\alpha - \gamma)(x - \beta)}{(\alpha - \beta)(x - \gamma)}} \quad \wedge \quad \psi(x) = \sqrt{\frac{\alpha - x}{\alpha - \beta}},$$

where the parameters  $\alpha, \beta$  and  $\gamma$  are determined with (3) and where  $x \in [\beta, \alpha]$ . In the next theorem, we will use the following lemma.

**Lemma 1.**

$$\varphi(x) = \psi(x) \quad \iff \quad x = -w \vee x = -\frac{a^2}{w}.$$

The following statement is true:

**Theorem 2.** *It holds that*

$$(5) \quad \mathcal{A}_{egg} = \mathcal{A}_1 + \mathcal{A}_2$$

where

$$\begin{aligned} \mathcal{A}_1 = & \frac{4}{3}H\sqrt{\alpha-\gamma} \left( (\alpha + \beta - 2\gamma) \cdot E(\kappa, p) - 2(\beta - \gamma) \cdot F(\kappa, p) \right) + \\ & \frac{4}{3}H(\mathbf{u} + \gamma - \alpha - \beta) \sqrt{\frac{(\alpha - \mathbf{u})(\mathbf{u} - \beta)}{\mathbf{u} - \gamma}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_2 = & \frac{4}{3}H\sqrt{\alpha-\gamma} \left( (\alpha + \beta - 2\gamma) \cdot E(\lambda, p) - 2(\beta - \gamma) \cdot F(\lambda, p) \right) - \\ & \frac{4}{3}H\sqrt{(\alpha - \mathbf{u})(\mathbf{u} - \beta)(\mathbf{u} - \gamma)} \end{aligned}$$

with the parameters:

$$p = \sqrt{\frac{\alpha - \beta}{\alpha - \gamma}}, \quad \kappa = \arcsin \varphi(\mathbf{u}), \quad \lambda = \arcsin \psi(\mathbf{u})$$

and where  $\mathbf{u}$  is the abscissa of the maximum of function  $y = f_1(x)$  (see eq. (4)).

*Proof.* Note that  $\alpha > \beta > \gamma$  and it follows that for  $p$  it holds

$$0 < p^2 = \frac{\alpha - \beta}{\alpha - \gamma} = 1 - \frac{\beta - \gamma}{\alpha - \gamma} < 1.$$

Starting from  $\mathbf{u}$  as the abscissa of the maximum of the function for parameters  $\kappa = \arcsin \varphi(\mathbf{u})$  and  $\lambda = \arcsin \psi(\mathbf{u})$  it holds that

$$\kappa = \lambda = \arcsin \sqrt{\frac{\alpha - \mathbf{u}}{\alpha - \beta}},$$

in accordance with Theorem 1. and Lemma 1. Additionally, the expression

$$0 < \sin \lambda = \sqrt{\frac{\alpha - \mathbf{u}}{\alpha - \beta}} = \sqrt{1 - \frac{\mathbf{u} - \beta}{\alpha - \beta}} < 1,$$

is satisfied, because  $\alpha > \mathbf{u} > \beta > \gamma$ . Because of this, the inequality

$$0 < \sin \kappa < 1.$$

is also correct. First, let us consider

$$I_1 = \int_{\beta}^{\mathbf{u}} f_1(t) dt = H \int_{\beta}^{\mathbf{u}} \sqrt{\frac{(\alpha - t)(t - \beta)}{t - \gamma}} dt.$$

Then, from [12] according to formula 3.141/34 it holds that

$$I_1 = \frac{2}{3} H \sqrt{\alpha - \gamma} \left( (\alpha + \beta - 2\gamma) \cdot E(\kappa, p) - 2(\beta - \gamma) \cdot F(\kappa, p) \right) + \frac{2}{3} H (\mathbf{u} + \gamma - \alpha - \beta) \sqrt{\frac{(\alpha - \mathbf{u})(\mathbf{u} - \beta)}{\mathbf{u} - \gamma}}$$

because the condition  $\alpha \geq \mathbf{u} > \beta > \gamma$  is satisfied. Second, let us consider

$$I_2 = \int_{\mathbf{u}}^{\alpha} f_1(t) dt = H \int_{\mathbf{u}}^{\alpha} \sqrt{\frac{(\alpha - t)(t - \beta)}{t - \gamma}} dt.$$

Then, from [12] according to formula 3.141/35 it holds that

$$I_2 = \frac{2}{3} H \sqrt{\alpha - \gamma} \left( (\alpha + \beta - 2\gamma) \cdot E(\lambda, p) - 2(\beta - \gamma) \cdot F(\lambda, p) \right) - \frac{2}{3} H \sqrt{(\alpha - \mathbf{u})(\mathbf{u} - \beta)(\mathbf{u} - \gamma)}$$

because the condition  $\alpha > \mathbf{u} \geq \beta > \gamma$  is satisfied. Therefore, the claim of the theorem follows from

$$\mathcal{A}_1 = 2I_1 \wedge \mathcal{A}_2 = 2I_2 \quad \implies \quad \mathcal{A}_{egg} = \mathcal{A}_1 + \mathcal{A}_2.$$

Also see Figure 3.

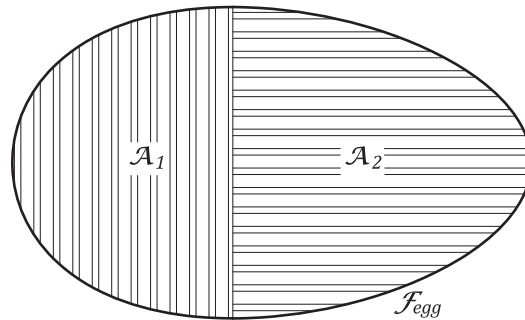


Figure 3: Areas  $\mathcal{A}_1$  and  $\mathcal{A}_2$  of the Hügelschäffer egg curve

□

Let us emphasize that if  $\mathbf{u} = -w$ , then

$$(6) \quad \mathcal{A}_1 = \frac{2(a+w)b}{3w} \left( \frac{a^2+w^2}{w} E \left( \arcsin \sqrt{\frac{a+w}{2a}}, \frac{2\sqrt{aw}}{a+w} \right) - \frac{(a-w)^2}{w} F \left( \arcsin \sqrt{\frac{a+w}{2a}}, \frac{2\sqrt{aw}}{a+w} \right) \right) - \frac{2b}{3w} (a^2 + 3w^2)$$

and

$$(7) \quad \mathcal{A}_2 = \frac{2(a+w)b}{3w} \left( \frac{a^2+w^2}{w} E \left( \arcsin \sqrt{\frac{a+w}{2a}}, \frac{2\sqrt{aw}}{a+w} \right) - \frac{(a-w)^2}{w} F \left( \arcsin \sqrt{\frac{a+w}{2a}}, \frac{2\sqrt{aw}}{a+w} \right) \right) - \frac{2b}{3w} (a^2 - w^2).$$

Also, let us emphasize that if  $\mathbf{u} = -\frac{a^2}{w}$ , then

$$(8) \quad \mathcal{A}_1 = \frac{2(a+w)b}{3w} \left( \frac{a^2+w^2}{w} E \left( \arcsin \sqrt{\frac{a+w}{2w}}, \frac{2\sqrt{aw}}{a+w} \right) - \frac{(a-w)^2}{w} F \left( \arcsin \sqrt{\frac{a+w}{2w}}, \frac{2\sqrt{aw}}{a+w} \right) \right) - \frac{2ab}{3w^2} (3a^2 + w^2)$$

and

$$(9) \quad \mathcal{A}_2 = \frac{2(a+w)b}{3w} \left( \frac{a^2+w^2}{w} E \left( \arcsin \sqrt{\frac{a+w}{2w}}, \frac{2\sqrt{aw}}{a+w} \right) - \frac{(a-w)^2}{w} F \left( \arcsin \sqrt{\frac{a+w}{2w}}, \frac{2\sqrt{aw}}{a+w} \right) \right) - \frac{2ab}{3w^2} (-a^2 + w^2).$$

Notice that from (6) and (7) it follows

$$\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_{egg} \quad \text{and} \quad \mathcal{A}_2 - \mathcal{A}_1 = \frac{8}{3}bw;$$

and that from (8) and (9) it follows

$$\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_{egg} \quad \text{and} \quad \mathcal{A}_2 - \mathcal{A}_1 = \frac{8}{3} \frac{a^3b}{w^2}.$$

## 2.2. Hügelschäffer's surface

1. *The implicit form of the HÜGELSCHÄFFER's surface.* The 3D surface generated by rotating the curve  $\mathcal{F}$  given by (1) around the  $x$ -axis for  $360^\circ$  has the following implicit form

$$(10) \quad \frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} g(x) = 1, \quad g(x) = 1 + \frac{2wx + w^2}{a^2}.$$

In Figure 4, a 3D surface generated using the cubic curve  $\mathcal{F}$  with appropriate parameters  $a$ ,  $b$  and  $w$  is shown.

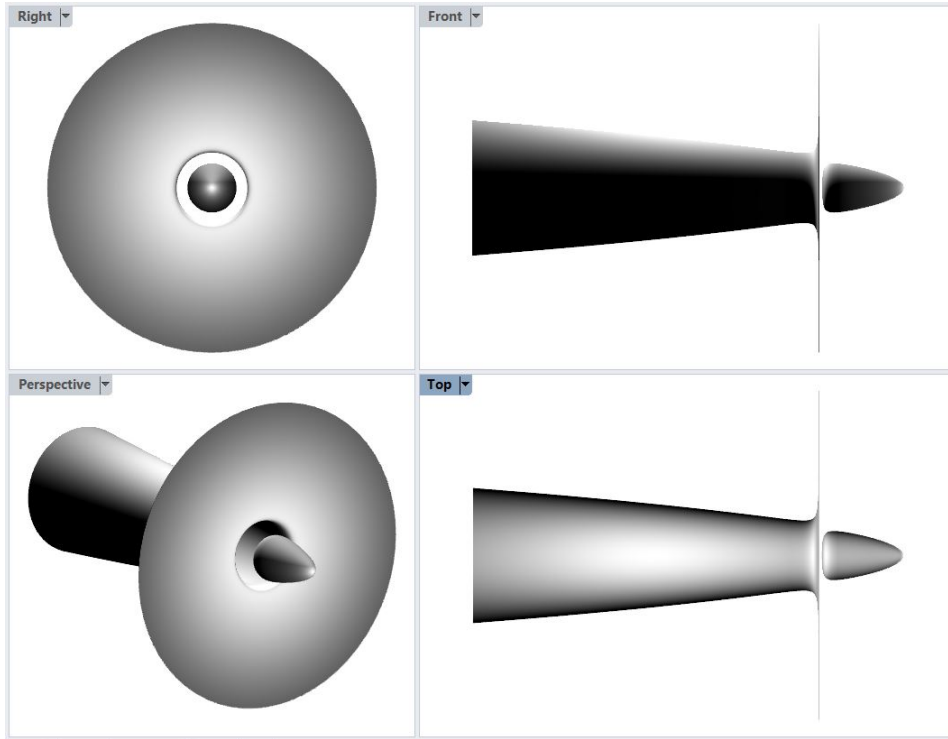


Figure 4: A Hügelschäffer surface (top, front, right and axonometric view)

2. *The explicit form of the HÜGELSCHÄFFER surface.* Solving the equation (10) for  $z$ , the explicit form of the HÜGELSCHÄFFER surface is obtained

$$z = z(x, y) = \pm \sqrt{\frac{b^2(a^2 - x^2)}{2wx + w^2 + a^2} - y^2},$$

with domain  $D_{xy} = \{(x, y) \in \mathbb{R}^2 \mid f_1(x) \leq y \leq f_2(x) : x \in (-\infty, \gamma) \cup [-a, a]\}$ .



3. *The volume of the surface generated by rotating the curve  $\mathcal{F}_{egg}$  around the  $x$ -axis.* It is simple to show that the volume of the surface generated by rotating the curve  $\mathcal{F}_{egg}$  around the  $x$ -axis is given by

$$(11) \quad V_{egg} = \pi \int_{\beta}^{\alpha} f_1^2(x) dx = \frac{\pi b^2}{4w^3} \left( (a^2 - w^2)^2 \ln \left| \frac{a-w}{a+w} \right| + 2aw(a^2 + w^2) \right).$$

For the previous expression for surface volume it holds:

(i) if observing  $w$  in the following limit process, the volume of a spheroid is obtained

$$V_{spheroid} = \lim_{w \rightarrow 0^+} V_{egg} = \frac{4}{3} ab^2 \pi.$$

Note that, from this, it is obtained, for the value  $b = a$ :

$$V_{sphere} = \frac{4}{3} a^3 \pi,$$

which represents the formula for the volume of a sphere of radius  $a$ .

(ii) if  $w = a$  and  $a \neq 0$ , then the volume of a paraboloid is obtained:

$$V_{paraboloid} = ab^2 \pi.$$

4. *The surface area of the surface generated by rotating the curve  $\mathcal{F}_{egg}$  around the  $x$ -axis for  $360^\circ$ .* The egg-shaped surface has the following formula for calculating the area of the 3D surface:

$$\mathcal{S}_{egg} = 2\pi \int_{\beta}^{\alpha} f_1(x) \sqrt{1 + (f_1'(x))^2} dx.$$

Substituting  $x = t + \gamma$  it is obtained:

$$\mathcal{S}_{egg} = \frac{b\pi}{4w^2} \int_{-a-\gamma}^{a-\gamma} \frac{\sqrt{Q_5(t)}}{t^2} dt,$$

with the polynomial of the fifth degree

$$Q_5(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

and the coefficients

$$a_5 = -32w^3$$

$$a_4 = 4w^2(8a^2 + b^2 + 8w^2)$$

$$a_3 = -8w(a^2 - w^2)^2$$

$$a_2 = -2b^2(a^2 - w^2)^2$$

$$a_1 = 0$$

$$a_0 = \frac{b^2}{4w^2}(a^2 - w^2)^4.$$

Based on the previous, it follows that

$$(12) \mathcal{S}_{egg} = \frac{b\pi}{4w^2} \int_{\frac{(a-w)^2}{2w}}^{\frac{(a+w)^2}{2w}} \frac{\sqrt{-32w^3t^5 + 4w^2(8a^2 + b^2 + 8w^2)t^4 - 8w(a^2 - w^2)^2t^3 - 2b^2(a^2 - w^2)^2t^2 + \frac{b^2}{4w^2}(a^2 - w^2)^4}}{t^2} dt.$$

Let us consider some approximative methods for calculating the integral (12). Utilizing the Simpson's quadrature rule for  $n = 3$  points:

$$\begin{aligned} \mathcal{S}_{egg} &\approx \mathcal{S}_{egg_{n=3}} = \frac{a}{3} \frac{b\pi}{4w^2} \left( \frac{\sqrt{Q_5(-a-\gamma)}}{(-a-\gamma)^2} + 4 \frac{\sqrt{Q_5(-\gamma)}}{\gamma^2} + \frac{\sqrt{Q_5(a-\gamma)}}{(a-\gamma)^2} \right) = \\ &= \frac{\pi a}{3} \left( \frac{2ab^2}{(a-w)^2} + \frac{8ab}{(a^2+w^2)^2} \sqrt{(a^2+w^2)^3 + a^2b^2w^2} + \frac{2ab^2}{(a+w)^2} \right) \end{aligned}$$

an approximative formula for the area of the HÜGELSCHÄFFER egg-shaped surface is obtained.

For the previous expression for the area of the surface, it holds that:

(i) if observing  $a \neq 0$  and  $w \rightarrow 0+$  an approximative formula for the area of a spheroid is obtained:

$$\mathcal{S}_{spheroid} \approx \frac{4b\pi}{3}(b+2a).$$

This approximative formula is also given in [42]. Note that for the value  $b = a$  it holds that

$$\mathcal{S}_{sphere} = 4a^2\pi,$$

which represents the formula for the area of a sphere of radius  $a$ .

(ii) if observing  $w = a$  and  $a \neq 0$ , from (12) a well-known formula for the area of a paraboloid is obtained:

$$\mathcal{S}_{paraboloid} = \frac{b\pi}{2a^2} \int_0^{2a} \frac{\sqrt{-8a^3t^5 + a^2(8a^2 + b^2 + 8a^2)t^4}}{t^2} dt = \frac{b\pi}{24a^2} ((16a^2 + b^2)^{3/2} - b^3).$$

### 3. APPLICATIONS

The construction of the HÜGELSCHÄFFER egg-shaped curve model was established in the mid-20th century for the purposes of aero-engineering [8], [43], [44]. Even today, this model is used for the fuselage cross-sections of gliders, [4]: "The fuselage cross-sections of the ASW27 are described by a so-called 'Hügelschäffer-Egg-Curve' which has the special feature of a continuous curvature on its circumference, a prerequisite for a smooth pressure distribution and undisturbed boundary layer development" and according to [5]: "For an undisturbed boundary layer development, continuity of curvature is required in flow direction. This is guaranteed by deriving the top, bottom, and line of largest width from airfoil shapes, and using Hügelschäffer curves (deformed ellipses) for the fuselage cross-sections".

In the papers [7], [19], [34], [35], [36] and [40] some mathematical properties of the HÜGELSCHÄFFER curve are considered, as well as some of its applications in architecture.

The applications of egg-shaped curves are also significant in civil engineering, and especially in hydrology, [1], [3], [13], [41], [47], [48], [49], [54]. Sewer Systems with Egg-Shaped Pipe Cross Sections are gaining traction in cities because of the availability of modern technologies. Egg-shaped or ovoid cross-sections are also common in old brick-built sewerage systems. Several cities around the world including London, Paris, Hamburg, New York, Los Angeles, Cleveland, Minneapolis, Newark, Kolkata, Mumbai and Delhi have such sewer systems. The egg-shaped cross section of these pipes is achieved by the composition of circular arcs. Note that in India, according to a standard from 1976, sewer pipes with an egg-shaped trifocal curve profile (eggclipse section) are used, [6].

In the next part of this section, we state some numerical examples.

**Numerical example 1:** For the city of Cleveland, Ohio, USA, in [16], a graphical model of egg-shaped curves for sewer channels with length  $L = H$ , shape index  $B/L = \frac{2+\sqrt{3}}{3+\sqrt{3}}$  and offset from the bottom to the point where the channel is widest  $S = \frac{5}{2} \frac{L}{3+\sqrt{3}}$  is given. The profile of the sewer pipe is an egg-shaped curve constructed through composition of circular arcs. Based on this model, a HÜGELSCHÄFFER egg-shaped curve model can be formed with parameters  $L$ ,  $B$  and  $w = \frac{2-\sqrt{3}}{3+\sqrt{3}} \frac{L}{2}$ . In the following table, a comparison of values for area  $A$  according to [16] (see Table: Egg-Shaped Sewers; File No. 73M) and values for the area of the corresponding HÜGELSCHÄFFER curve  $\mathcal{A}$  calculated according to our paper (see Table 1: column 7) is given. Let us emphasize that based on this data, the coefficient of determination is  $R^2 = 0.999179\dots$ . The value of the coefficient of determination of  $R^2 \approx 1$  confirms that this HÜGELSCHÄFFER egg-shaped curve model can be a valid approximation of the egg-shaped curve considered in [16].

Analogous to the previous example, a HÜGELSCHÄFFER egg-shaped curve model can be generated according to the profiles of egg-shaped pipe sewers obtained through composition of circular arcs with the shape index  $B/L = 2/3$  or  $B/L = 3/4$  as given in [51].

Table 1: Cross section areas of Egg-Shaped Sewers with respective parameters

No.	$L(ft)$	$B(ft)$	$S(ft)$	$A(ft^2)$	$w(ft)$	$\mathcal{A}(ft^2)$
2	2.25	1.94	1.28	3.41	0.155	3.420108855
3	2.75	2.23	1.64	4.75	0.265	4.793986508
4	3.23	2.55	1.71	6.37	0.095	6.46613264
5	3.74	2.95	1.98	8.54	0.110	8.66154837
6	4.23	3.34	2.24	10.93	0.125	11.09141523
7	4.69	3.70	2.48	13.43	0.135	13.62336584
8	5.12	4.04	2.71	16.01	0.150	16.23882865
9	5.54	4.38	2.93	18.74	0.160	19.04989185
10	5.94	4.69	3.14	21.55	0.170	21.87112886
11	6.33	5.00	3.35	24.47	0.185	24.84723069
12	6.71	5.30	3.55	27.50	0.195	27.9193156
13	7.08	5.59	3.75	30.61	0.210	31.0701810
14	7.44	5.88	3.94	33.80	0.220	34.3439429
15	7.79	6.15	4.12	37.06	0.225	37.6115462
16	8.13	6.42	4.30	40.37	0.235	40.9764108
17	8.47	6.69	4.48	43.81	0.245	44.4854110
18	8.79	6.94	4.65	47.19	0.255	47.8911598
19	9.12	7.20	4.83	50.79	0.270	51.5497738
20	9.43	7.45	4.55	54.31	0.275	55.1534978

□

In recent years, the application of HÜGELSCHÄFFER models, as well as other mathematical models for describing the shape of eggs - ovoid forms (curves and surfaces) in the poultry industry and food engineering is on the rise, as evident from papers [2], [14], [15], [17], [20], [22] – [32].

The reason for the use of the HÜGELSCHÄFFER egg-shaped curve model is its presentation as a non-destructive oomorphological model when calculating the area of a planar curve  $\mathcal{A}$  and the volume  $V$  and the surface area  $S$  of a 3D egg surface [26], [30], [31]. Note that "oomorphological models" is used as a substitute for "egg-shaped models" [18]. The authors Narushin et al. used 2D digital images of egg contours to obtain concrete values for the parameters  $L = 2a$ ,  $B = 2b$  and  $w$ , [25], [26]. According to [26] the typical length of a hen egg varies between 5 and 7 cm, and the ratio of parameters  $B$  and  $L$  which is called the shape index  $SI = B/L$  varies between 0.70 and 0.78. Based on this data and digital imaging

of hen eggs Narushin et al. reached the conclusion that the parameter  $w$  has a minimum value of 0.021 and a maximum value of 0.249 ([26], see Table 1). Note that in paper [20] and [46], the authors consider other ranges for the values of parameters  $B$  and  $L$ .

The formulae from the second section of this paper produce correct results for the area of a planar curve  $\mathcal{A}$  – formula (5), the volume  $V$  – formula (11) and the surface area  $\mathcal{S}$  of a 3D egg surface – formula (12). In the following several numerical examples, a comparison of these formulae with approximative formulae from works [25], [26] and [27] is given.

**Numerical example 2:** In the paper [27], the case where  $L = 2a = 6\text{ cm}$ ,  $B/L = 0.775$  and  $w/L = 0.125$  was considered. In this case, the parameters  $a$ ,  $b$  and  $w$  for the HÜGELSCHÄFFER egg-shaped curve model have the following values:  $a = L/2 = 3\text{ cm}$ ,  $b = B/2 = 4.65/2 = 2.325\text{ cm}$  and  $w = L/8 = 0.75\text{ cm}$ . Based on the approximative formula given by the authors of [27]:  $Area = 0.118B^2 + 0.637LB + 0.014L^2$  the area of the plane curve  $Area = 20.827755\text{ cm}^2$  was obtained. The same authors, in paper [26], utilizing Simpson’s rule with three points, obtained an approximate value for the area of the plane curve  $\mathcal{A}_{Simpson} = 21.062316\text{ cm}^2$ . Comparing these values with the values obtained using formulae (6) and (7) as exact results:

$$\mathcal{A}_1 = 8.545026\dots\text{ cm}^2, \mathcal{A}_2 = 13.195026\dots\text{ cm}^2 \Rightarrow \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 = 21.740052\dots\text{ cm}^2,$$

we can conclude that

$$|\mathcal{A} - Area| = 0.912297\dots\text{ cm}^2 \quad \text{and} \quad |\mathcal{A} - \mathcal{A}_{Simpson}| = 0.677736\dots\text{ cm}^2. \quad \square$$

**Numerical example 3:** Based on data from Table 2 obtained by the authors of work [25] through digitally measuring 40 chicken eggs, the mean values of the relevant parameters are:  $L = 2a = 405.81\text{ pixels}$ ,  $B = 312.65\text{ pixels}$  and  $A = 98984.1\text{ pixels}$  i.e.  $A = 19.05\text{ cm}^2$ . Note that in [25] the following graphical conversion was used:  $72.09\text{ pixels}$  in  $1\text{ cm}$  and  $5197.03\text{ pixels}$  in  $1\text{ cm}^2$ . Furthermore, from formula  $Area = 0.118B^2 + 0.637LB + 0.014L^2$  of the aforementioned work, the value of the area of the plane curve  $Area = 94660.37552\text{ pixels}$  was obtained, i.e.  $Area = 18.214\text{ cm}^2$ .

Observing the equation  $\mathcal{A} = A$  it can be concluded that the respective HÜGELSCHÄFFER egg-shaped curve model has an unique value for the parameter  $w$  given by  $w_0 = 46.678275\dots\text{ pixels}$  i.e.  $w_0 = 0.6475\dots\text{ cm}$ . In the work [26] for the observed data, a mean value of parameter  $w = 0.12\text{ cm}$  is given. Based on the fact that  $|w_0 - w| > 0.5\text{ cm}$  it can be concluded that, for these values of parameters  $w$  and  $w_0$ , the contours of the respective egg-shaped curves, despite the area in the HÜGELSCHÄFFER models being the same, are, geometrically, significantly different.  $\square$

**Numerical example 4:** Based on data from Table 1 which authors of the work [45], Sedghi and Ghaderi, obtained by digitally measuring 177 hen eggs at

60 weeks of age, their average values of the relevant parameters are:  $L = 2a = 5.708 \text{ cm}$ ,  $B = 4.431 \text{ cm}$  and  $V = 57.458 \text{ cm}^3$  ([45], see Table 1). Using the following values in our egg-shaped surface model:  $a = L/2 = 2.854 \text{ cm}$ ,  $b = B/2 = 2.2155 \text{ cm}$  and  $V_{egg} = V (= 57.458 \text{ cm}^3)$ , then, according to formula (11) for the volume of the HÜGELSCHÄFFER egg-shaped surface model, an unique positive value of parameter  $w = 0.9138298 \dots \text{ cm}$  can be determined, for which  $V_{egg} = V$ . For these values of parameters  $a$ ,  $b$  and  $w$  the surface area of the HÜGELSCHÄFFER model is  $\mathcal{S}_{egg} = 73.61192 \text{ cm}^2$  which corresponds to some predicted means for  $S$  ([45], see Table 3, Code A-K).  $\square$

The numerical values in this paper were determined using the computer algebra system Maple. For the purposes of this paper, a web applet dedicated to all the numerical values considered here was also developed [21].

#### 4. CONCLUSION

Egg-shaped curves and surfaces have wide applications in engineering (air, civil, food). For the calculation of  $A$ ,  $V$  and  $S$  of egg-shaped curves and surfaces, mostly numerical approximations were used, such as the ones presented in works [17], [25], [28], [37], [45], [50], [53]. In this paper, we determined the formulae for calculating  $A$ ,  $V$  and  $S$  in a HÜGELSCHÄFFER model and we expect that the obtained result will be applicable in various engineering areas.

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