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MODIFICATION EXPONENTIAL EULER TYPE SPLINES DERIVED FROM APOSTOL-EULER NUMBERS AND POLYNOMIALS OF COMPLEX ORDER

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The purpose of this paper is to give formulas and Recurrence relations for the Apostol-Euler numbers and polynomials of order with complex numbers with the aid of the Euler operator and partial derivatives of the generating function. Relations among the these numbers and polynomials of neqative integer order, the beta-type rational functions, finite combinatorial sums, the Stirling numbers, and the Lah numbers are given. Finally, new classes of polynomials and modification exponential Euler type splines are constructed.

1. INTRODUCTION

More recently, using not only generating functions with their functional equations, but also other methods, many researches have studied several properties and relationships involving the higher-order of Apostol-Euler numbers and polynomials. The motivation of this article is to give not only several general properties and relationships including the Apostol-Euler numbers and polynomials of higher-order, the Stirling numbers of the second kind, and the Catalan numbers, but also to construct genearlized exponential Euler type splines. Some sepcial values and numerical examplaes of these results are given. Relations between the Apoatol-Euler polynomials and exponential Euler type splines are given.

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Isaac Jacob Schoenberg [**15**]-[**16**], who was known as father of splines, constructed the cardinal splines, the exponential splines, the exponential Euler type splines, and other splines. Splines have many application in several areas of the many different sciences involving computer geometric modeling, signal processing, data analysis, visualization, numerical simulation, probability, quadrature formulae, approximation theory, and other sciences.

In this section, we also present some well-known families of the special numbers and polynomials with their generating functions. These numbers and polynomials will be used in the next sections.

Let $\alpha, \lambda \in \mathbb{C}$. The Apostol-Euler numbers and polynomials of order α are respectively defined by

$$F_{\alpha}(t;\lambda) = \frac{2^{\alpha}}{(\lambda e^{t}+1)^{\alpha}} = \sum_{n=0}^{\infty} \mathcal{E}_{n}^{(\alpha)}(\lambda)\frac{t^{n}}{n!} \tag{1}$$

and

$$F_{\alpha}(t,x;\lambda) = e^{xt}F_{\alpha}(t;\lambda) = \sum_{n=0}^{\infty} \mathcal{E}_{n}^{(\alpha)}(x;\lambda)\frac{t^{n}}{n!}, \tag{2}$$

where $|t| < \pi$ when $\lambda = 1$; $|t| < |\log(-\lambda)|$ when $\lambda \neq 1$, and also $1^{\alpha} = 1$ (*cf.* [**5**, p. 253], [**10**], [**11**], [**12**], [**24**, p. 93]).

When $x = 0$, (2) reduces to (1). That is, $\mathcal{E}_{n}^{(\alpha)}(\lambda) = \mathcal{E}_{n}^{(\alpha)}(0;\lambda)$. Putting $\lambda = 1$ in (2), we have the Euler polynomials $E_{n}^{(\alpha)}(x)$ of order α. It clear that for $\alpha = 1$, we have the following Euler polynomials and numbers respectively

$$E_{n}(x) = E_{n}^{(1)}(x)$$

and

$$E_{n} = E_{n}(0)$$

(*cf.* [**1**]-[**24**, p. 93]).

Substituting $\alpha = 1$ into (1), we have the Apostol-Euler numbers

$$\mathcal{E}_{n}(\lambda) = \mathcal{E}_{n}^{(1)}(\lambda)$$

(*cf.* [**1**]-[**24**, p. 93]).

By using (1) and (2), one can easily get the following known formulas:

$$\mathcal{E}_{n}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\mathcal{E}_{k}^{(\alpha)}(\lambda) \tag{3}$$

and

$$\mathcal{E}_{n}^{(\beta+\gamma)}(\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_{k}^{(\beta)}(\lambda)\,\mathcal{E}_{n-k}^{(\gamma)}(\lambda) \tag{4}$$

(*cf.* [**11**], [**12**], [**24**, p. 93]).

Let λ be a complex number with $\lambda \neq 1$. The Frobenius-Euler numbers $H_n^{(\alpha)}(\lambda)$ are defined by means of the following generating function:

$$F_H(t;\lambda,\alpha) = \left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(\lambda)\frac{t^n}{n!}$$

(*cf.* [**2**], [**9**], [**16**], [**18**], [**22**], [**24**]).

Setting $\alpha = 1$ in (2), we have

$$2x^n = \lambda \sum_{j=0}^{n} \binom{n}{j} \sum_{v=0}^{j} \binom{j}{v} x^{j-v} \mathcal{E}_v(\lambda) + \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \mathcal{E}_j(\lambda). \tag{5}$$

When $x = 0$, the equation (5) reduces to the following known results:

$$\mathcal{E}_0(\lambda) = \frac{2}{\lambda+1}$$

and for $n > 1$, we have

$$\mathcal{E}_n(1;\lambda) = -\frac{1}{\lambda}\mathcal{E}_n(\lambda).$$

By using (1), for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we have

$$\mathcal{E}_n^{(\alpha)}(\lambda) = \frac{2^{\alpha}}{(\lambda+1)^{\alpha}} H_n^{(\alpha)}\left(-\frac{1}{\lambda}\right)$$

(*cf.* [**15**], [**17**], [**24**]). Substituting $\alpha = \lambda = 1$ into the above relation, the numbers $\mathcal{E}_n^{(1)}(1)$ reduces to the Euler numbers:

$$E_n = H_n(-1) = \mathcal{E}_n^{(1)}(1)$$

(*cf.* [**15**]).

By using (1), one has the following known formula:

$$\mathcal{E}_n^{(\alpha)}(\lambda) = \frac{\partial^n}{\partial t^n}\{F_{\alpha}(t;\lambda)\}\Big|_{t=0}. \tag{6}$$

Let $k \in \mathbb{N}_0$. The Stirling numbers of the second kind, $S_2(n,k)$ are defined by

$$F_S(t;k) = \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}$$

(*cf.* [**3**], [**10**], [**11**], [**18**], [**24**]).

We also need the following notations and definitions: The falling factorial is given by

$$x_{(n)} = \begin{cases} x(x-1)\dots(x-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

the rising factorial (the Pochhammer symbol) is given by

$$(x)^{(n)} = \begin{cases} x\,(x+1)\ldots(x+n-1)\,, & n \in \mathbb{N} \\ 1, & n = 0; \end{cases}$$

and using the Pochhammer symbol, we have

$$(x)^{(n)} = \sum_{k=0}^{n} |S_1(n,k)|\, x^k = \sum_{k=1}^{n} L(n,k)\, x_{(k)}, \tag{7}$$

where $L(n,k)$ the note the Lah numbers which are defined by:

$$L(n,k) \quad = \quad (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}$$

and

$$|L(n,k)| \quad = \quad \frac{n!}{k!}\binom{n-1}{k-1}$$

(*cf.* [**1**], [**4**], [**14**], [**22**], [**24**]).

We briefly summarize the results which will be given in next sections on newly constructed splines involving the Apostol-Euler numbers and polynomials of complex order.

In Section 2, using the Euler operator, we give some novel computational formulas for the numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$ the polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$.

In Section 3, we give some new formulas and recurrence relations with the aid of partial derivatives equations for the function $F_\alpha(t;\lambda)$.

In Section 4, we present some combinatorial sums involving beta-type rational functions, the Apostol-Euler numbers of order $-m$ and the Stirling numbers of the second kind.

In Section 5, we construct new classes polynomials and new family of modification exponential Euler type splines of degree n with order α. We give some properties of these polynomials and splines. After that, the last section of the article is the conclusion section.

2. COMPUTATIONAL FORMULAS FOR THE NUMBERS $\mathcal{E}_n^{(\alpha)}(\lambda)$ AND THE POLYNOMIALS $\mathcal{E}_n^{(\alpha)}(x;\lambda)$

In this section, we assume that $|\lambda| < 1$. By applying the Umbral calculus convention method to the generating function for the numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$ and binomial series in the equation (1), we present some novel computational formulas for the numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$. By using these formulas, we also give computational formulas for the polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$.

$$2^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}\lambda^{k}\sum_{n=0}^{\infty}\left(\mathcal{E}^{(\alpha)}\left(\lambda\right)+k\right)^{n}\frac{t^{n}}{n!}.$$

After equalizing the coefficients of t^n in the previous equation and making the necessary calculations, the following results are obtained:

For $n=0$, we have

$$2^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}\lambda^{k}\left(\mathcal{E}^{(\alpha)}\left(\lambda\right)+k\right)^{0},$$

which implies

$$\mathcal{E}_{0}^{(\alpha)}\left(\lambda\right)=\frac{2^{\alpha}}{\left(1+\lambda\right)^{\alpha}}.$$

For $n\geq 1$, we have

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\lambda^{k}\left(\mathcal{E}^{(\alpha)}\left(\lambda\right)+k\right)^{n}=0,$$

where we mention that after expanding binomial expansion in the previous equation, each index of $\left(\mathcal{E}^{(\alpha)}\left(\lambda\right)\right)^{n}$ is to be replaced by the corresponding suffix: $\mathcal{E}_{n}^{(\alpha)}\left(\lambda\right)$, which represented the Apostol-Euler numbers of order α. Therefore, we have

$$\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_{n-j}^{(\alpha)}\left(\lambda\right)\sum_{k=0}^{\infty}\binom{\alpha}{k}\lambda^{k}k^{j}=0, \tag{8}$$

where $|\lambda|<1$. By combining the following well-known Euler operator (*cf.* [**13**])

$$\vartheta=\lambda\frac{d}{d\lambda}$$

to the function $(1+\lambda)^{\alpha}$, we have the following well-known formula:

$$\vartheta^{j}\left\{(1+\lambda)^{\alpha}\right\}=\sum_{k=0}^{j}S_{2}(j,k)\lambda^{k}\frac{d^{k}}{d\lambda^{k}}\left\{(1+\lambda)^{\alpha}\right\} \tag{9}$$

after that combining the above formula with the equation (8), we arrive at the following theorem:

Theorem 1. *Let $n\in\mathbb{N}_0$. We have*

$$\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_{n-j}^{(\alpha)}\left(\lambda\right)\sum_{k=0}^{j}S_{2}(j,k)\lambda^{k}\frac{d^{k}}{d\lambda^{k}}\left\{(1+\lambda)^{\alpha}\right\}=0.$$

After performing the above similar operations, the following presumably well-known result is obtained for the Apostol-Euler polynomials, $\mathcal{E}_{n}^{(\alpha)}\left(x;\lambda\right)$:

Corollary 2. *Let $n \in \mathbb{N}_0$. We have*

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} \lambda^k \left(\mathcal{E}^{(\alpha)}(x;\lambda) + k\right)^n = 2^{\alpha} x^n,$$

where $|\lambda| < 1$.

By using the above result, the following corollary is also given for the Apostol-Euler numbers, $\mathcal{E}_n^{(\alpha)}(\lambda)$:

Corollary 3. *Let $n \in \mathbb{N}_0$. We have*

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} \lambda^k \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \sum_{v=0}^{j} \binom{j}{v} x^{j-v} \mathcal{E}_v^{(\alpha)}(\lambda) = 2^{\alpha} x^n,$$

where $|\lambda| < 1$.

Remark 4. *We note that Bayad et al.* [**1**], [**2**], *gave integral representation with analytic continuation for the generalized Hurwitz-Lerch zeta functions of complex number order. Using these functions, they gave some different formulas involving some interesting the Apostol-Euler-Nörlund polynomials of complex number order related to the numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$ and the polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$.*

3. FORMULAS AND RECURRENCE RELATIONS ARISING FROM PARTIAL DERIVATIVES EQUATIONS FOR THE FUNCTION $F_{\alpha}(t;\lambda)$

In this section, we can give successive partial derivatives of the function $F_{\alpha}(t;\lambda)$ with respect to t. By using these partial derivatives, we derive partial derivatives equations for the functions $F_{\alpha}(t;\lambda)$ and $F_{\alpha}(t,x;\lambda)$.

We now give the following successive partial derivatives of the function $F_{\alpha}(t;\lambda)$ with respect to t, we get the following partial derivatives equations for the function $F_{\alpha}(t;\lambda)$:

$$\frac{\partial}{\partial t}\{F_{\alpha}(t;\lambda)\} = -\frac{1}{2}\lambda\alpha F_{\alpha+1}(t,1;\lambda), \tag{10}$$

$$\frac{\partial^2}{\partial t^2}\{F_{\alpha}(t;\lambda)\} = -\frac{1}{2}\lambda\alpha F_{\alpha+1}(t,1;\lambda) + \frac{1}{4}\lambda^2\alpha(\alpha+1)F_{\alpha+2}(t,2;\lambda), \tag{11}$$

$$\begin{aligned}\frac{\partial^3}{\partial t^3}\{F_{\alpha}(t;\lambda)\} &= -\frac{1}{2}\lambda\alpha F_{\alpha+1}(t,1;\lambda) \\ &\quad +\frac{3}{4}\lambda^2\alpha(\alpha+1)F_{\alpha+2}(t,2;\lambda) \\ &\quad -\frac{1}{8}\lambda^3\alpha(\alpha+1)(\alpha+2)F_{\alpha+3}(t,3;\lambda)\end{aligned} \tag{12}$$

and

$$\frac{\partial^4}{\partial t^4}\{F_\alpha(t;\lambda)\} = -\frac{1}{2}\lambda\alpha F_{\alpha+1}(t,1;\lambda) + \frac{7}{4}\lambda^2\alpha(\alpha+1)F_{\alpha+2}(t,2;\lambda) - \frac{6}{8}\lambda^3\alpha(\alpha+1)(\alpha+2)F_{\alpha+3}(t,3;\lambda) + \frac{1}{16}\lambda^4\alpha(\alpha+1)(\alpha+2)(\alpha+3)F_{\alpha+4}(t,4;\lambda). \tag{13}$$

By using the above equations, we obtain the following recurrence relations, respectively:

$$\mathcal{E}_{n+1}^{(\alpha)}(\lambda) = -\frac{\lambda\alpha}{2}\mathcal{E}_n^{(\alpha+1)}(1;\lambda), \tag{14}$$

$$\mathcal{E}_{n+2}^{(\alpha)}(\lambda) = -\frac{\lambda\alpha}{2}\mathcal{E}_n^{(\alpha+1)}(1;\lambda) + \frac{1}{4}\lambda^2\alpha(\alpha+1)\mathcal{E}_n^{(\alpha+2)}(2;\lambda), \tag{15}$$

$$\mathcal{E}_{n+3}^{(\alpha)}(\lambda) = -\frac{\lambda\alpha}{2}\mathcal{E}_n^{(\alpha+1)}(1;\lambda) + \frac{3}{4}\lambda^2\alpha(\alpha+1)\mathcal{E}_n^{(\alpha+2)}(2;\lambda) - \frac{1}{8}\lambda^3\alpha(\alpha+1)(\alpha+2)\mathcal{E}_n^{(\alpha+3)}(3;\lambda) \tag{16}$$

and

$$\mathcal{E}_{n+4}^{(\alpha)}(\lambda) = -\frac{\lambda\alpha}{2}\mathcal{E}_n^{(\alpha+1)}(1;\lambda) + \frac{7}{4}\lambda^2\alpha(\alpha+1)\mathcal{E}_n^{(\alpha+2)}(2;\lambda) - \frac{6}{8}\lambda^3\alpha(\alpha+1)(\alpha+2)\mathcal{E}_n^{(\alpha+3)}(3;\lambda) + \frac{1}{16}\lambda^4\alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathcal{E}_n^{(\alpha+4)}(4;\lambda). \tag{17}$$

In order to understand how can the above formulas can be proved, we briefly explain the proof of equation (17). The proof of equation (14), (15), and (16) follow along the same ways as the proof of (17), and so we skip them.

Combining (1) with (13), we get

$$\sum_{n=4}^{\infty}\mathcal{E}_n^{(\alpha)}(\lambda)\frac{t^{n-4}}{(n-4)!} = -\frac{1}{2}\lambda\alpha\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha+1)}(1;\lambda)\frac{t^n}{n!} + \frac{7}{4}\lambda^2\alpha(\alpha+1)\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha+2)}(2;\lambda)\frac{t^n}{n!} - \frac{6}{8}\lambda^3\alpha(\alpha+1)(\alpha+2)\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha+3)}(3;\lambda)\frac{t^n}{n!} + \frac{1}{16}\lambda^4\alpha(\alpha+1)(\alpha+2)(\alpha+3)\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha+4)}(4;\lambda)\frac{t^n}{n!}.$$

After some elementary calculations and then comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at equation (17).

By examining partial derivatives equations from (14) to (17), we arrive at the following question:

$$\begin{aligned}\mathcal{E}_{n+d}^{(\alpha)}(\lambda) &= -\frac{\lambda\alpha}{2}\mathcal{E}_{n}^{(\alpha+1)}(1;\lambda)+x_1\frac{\lambda^2}{2^2}(\alpha)^{(2)}\mathcal{E}_{n}^{(\alpha+2)}(2;\lambda)\\ &+x_2\frac{\lambda^3}{2^3}(\alpha)^{(3)}\mathcal{E}_{n}^{(\alpha+3)}(3;\lambda)+\dots\\ &+x_k\frac{\lambda^{k+1}}{2^{k+1}}(\alpha)^{(k+1)}\mathcal{E}_{n}^{(\alpha+k+1)}(k+1;\lambda)+\dots\\ &+x_{d-1}\frac{\lambda^d}{2^d}(\alpha)^{(d)}\mathcal{E}_{n}^{(\alpha+d)}(d;\lambda).\end{aligned}\tag{18}$$

How can compute the coefficients x_1, x_2,…,x_{d-1} in the above equation?

In order to prove the above question, we give the partial derivative of the function $\mathcal{F}_\alpha(t;\lambda)$ with respect to t to derive the following higher order partial differential equation:

Theorem 5. *Let $d \in \mathbb{N}$. Then we have*

$$\frac{\partial^d}{\partial t^d}\{\mathcal{F}_\alpha(t;\lambda)\}=\sum_{j=1}^{d}(-1)^j\frac{\lambda^j}{2^j}(\alpha)^{(j)}S_2(d,j)F_{\alpha+j}(t,j;\lambda).\tag{19}$$

Proof. With the aid of the mathematical induction method, higher order partial differential equation formula (19) follows immediately from (10) to (13) and also (9). □

Substituting (7) into (19) yields the following higher order partial differential formula for the function $\mathcal{F}_\alpha(t;\lambda)$:

Corollary 6. *Let $d \in \mathbb{N}$. Then we have*

$$\frac{\partial^d}{\partial t^d}\{\mathcal{F}_\alpha(t;\lambda)\}=\sum_{j=1}^{d}(-1)^j\frac{\lambda^j}{2^j}S_2(d,j)F_{\alpha+j}(t,j;\lambda)\sum_{k=1}^{j}|L(j,k)|\,\alpha_{(k)}.\tag{20}$$

Theorem 7. *(Recurrence relation) Let $d \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_{n+d}^{(\alpha)}(\lambda)=\sum_{j=0}^{d}(-1)^{j+1}\left(\frac{\lambda}{2}\right)^{j+1}(\alpha)^{(j+1)}S_2(d,j+1)\mathcal{E}_{n}^{(\alpha+j+1)}(j+1;\lambda).\tag{21}$$

Proof. By combining (1) with (19) and (2), we obtain

$$\sum_{n=d}^{\infty}\mathcal{E}_{n}^{(\alpha)}(\lambda)\frac{t^{n-d}}{(n-d)!}=\sum_{j=1}^{d}(-1)^j\frac{\lambda^j}{2^j}(\alpha)^{(j)}S_2(d,j)\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(\alpha+j)}(j,\lambda)\frac{t^n}{n!}.$$

After some elementary calculations and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 8. *Proof of* (21) *is also given by the mathematical induction method, which follow along the same ways as the proof of* (19)*, and so we skip them.*

It is time to give solution of the question with the aid of (21). By comparing the coefficients of $\mathcal{E}_n^{(\alpha+k+1)}(k+1;\lambda)$ on right hand sides of the equation (21) and (18), we obtain

$$\begin{aligned} x_1 &= S_2(d,2), \\ x_2 &= S_2(d,3), \\ &\vdots \\ x_{d-1} &= S_2(d,d). \end{aligned}$$

Putting $\alpha = \frac{1}{2}$ in (21), we get

$$\mathcal{E}_{n+d}^{\left(\frac{1}{2}\right)}(\lambda) = \sum_{j=0}^{d} (-1)^{j+1} \left(\frac{\lambda}{2}\right)^{j+1} \left(\frac{1}{2}\right)^{(j+1)} S_2(d,j+1)\, \mathcal{E}_n^{\left(\frac{3}{2}+j\right)}(j+1;\lambda).$$

Since

$$\left(\frac{1}{2}\right)^{(j)} = \frac{(j+1)!}{4^j} C_j,$$

where C_j denotes the well-known Catalan numbers, which are given by

$$C_j = \frac{1}{j+1}\binom{2j}{j}, \tag{22}$$

after some elementary calculation, we obtain the following corollary:

Corollary 9. *Let $d \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_{n+d}^{\left(\frac{1}{2}\right)}(\lambda) = \sum_{j=0}^{d} (-1)^{j+1} \left(\frac{\lambda}{2}\right)^{j+1} \frac{(j+2)!}{4^{j+1}} C_{j+1} S_2(d,j+1)\, \mathcal{E}_n^{\left(j+\frac{3}{2}\right)}(j+1;\lambda). \tag{23}$$

By using equations (22) and (23), we get

$$\mathcal{E}_{n+d}^{\left(\frac{1}{2}\right)}(\lambda) = \sum_{j=0}^{d} (-1)^{j+1} \left(\frac{\lambda}{2}\right)^{j+1} \frac{(j+1)!}{4^{j+1}} \binom{2j+2}{j+1} S_2(d,j+1)\, \mathcal{E}_n^{\left(j+\frac{3}{2}\right)}(j+1;\lambda).$$

When $n = 0$, equation (23) reduces to the following result:

Corollary 10. *Let $d \in \mathbb{N}$. Then we have*

$$\mathcal{E}_{d}^{\left(\frac{1}{2}\right)}(\lambda) = \sum_{j=1}^{d} (-1)^{j} \lambda^{j} \frac{(j+1)!}{2^{2j-\frac{1}{2}} (\lambda+1)^{j+\frac{1}{2}}} C_j S_2(d,j).$$

Theorem 11. *Let $d \in \mathbb{N}$. Then we have*

$$\mathcal{E}_d^{(\alpha)}(\lambda) = 2^{\alpha} \sum_{j=1}^{d} (-1)^j \lambda^j S_2(d,j) \frac{(\alpha)^{(j)}}{(\lambda+1)^{\alpha+j}}. \tag{24}$$

Proof. It is known that

$$\frac{\partial^d}{\partial t^d} \{F_\alpha(t;\lambda)\}|_{t=0} = \mathcal{E}_d^{(\alpha)}(\lambda). \tag{25}$$

Substituting $t = 0$ into (19) and combining the final equation with (25), we get

$$\frac{\partial^d}{\partial t^d} \{F_\alpha(t;\lambda)\}|_{t=0} = \mathcal{E}_d^{(\alpha)}(\lambda) = \sum_{j=1}^{d} (-1)^j \frac{\lambda^j}{2^j} (\alpha)^{(j)} S_2(d,j) F_{\alpha+j}(0,j;\lambda). \tag{26}$$

Substituting $t = 0$ into (1) and assuming $0^0 = 1$, we have

$$F_{\alpha+j}(0,j;\lambda) = \frac{2^{\alpha+j}}{(\lambda+1)^{\alpha+j}} = \mathcal{E}_0^{(\alpha+j)}(\lambda).$$

Combining the above equation with (26), we get

$$\mathcal{E}_d^{(\alpha)}(\lambda) = \sum_{j=1}^{d} (-1)^j \frac{\lambda^j}{2^j} (\alpha)^{(j)} S_2(d,j) \mathcal{E}_0^{(\alpha+j)}(\lambda).$$

After some calculations, we arrive at the desired result. □

Remark 12. *Here we note that different proof of the equation* (24) *was also given by* [**10**] *and see also* [**24**].

For $n = 0,1,2,3,4,5$, and $\alpha \in \mathbb{C}$, using (24), we give few values of the numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$ as follows:

$$\mathcal{E}_0^{(\alpha)}(\lambda) = \frac{2^\alpha}{(\lambda+1)^\alpha},$$

$$\mathcal{E}_1^{(\alpha)}(\lambda) = \frac{-\alpha\lambda 2^\alpha}{(\lambda+1)^{\alpha+1}},$$

$$\mathcal{E}_2^{(\alpha)}(\lambda) = \frac{-\alpha\lambda 2^\alpha}{(\lambda+1)^{\alpha+1}} + \frac{\alpha(\alpha+1)\lambda^2 2^\alpha}{(\lambda+1)^{\alpha+2}},$$

$$\mathcal{E}_3^{(\alpha)}(\lambda) = \frac{-\alpha\lambda 2^\alpha}{(\lambda+1)^{\alpha+1}} + \frac{3\alpha(\alpha+1)\lambda^2 2^\alpha}{(\lambda+1)^{\alpha+2}} - \frac{\alpha(\alpha+1)(\alpha+2)\lambda^3 2^\alpha}{(\lambda+1)^{\alpha+3}},$$

$$\begin{aligned}
\mathcal{E}_4^{(\alpha)}(\lambda) &= \frac{-\alpha\lambda 2^{\alpha}}{(\lambda+1)^{\alpha+1}} + \frac{7\alpha(\alpha+1)\lambda^2 2^{\alpha}}{(\lambda+1)^{\alpha+2}} - \frac{6\alpha(\alpha+1)(\alpha+2)\lambda^3 2^{\alpha}}{(\lambda+1)^{\alpha+3}} \\
&+ \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\lambda^4 2^{\alpha}}{(\lambda+1)^{\alpha+4}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_5^{(\alpha)}(\lambda) &= \frac{-\alpha\lambda 2^{\alpha}}{(\lambda+1)^{\alpha+1}} + \frac{15\alpha(\alpha+1)\lambda^2 2^{\alpha}}{(\lambda+1)^{\alpha+2}} - \frac{25\alpha(\alpha+1)(\alpha+2)\lambda^3 2^{\alpha}}{(\lambda+1)^{\alpha+3}} \\
&\quad + \frac{10\alpha(\alpha+1)(\alpha+2)(\alpha+3)\lambda^4 2^{\alpha}}{(\lambda+1)^{\alpha+4}} \\
&\quad - \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\lambda^5 2^{\alpha}}{(\lambda+1)^{\alpha+5}},
\end{aligned}$$

and so on.

For $n = 0, 1$ and using (24), we give few values of the numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$ as follows:

$$\begin{aligned}
\mathcal{E}_0^{(0)}(\lambda) &= 1, \\
\mathcal{E}_0^{(1)}(\lambda) &= \mathcal{E}_0(\lambda) = \frac{2}{\lambda+1}, \\
\mathcal{E}_0^{(2)}(\lambda) &= \frac{4}{(\lambda+1)^2}, \\
\mathcal{E}_0^{(3)}(\lambda) &= \frac{8}{(\lambda+1)^3}, \cdots,
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_0^{(-1)}(\lambda) &= \frac{\lambda+1}{2}, \\
\mathcal{E}_0^{(-2)}(\lambda) &= \frac{(\lambda+1)^2}{4}, \\
\mathcal{E}_0^{(-3)}(\lambda) &= \frac{(\lambda+1)^3}{8}, \cdots,
\end{aligned}$$

$$\mathcal{E}_0^{\left(\frac{2}{3}\right)}(\lambda) = \frac{2^{\frac{2}{3}}}{(\lambda+1)^{\frac{2}{3}}},$$

$$\begin{aligned}
\mathcal{E}_1^{(0)}(\lambda) &= 0, \\
\mathcal{E}_1(\lambda) &= \frac{-2\lambda}{(\lambda+1)^2}, \\
\mathcal{E}_1^{(2)}(\lambda) &= \frac{-8\lambda}{(\lambda+1)^3}, \\
\mathcal{E}_1^{(3)}(\lambda) &= \frac{-24\lambda}{(\lambda+1)^4},
\end{aligned}$$

and so on.

Example 13. *Substituting $\alpha = i$ and $\lambda = -1+i$ with $i^2=-1$ into (24), we get*

$$\mathcal{E}_d^{(i)}(i-1) = 2^i \sum_{j=1}^{d} (-1)^j (i-1)^j S_2(d,j) \frac{(i)^{(j)}}{i^{i+j}}.$$

Since

$$(i)^{(j)} = \sum_{k=0}^{j} |S_1(j,k)| \left(\cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)\right) = \sum_{k=0}^{j} |S_1(j,k)|\, e^{\frac{k\pi i}{2}}$$

and

$$(i-1)^j = e^{j\left(\ln\sqrt{2}+\frac{3\pi i}{4}\right)},$$

we obtain

$$\begin{aligned}\mathcal{E}_d^{(i)}(i-1) &= \sum_{j=1}^{d}\sum_{k=0}^{j} (-1)^j S_2(d,j)\, |S_1(j,k)|\, e^{\left(j\ln\sqrt{2}+\frac{\pi}{2}\right)} \\ &\quad \times \left(\cos\left(\frac{(2k+j)\pi}{4}\right) + i\sin\left(\frac{(2k+j)\pi}{4}\right)\right).\end{aligned}$$

4. COMBINATORIAL SUMS INVOLVING SPECIAL NUMBERS AND FUNCTIONS

In this section, we give some combinatorial sums involving beta-type rational functions, the Apostol-Euler numbers of order $-m$ and the Stirling numbers of the second kind.

Substituting $\alpha = -m$ $(m \in \mathbb{N})$ into (24), we get

$$\mathcal{E}_d^{(-m)}(\lambda) = 2^{-m} \sum_{j=1}^{d} (-1)^j \lambda^j S_2(d,j) \frac{(-m)^{(j)}}{(\lambda+1)^{-m+j}}. \tag{27}$$

Combining (27) with the following beta-type rational functions

$$\mathfrak{M}_{j,m}(\lambda) = \lambda^j (\lambda+1)^{m-j}, \tag{28}$$

where $j,m \in \mathbb{N}_0$ (*cf.* [**19**, Definition 1.1]), we arrive at the following result.

Corollary 14. *Let $d,m \in \mathbb{N}$. Then we have*

$$\mathcal{E}_d^{(-m)}(\lambda) = 2^{-m} \sum_{j=1}^{d} \binom{m}{j} j!\, S_2(d,j)\, \mathfrak{M}_{j,m}(\lambda). \tag{29}$$

It is known that $\mathcal{E}_d^{(-m)}(\lambda)$ is a rational function with variable λ. Integrating both sides of the equation (29) from -1 to 0 with respect to λ, we obtain

$$\int_{-1}^{0} \mathcal{E}_d^{(-m)}(\lambda)\, d\lambda = 2^{-m} \sum_{j=1}^{d} \binom{m}{j} j! S_2(d,j) \int_{-1}^{0} \mathfrak{M}_{j,m}(\lambda) d\lambda.$$

Combining the above equation with the following integral formulas for the function $\mathfrak{M}_{j,m}(\lambda)$, which given by Simsek [**19**, Eqs. (18)-(19)],

$$\int_{-1}^{0} \mathfrak{M}_{j,m}(\lambda) d\lambda = \sum_{k=0}^{m-j} (-1)^{m-k} \binom{m-j}{k} \frac{1}{m+1-k}$$

and

$$\int_{-1}^{0} \mathfrak{M}_{j,m}(\lambda) d\lambda = (-1)^j \frac{1}{(m+1)\binom{m}{j}},$$

we get

$$\int_{-1}^{0} \mathcal{E}_d^{(-m)}(\lambda)\, d\lambda = 2^{-m} \sum_{j=1}^{d} \binom{m}{j} j! S_2(d,j) \sum_{k=0}^{m-j} (-1)^{m-k} \binom{m-j}{k} \frac{1}{m+1-k} \tag{30}$$

and

$$\int_{-1}^{0} \mathcal{E}_d^{(-m)}(\lambda)\, d\lambda = \frac{1}{2^m(m+1)} \sum_{j=1}^{d} (-1)^j j! S_2(d,j). \tag{31}$$

By substituting the following well-known formula:

$$\sum_{j=1}^{d} (-1)^j j! S_2(d,j) = (-1)^d$$

(*cf.* [**6**, P. 117, Eq. (9.12)]), into (31), we get

$$\int_{-1}^{0} \mathcal{E}_d^{(-m)}(\lambda)\, d\lambda = \frac{(-1)^d}{2^m(m+1)}.$$

Combining (30) and (31) yields the following theorem:

Theorem 15. *Let $d, m \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{d} \sum_{k=0}^{m-j} (-1)^{m-k} \binom{m}{j} \binom{m-j}{k} \frac{j! S_2(d,j)}{m+1-k} = \frac{(-1)^d}{m+1}.$$

Substituting $\alpha=\beta+\gamma$ into (24), we get

$$\mathcal{E}_d^{(\beta+\gamma)}(\lambda)=2^{\beta+\gamma}\sum_{j=1}^{d}(-1)^j\lambda^jS_2(d,j)\frac{(\beta+\gamma)^{(j)}}{(\lambda+1)^{\beta+\gamma+j}}.$$

By substituting the following Vandermonde's convolution formula

$$(\beta+\gamma)^{(j)}=\sum_{v=0}^{j}\binom{j}{v}(\beta)^{(v)}(\gamma)^{(j-v)}$$

into the above equation, we get the following corollary:

Corollary 16. *Let $\beta,\gamma\in\mathbb{C}$, $d\in\mathbb{N}$ and $n\in\mathbb{N}_0$. Then we have*

$$\mathcal{E}_d^{(\beta+\gamma)}(\lambda)=2^{\beta+\gamma}\sum_{j=1}^{d}\sum_{v=0}^{j}(-1)^j\binom{j}{v}\frac{(\beta)^{(v)}(\gamma)^{(j-v)}\lambda^jS_2(d,j)}{(\lambda+1)^{\beta+\gamma+j}}. \tag{32}$$

By combining (4) with (32) yields the following theorem:

Theorem 17. *Let $\beta,\gamma\in\mathbb{C}$, $d\in\mathbb{N}$ and $n\in\mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k^{(\beta)}(\lambda)\,\mathcal{E}_{n-k}^{(\gamma)}(\lambda)\\ =&\sum_{j=1}^{d}\sum_{v=0}^{j}(-1)^j\binom{j}{v}2^{-j}(\beta)^{(v)}(\gamma)^{(j-v)}\lambda^jS_2(d,j)\,\mathcal{E}_0^{(\beta+\gamma+j)}(\lambda).\end{aligned}$$

Combining (3) with (24), we arrive at the following theorem:

Theorem 18. *Let $n\in\mathbb{N}_0$. Then we have*

$$\mathcal{E}_n^{(\alpha)}(x,\lambda)=2^{\alpha}\sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^j\binom{n}{k}\frac{\lambda^jS_2(k,j)(\alpha)^{(j)}}{(\lambda+1)^{\alpha+j}}x^{n-k}. \tag{33}$$

Remark 19. *Here we note that different proof of the equation* (33) *was also given by* [**10**] *and see also* [**24**].

5. A NEW CLASS OF MODIFICATION EXPONENTIAL EULER TYPE SPLINES OF DEGREE n ORDER α

In this section, we construct a new class of exponential Euler type splines of degree n and order α ($\alpha\in\mathbb{C}$).

Let α and λ be arbitrary real or complex parameters. We define monic polynomials $\mathfrak{u}_n^{(\alpha)}(x;\lambda)$ with real variable x as follows:

$$h(t,x;\lambda,\alpha)=\left(\frac{1+\lambda}{\lambda e^t+1}\right)^{\alpha}e^{tx}=\sum_{n=0}^{\infty}\mathfrak{u}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{34}$$

where $|t|<\pi$ when $\lambda=1$; $|t|<|\log(-\lambda)|$ when $\lambda\neq 1$, and also $1^{\alpha}=1$. When $\alpha=0$, we have

$$\mathfrak{u}_n^{(0)}(x;\lambda)=x^n.$$

We have the following functional equations:

$$h(t,x;\lambda,\alpha)=\left(\frac{1+\lambda}{2}\right)^{\alpha}F_{\alpha}(t,x;\lambda)$$

and

$$h(t,0;\lambda,\alpha)=\left(\frac{1+\lambda}{2}\right)^{\alpha}F_{\alpha}(t;\lambda).$$

Using the above equations, we have

$$\mathfrak{u}_n^{(\alpha)}(x;\lambda)=\left(\frac{1+\lambda}{2}\right)^{\alpha}\mathcal{E}_n^{(\alpha)}(x;\lambda) \tag{35}$$

and

$$\mathfrak{u}_n^{(\alpha)}(\lambda)=\left(\frac{1+\lambda}{2}\right)^{\alpha}\mathcal{E}_n^{(\alpha)}(\lambda).$$

A relation between the polynomials $\mathfrak{u}_n^{(\alpha)}(x;\lambda)$ and $H_n^{(\alpha)}(x;\lambda)$ is given by

$$\mathfrak{u}_n^{(\alpha)}(x;\lambda)=H_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right).$$

We also define

$$\mathfrak{q}_n(\lambda;\alpha)=(1+\lambda)^n\,\mathfrak{u}_n^{(\alpha)}(\lambda). \tag{36}$$

By using (36), we give few values of the polynomial

$$\mathfrak{q}_0(\lambda;\alpha)=1,$$
$$\mathfrak{q}_1(\lambda;\alpha)=-\alpha\lambda,$$
$$\mathfrak{q}_2(\lambda;\alpha)=-\alpha\lambda(\lambda+1)+\alpha(\alpha+1)\lambda^2,$$
$$\mathfrak{q}_3(\lambda;\alpha)=-\alpha\lambda(\lambda+1)^2+3\alpha(\alpha+1)\lambda^2(\lambda+1)-\alpha(\alpha+1)(\alpha+2)\lambda^3,$$

$$\begin{aligned}\mathfrak{q}_4(\lambda;\alpha)\;=\;&-\alpha\lambda(\lambda+1)^3+7\alpha(\alpha+1)\lambda^2(\lambda+1)^2-6\alpha(\alpha+1)(\alpha+2)\lambda^3(\lambda+1)\\&+\alpha(\alpha+1)(\alpha+2)(\alpha+3)\lambda^4,\end{aligned}$$

$$\begin{aligned}\mathfrak{q}_5(\lambda;\alpha) &= -\alpha\lambda(\lambda+1)^4+15\alpha(\alpha+1)\lambda^2(\lambda+1)^3\\ &\quad -25\alpha(\alpha+1)(\alpha+2)\lambda^3(\lambda+1)^2\\ &\quad +10\alpha(\alpha+1)(\alpha+2)(\alpha+3)\lambda^4(\lambda+1)\\ &\quad -\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\lambda^5,\end{aligned}$$

and so on.

The $\mathfrak{q}_n(\lambda;\alpha)$ polynomial is related to beta-type rational functions. That is,

$$\mathfrak{q}_0(\lambda;\alpha)=\mathfrak{M}_{0,0}(\lambda),$$

$$\mathfrak{q}_1(\lambda;\alpha)=-\alpha\mathfrak{M}_{1,1}(\lambda),$$

$$\mathfrak{q}_2(\lambda;\alpha)=-\alpha\mathfrak{M}_{1,2}(\lambda)+\alpha(\alpha+1)\mathfrak{M}_{2,2}(\lambda),$$

$$\mathfrak{q}_3(\lambda;\alpha)=-\alpha\mathfrak{M}_{1,3}(\lambda)+3\alpha(\alpha+1)\mathfrak{M}_{2,3}(\lambda)-\alpha(\alpha+1)(\alpha+2)\mathfrak{M}_{3,3}(\lambda),$$

$$\begin{aligned}\mathfrak{q}_4(\lambda;\alpha) &= -\alpha\mathfrak{M}_{1,4}(\lambda)+7\alpha(\alpha+1)\mathfrak{M}_{2,4}(\lambda)-6\alpha(\alpha+1)(\alpha+2)\mathfrak{M}_{3,4}(\lambda)\\ &\quad +\alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathfrak{M}_{4,4}(\lambda),\end{aligned}$$

$$\begin{aligned}\mathfrak{q}_5(\lambda;\alpha) &= -\alpha\mathfrak{M}_{1,5}(\lambda)+15\alpha(\alpha+1)\mathfrak{M}_{2,5}(\lambda)-25\alpha(\alpha+1)(\alpha+2)\mathfrak{M}_{3,5}(\lambda)\\ &\quad +10\alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathfrak{M}_{4,5}(\lambda)\\ &\quad -\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\mathfrak{M}_{5,5}(\lambda),\end{aligned}$$

and so on.

In the above equations, we observe that in the definition $\mathfrak{M}_{j,m}(\lambda)$, $j=1,2,\ldots,m$. Thus the $\mathfrak{M}_{j,m}(\lambda)$ reduces to the beta polynomials of degree n, for detail see [**19**, Definition 1.2.]. Therefore, using the mathematical induction method, we arrive the following theorem involving explicit formula for the polynomials $\mathfrak{q}_n(\lambda;\alpha)$:

Theorem 20. *Let $n\in\mathbb{N}_0$. Then we have*

$$\mathfrak{q}_n(\lambda;\alpha)=\sum_{j=1}^{n}(-1)^j S_2(n,j)(\alpha)^{(n)}\mathfrak{M}_{j,n}(\lambda).$$

Let $n\in\mathbb{N}_0$. We define the following a new class of modification exponential Euler type splines of degree n with order α:

$$Y(x;\lambda;n,\alpha)=\frac{\mathfrak{u}_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right)}{\mathfrak{u}_n^{(\alpha)}\left(-\frac{1}{\lambda}\right)}. \tag{37}$$

We define

$$s_n(x;\lambda;\alpha)=Y(x;\lambda;n,\alpha) \tag{38}$$

if $0\le x<1$. Thus

$$Y(x;\lambda;n,\alpha)=\frac{H_n^{(\alpha)}(x,\lambda)}{H_n^{(\alpha)}(\lambda)}. \tag{39}$$

5.1 Some properties of the $Y(x; \lambda; n, \alpha)$

Here, we give some properties of the $Y(x; \lambda; n, \alpha)$. According to the equation (35), we have

$$Y(x; \lambda; n, \alpha) \in C^{n-1}(\mathbb{R})$$

and

$$Y(0; \lambda; n, \alpha) = 1.$$

Remark 21. *When $\alpha = 1$, the equation* (39) *reduces to the following exponential Euler splines of degree n:*

$$s_n(x; u) = Y(x; u; n, 1).$$

According to Schoneberg's work [**15**] *and* [**16**]*, extending definition in* (38) *to all real numbers x with the aid of the following functional equation*

$$s_n(x + 1; u) = us_n(x; u).$$

Since $s_n(x; u) \in C^{n-1}(\mathbb{R})$, it is clear to see that if $0 \leq x < 1$, then the polynomial

$$s_n(x; u) = \frac{H_n(x; u)}{H_n(0; u)}$$

is a member of the class of ordinary of order n, that is exponential Euler splines of degree n with knots at the integer v. For $v \in \mathbb{Z}$, $s_n(v; u) = u^v$ (cf. for detail, see [**15**]*,* [**16**]*).*

6. CONCLUSION

In this article, using functional equations and partial derivatives equations of generating functions for the Apostol-Euler numbers and polynomials of complex order, some new and applicable formulas and relations were given. In addition to these, partial derivative equations including the Stirling numbers and the Lah numbers of the second kind were found with the help of successive partial derivatives of generating functions for these numbers and polynomials. By using these partial differential equations, novel recurrence relations for the Apostol-Euler numbers were given. With these aid of relations, the relations among beta type rational functions, finite combinatorial sums, and the Stirling numbers were also found. In addition, new polynomial classes containing Stirling numbers and beta type rational functions and exponential Euler type splines of degree n involving the Apostol-Euler numbers and polynomials of complex order were constructed. Then, some properties of these polynomials and splines were studied. The results of this article were examined in relation to previous studies.

Since the results obtained in this article have the potential to contribute to fields such as Applicable Analysis, Differential and Difference Equations, Special Functions, Combinatorics and also Discrete Mathematics, we will investigate future applications of the exponential the exponential Euler type splines.

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