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ANISOTROPIC DISCRETE BOUNDARY VALUE PROBLEMS

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For an anisotropic discrete nonlinear problem with variable exponent, we demonstrate both the existence and multiplicity of nontrivial solutions in this study. The variational principle and critical point theory are the key techniques employed here.

1. INRODUCTION

For the following discrete problem, our primary focus in this research is on existence and multiplicity results,

$$(P) \quad \begin{cases} -\Delta(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) = g(t,u(t)), & t \in [1,N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

where $N \geq 2$ is an integer, $[1,N]_{\mathbb{Z}}$ is the discrete interval $\{1,2,3...,N\}$, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$ and $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, i.e. for any fixed $t \in [1,N]_{\mathbb{Z}}$ a function $g(t,.)$ is continuous. For the function $p : [0,N]_{\mathbb{Z}} \longrightarrow [2,\infty[$ denote

$$p^+ = \max_{t\in[0,N]_{\mathbb{Z}}} p(t) \quad \text{and} \quad p^- = \min_{t\in[0,N]_{\mathbb{Z}}} p(t).$$

As usual, a solution of (P) is a function $u : [0,N+1]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (P).

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We would like to point out that issue (P) is a discrete equivalent of the variable exponent anisotropic problem

$$(1) \qquad \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = g(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{cases}$$

Where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is bounded domain with smooth boundary, $g \in C\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ is given function that satisfy certain properties, $p_i(x)$ are a continuous functions on $\overline{\Omega}$, with $p_i(x) \geq 2$ for $(i,x) \in [1,N]_{\mathbb{Z}} \times \Omega$.

Difference equations naturally emerge as discrete analogues, numerical solutions, and delay differential equations that model a variety of distinct processes in statistics, computing, mechanical engineering, control systems, artificial or biological neural networks, and economics (for example [**1, 16, 17**]). The existence and multiplicity solutions to boundary value issues for difference equations with the $p(.)$-Laplacian operator have recently attracted more attention. Fixed point theorems in cones are typically used to get this results on this issue (see [**2, 3, 18, 19**] and references therein). Upper and lower solution method is yet another instrument in the study of nonlinear difference equations (see, for instance [**5, 10, 11**] and references therein). It is widely recognized that, critical point theory, variational methods and also monotonicity methods are powerful tools to investigate the existence and multiplicity of solutions of various problems, see the monographs [**4, 6, 7, 8, 13, 14, 20, 22, 23, 25, 26, 27, 28, 29, 30**].

The authors S. Heidarkhani et al. [**26**] proved the existence of at least one solution for the following problem

$$(P_\alpha) \quad \begin{cases} -\Delta(\alpha(t)|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) = f(t,u(t)), \quad t \in [1,N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

where $\alpha : [1, N+1]_{\mathbb{Z}} \longrightarrow (0,+\infty)$,and its parametric version by employing Recceri's variational principe, requiring an algebraic condition on the nonlinear term f. In [**27**], the established the existence of at least three solutions by applying Bonanno's theorem, for the problem $(P^{f,g}_{\lambda,\mu})$ in which two parameters are involved, where

$$(P^{f,g}_{\lambda,\mu}) \begin{cases} -\Delta(\alpha(t)|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) = \lambda f(t,u(t)) + \mu g(t,u(t)), t \in [1,N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

with $\alpha > 0$, $\mu \geq 0$ and $f, g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions.

The objective of the present paper is to establish first the existence of at least two solutions of (P), and leter, the existence of $2N$ nontrivial solutions by employing (Mountain Pass Lemma [**24**]) and (Lemma 2.11 [**2**]). Precisely, in Theorem 1 we establish the existence of at least two non-negative and non-positive solutions for the problem (P), by employing variational methods, requiring an algebraic condition on the nonlinear term g. Also in Theorem 3, the existence of two nontrivial solutions are established by assuming the suitable conditions on nonlinear terms.
Finally in Theorem 5, we obtain $2N$ nontrivial solutions.

We present Examples 2, 4, 6 in which the hypotheses of Theorems 1, 3, 5 are fulfilled, respectively.

Through the use of min-max techniques and the Mountain Pass Theorem, we will investigate the existence and multiplicity of nontrivial solutions to the equation (P).

Let

$$G(t,x)=\int_0^x g(t,s)ds \quad \text{for} \quad (t,x)\in[1,N]_{\mathbb{Z}}\times\mathbb{R}.$$

We set the following conditions in order to state our main findings:

(G_1) There exists η with $\eta<\dfrac{1}{\left(\sqrt{N}(N+1)\right)^{p^-}}$ such that

$$\limsup_{|x|\to\infty}\frac{p^+G(t,x)}{|x|^{p^-}}\leq\eta,\ \forall t\in[1,N]_{\mathbb{Z}}.$$

(G_2) $\displaystyle\lim_{|x|\to\infty}\left(G(t,x)-\frac{p^-}{(p^+)^2}\lambda_1|x|^{p^+}\right)=-\infty,\quad\forall t\in[1,N]_{\mathbb{Z}}$ where

$$\lambda_1=\inf\left\{\frac{\sum_{t=1}^{N+1}|\Delta u(t-1)|^{p^-}}{\sum_{t=1}^{N}|u(t)|^{p^+}}\ \Big|\ u\in H_N\diagdown\{0\}\right\}, \tag{2}$$

with

$$H_N=\{u:[0,N+1]_{\mathbb{Z}}\longrightarrow\mathbb{R}\ \mid u(0)=u(N+1)=0\ \}, \tag{3}$$

It is easy to see that $0<\lambda_1<\infty$.

(G_3) There exists $\delta>2^{p^+}(N+1)^{\frac{p^+}{2}}$ such that

$$\liminf_{|x|\to\infty}\frac{p^-G(t,x)}{|x|^{p^+}}\geq\delta,\ \forall t\in[1,N]_{\mathbb{Z}}.$$

(G_4) $\displaystyle\lim_{|x|\to 0}\frac{G(t,x)}{|x|^{p^+}}=0,\ \ \forall t\in[1,N]_{\mathbb{Z}}.$

(G_5) $\displaystyle G_\star=\liminf_{x\to 0}\frac{p^-G(t,x)}{|x|^{p^-}}>2^{p^-}(N+1)^{\frac{p^-}{2}},\ \forall t\in[1,N]_{\mathbb{Z}}.$

(G_6) $g(t,x)$ is odd in x, i.e., $g(t,-x)=-g(t,x)$ for $(t,x)\in[1,N]_{\mathbb{Z}}\times\mathbb{R}$.

The following theorems are the key findings of this study.

Theorem 1. *Suppose that* (G_1) *and* $g(t,0)=0$ *for any* $t\in[1,N]_{\mathbb{Z}}$*, hence, there are at least two non-negative and non-positive solutions to the problem* (P)*.*

Example 2. *Put* $g:[1,N]_{\mathbb{Z}}\times\mathbb{R}\longrightarrow\mathbb{R}$ *by the formula*

$$g(t,x)=\frac{p^-|\sin t|}{2p^+\left(\sqrt{N}(N+1)\right)^{p^-}}|x|^{p^--2}x.$$

Clearly,

$$G(t,x)=\frac{|\sin t|}{2p^+\left(\sqrt{N}(N+1)\right)^{p^-}}|x|^{p^-},\ \textit{for all } (t,x)\in[1,N]_{\mathbb{Z}}\times\mathbb{R}.$$

It's simple to observe that G *satisfy the condition* (G_1) *with* $\eta=\dfrac{1}{2\left(\sqrt{N}(N+1)\right)^{p^-}}$ *and* $g(t,0)=0,\ \forall t\in[1,N]_{\mathbb{Z}}$*.*

Theorem 3. *Suppose that* (G_2) *and* (G_5) *hold, then the problem* (P) *has at least two nontrivial solutions.*

Example 4. *Consider the continuous function* $g:[1,N]_{\mathbb{Z}}\times\mathbb{R}\longrightarrow\mathbb{R}$ *represented by the following formula*

$$g(t,x)=2^{p^-}(N+1)^{\frac{p^-}{2}}e^t|x|^{p^--2}x.$$

Clearly, we have

$$G(t,x)=\frac{2^{p^-}}{p^-}(N+1)^{\frac{p^-}{2}}e^t|x|^{p^-},\ \textit{for all } (t,x)\in[1,N]_{\mathbb{Z}}\times\mathbb{R}.$$

Direct calculations give

$$\lim_{|x|\to\infty}\left(G(t,x)-\frac{p^-}{(p^+)^2}\lambda_1|x|^{p^+}\right)=-\infty$$

and

$$\liminf_{x\to 0}\frac{p^-G(t,x)}{|x|^{p^-}}=2^{p^-}(N+1)^{\frac{p^-}{2}}e^t>2^{p^-}(N+1)^{\frac{p^-}{2}}.$$

Thus G *satisfies the conditions* (G_2) *and* (G_5)*.*

Theorem 5. *Assume that* (G_3)*,* (G_4) *and* (G_6) *are satisfied, then the problem* (P) *has at least* $2N$ *nontrivial solutions.*

Example 6. *Take the function* $g:[1,N]_{\mathbb{Z}}\times\mathbb{R}\longrightarrow\mathbb{R}$ *given by*

$$g(t,x)=\begin{cases}|\cos t|(1+p^+\ln|x|)|x|^{p^+-2}x, & |x|>1, t\in[1,N]_{\mathbb{Z}},\\ |\cos t||x|^{p^+-1}x, & |x|\le 1, t\in[1,N]_{\mathbb{Z}}.\end{cases}$$

By the expression of g, we have

$$G(t,x) = \begin{cases} |\cos t|(|x|^{p^+} \ln |x| + \frac{1}{p^+ + 1}), & |x| > 1, t \in [1,N]_{\mathbb{Z}}, \\ \frac{1}{p^+ + 1} |\cos t||x|^{p^+ + 1}, & |x| \leq 1, t \in [1,N]_{\mathbb{Z}}. \end{cases}$$

Direct calculations give

$$\lim \inf_{|x| \to \infty} \frac{G(t,x)}{|x|^{p^+}} = +\infty \text{ and } \lim_{|x| \to 0} \frac{G(t,x)}{|x|^{p^+}} = 0, \text{ for any } t \in [1,N]_{\mathbb{Z}}.$$

We see that (G_3), (G_4) and (G_6) are fulfilled.

The structure of this paper is as follows: The second part introduces the knowledge of space theory and related lemmas, the third part presents the main results and proofs.

2. VARIATIONAL STRUCTURE AND SOME LEMMAS

The vector space H_N defined in (3) is an N-dimensional Hilbert space with the inner product

$$\langle u, v \rangle = \sum_{t=1}^{N} \Delta u(t-1) \Delta v(t-1), \ \ \forall u, v \in H_N,$$

while the corresponding norm is given by

$$\|u\| = \left(\sum_{t=1}^{N+1} |\Delta u(t-1)|^2 \right)^{\frac{1}{2}}.$$

We recall some auxiliary results which we use through the paper.

Lemma 7. *(see [15]) For every $u \in H_N$, we have*

(A_1) $$\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)} \geq N^{\frac{p^+ - 2}{2}} \|u\|^{p^+}, \ \ \text{with } \|u\| \leq 1.$$

(A_2) $$\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)} \geq N^{\frac{2-p^-}{2}} ||u||^{p^-} - (N+1), \ \ \text{with } \|u\| > 1.$$

(A_3) $$\sum_{t=1}^{N} |u(t)|^m \leq N(N+1)^{m-1} \sum_{t=1}^{N+1} |\Delta u(t-1)|^m, \ \ \forall m > 1.$$

(A_4) $$\max_{t \in [1,N]_{\mathbb{Z}}} |u(t)| < (N+1)^{\frac{1}{q}} \left(\sum_{t=1}^{N+1} |\Delta u(t-1)|^p \right)^{\frac{1}{p}}, \ \ \forall p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

(A_5) $\sum_{t=1}^{N+1}|\Delta u(t-1)|^m \leq 2^m \sum_{t=1}^{N}|u(t)|^m, \ \ \forall m \geq 2.$

(A_6) $\sum_{t=1}^{N+1}|\Delta u(t-1)|^{p(t-1)} \leq (N+1)||u||^{p^+} + (N+1).$

(A_7) $\sum_{t=1}^{N+1}|\Delta u(t-1)|^m \leq (N+1)||u||^m, \ \ \forall m \geq 1.$

(A_8) $\sum_{t=1}^{N+1}|\Delta u(t-1)|^m \geq (N+1)^{\frac{2-m}{2}}||u||^m, \ \ \forall m \geq 2.$

Let $u \in H_N$, we consider the functional as follows

$$\Phi(u) = -\sum_{t=1}^{N+1}\frac{1}{p(t-1)}|\Delta u(t-1)|^{p(t-1)} + \sum_{t=1}^{N}G(t,u(t)). \tag{4}$$

It's simple to see that $\Phi \in C^1(H_N,\mathbb{R})$ and its derivative $\Phi'(u)$ at $u \in H_N$ is defined by

$$\Phi'(u).v = -\sum_{t=1}^{N+1}|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Delta v(t-1) + \sum_{t=1}^{N}g(t,u(t))v(t), \tag{5}$$

for any $v \in H_N$.
The summation by parts rule allows us to express Φ' as

$$\Phi'(u).v = \sum_{t=1}^{N}\Big[\Delta(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) + g(t,u(t))\Big]\,v(t),$$

for any $v \in H_N$.
Finding the solutions to the equation (P) is equal to getting the critical point of the functional Φ.

The truncated problem is what we'll focus at next.

$$(P_\pm) \quad \begin{cases} -\Delta(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) = g_\pm(t,u(t)), & t \in [1,N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

where

$$g_\pm(t,x) = \begin{cases} g(t,x) & , \quad \text{if} \pm x \geq 0, \\ 0 & , \quad \text{otherwise}. \end{cases} \tag{6}$$

The positive and negative components of u are denoted by $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$, respectively, for $u \in H_N$.

It is clear to see that $u^+ \geq 0$, $u^- \geq 0$, $u = u^+ - u^-$, $u^+.u^- = 0$, $u^\pm = \frac{1}{2}(|u| \pm u)$ and $u^\pm \leq |u|$.

Lemma 8. *All solutions of* (P_+) *(resp.* (P_-)*) are non-negative (resp. non positive) solutions of* (P)*.*

Proof.
Define
$$\Phi_\pm : H_N \longrightarrow \mathbb{R},$$

$$\begin{aligned}\Phi_\pm(u) &= -\sum_{t=1}^{N+1} \frac{1}{p(t-1)}|\Delta u(t-1)|^{p(t-1)} + \sum_{t=1}^{N} G_\pm(t,u(t))\\ &= -\sum_{t=1}^{N+1} \frac{1}{p(t-1)}|\Delta u(t-1)|^{p(t-1)} + \sum_{t=1}^{N} G(t,u^\pm(t)),\end{aligned}$$

where $G_\pm(t,x) = \int_0^x g_\pm(t,s)ds$.

Firstly we show that $\Delta u^+(t-1)\Delta u^-(t-1) \leq 0$ and

$$|\Delta u^-(t-1)| \leq |\Delta u(t-1)|.$$

Indeed,

$$\begin{aligned}\Delta u^+(t-1)\Delta u^-(t-1) &= (u^+(t)-u^+(t-1))(u^-(t)-u^-(t-1))\\ &= u^+(t)u^-(t) - u^+(t)u^-(t-1)\\ &\quad - u^+(t-1)u^-(t) + u^+(t-1)u^-(t-1)\\ &= -u^+(t)u^-(t-1) - u^+(t-1)u^-(t) \leq 0.\end{aligned}$$

And

$$\begin{aligned}|\Delta u^-(t-1)| &= |u^-(t)-u^-(t-1)|\\ &= |\frac{1}{2}(|u(t)|-u(t)) - \frac{1}{2}(|u(t-1)|-u(t-1))|\\ &= \frac{1}{2}||u(t)|-|u(t-1)|-(u(t)-u(t-1))|\\ &\leq \frac{1}{2}\left[||u(t)|-|u(t-1)||+|u(t)-u(t-1)|\right]\\ &\leq \frac{1}{2}\left[|u(t)-u(t-1)|+|u(t)-u(t-1)|\right]\\ &\leq |\Delta u(t-1)|.\end{aligned}$$

Let u be a solution of (P_+), or equivalently u be a critical point of Φ_+. Taking $v = u^-$ in
$\langle \Phi'_+(u), v\rangle = -\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Delta v(t-1) + \sum_{t=1}^{N} g_+(t,u(t))v(t),$

we have

$$\begin{aligned}\langle \Phi'_{+}(u), u^{-}\rangle &= -\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Delta u^{-}(t-1)\\ &= -\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta(u^{+}(t-1)-u^{-}(t-1))\Delta u^{-}(t-1)\\ &= -\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta u^{+}(t-1)\Delta u^{-}(t-1)\\ &-\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}(\Delta u^{-}(t-1))^2.\end{aligned}$$

Therefore, we deduce that

$$\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\left[-\Delta u^{+}(t-1)\Delta u^{-}(t-1)\right] + \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}(\Delta u^{-}(t-1))^2 = 0.$$

Since,

$$-\Delta u^{+}(t-1)\Delta u^{-}(t-1) \geq 0, \ \ \forall t \in [1, N+1]_{\mathbb{Z}},$$

then, we get

$$|\Delta u(t-1)|^{p(t-1)-2}(\Delta u^{-}(t-1))^2 = 0, \ \ \forall t \in [1, N+1]_{\mathbb{Z}}.$$

On the other hand

$$\begin{aligned}|\Delta u^{-}(t-1)|^{p(t-1)} &= |\Delta u^{-}(t-1)|^{p(t-1)-2}\left(\Delta u^{-}(t-1)\right)^2\\ &\leq |\Delta u(t-1)|^{p(t-1)-2}\left(\Delta u^{-}(t-1)\right)^2 = 0,\end{aligned}$$

for any $t \in [1, N+1]_{\mathbb{Z}}$.
So $u^{-} = 0$ and $u = u^{+}$ is also a critical point of Φ with critical value $\Phi(u) = \Phi_{+}(u)$.

In a similar manner, non-positive solutions of (P) are nontrivial critical points of Φ_{-}. Finished with the proof.

Lemma 9. *(see [20]) Let E be a reflexive Banach space. If a functional $\Phi \in C^1(E, \mathbb{R})$ is weakly lower semi continuous and anti-coercive, i.e. $\lim_{\|u\| \longrightarrow \infty} \Phi(u) = -\infty$, then there exists $\overline{u} \in E$ such that $\Phi(\overline{u}) = \sup_{u\in E} \Phi(u)$ and u is also a critical point of Φ, i.e $\Phi'(\overline{u}) = 0$. Moreover, if Φ is strictly convex, then a critical point is unique.*

Definition 1. *Let E be a real Banach space, and $\Phi \in C^1(E,\mathbb{R})$ is a continuously Fréchet differentiable functional. Φ is said to satisfy the Palais-Smale (PS) condition if every sequence $(u_n) \subset E$, such that $(\Phi(u_n))$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. The sequence (u_n) is called a (PS) sequence.*

Let B_r denote the open ball in E about 0 of radius r and let ∂B_r the denote its boundary.

Lemma 10. *(Mountain Pass Lemma ([24]))*
Let E be a real Banach space and $\Phi \in C^1(E,\mathbb{R})$ satisfy the (PS) condition. If $\Phi(0) = 0$ and

- σ_1) *there exist constants r, $\alpha > 0$ such that $\Phi|_{\partial B_r} \geq \alpha$,*
- σ_2) *there exist $e \in E \backslash B_r$ such that $\Phi(e) \leq 0$.*

Then Φ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \Phi(h(s)),$$

where

$$\Gamma = \{h \in C([0,1], E) \mid h(0) = 0,\ h(1) = e\}.$$

Lemma 11. *(see [2]) Let E be a real Banach space and $\Phi \in C^1(E,\mathbb{R})$ be even, bounded from below, and satisfy the (PS) condition. Suppose that $\Phi(0) = 0$ and there is a set $\Omega \subset E$ such that Ω is homeomorphic to S^{n-1} by an odd map and $\sup_{u\in\Omega}\Phi(u) < 0$, where S^{n-1} is the $n-1$ dimensional unit sphere. Then, Φ has at least n disjoint pairs of nontrivial critical points.*

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Since $\lim\sup_{|x|\to\infty} \dfrac{p^+G(t,x)}{|x|^{p^-}} \leq \eta$, there exists $R > 0$ such that

$$G(t,x) \leq \frac{1}{p^+}(\eta+\varepsilon)|x|^{p^-} \quad \text{for} \quad (t,|x|) \in [1,N]_{\mathbb{Z}} \times]R,+\infty[\,, \tag{7}$$

where

$$0 < \varepsilon < \frac{1}{\left(\sqrt{N}(N+1)\right)^{p^-}} - \eta.$$

Then, by (7) and the continuity of $x \longrightarrow G(t,x)$, there exists $c > 0$ such that

$$G(t,x) \leq \frac{1}{p^+}(\eta+\varepsilon)|x|^{p^-} + c, \quad \forall (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{8}$$

Let $u \in H_N$ with $\|u\| > 1$. According to (8), (A_3) and (A_7), we obtain

$$\begin{aligned}\sum_{t=1}^{N} G(t, u^+(t)) &\leq \frac{1}{p^+}(\eta+\varepsilon)\sum_{t=1}^{N}|u^+(t)|^{p^-} + cN \\ &\leq \frac{1}{p^+}(\eta+\varepsilon)\sum_{t=1}^{N}|u(t)|^{p^-} + cN \\ &\leq \frac{1}{p^+}(\eta+\varepsilon)N(N+1)^{p^-}\|u\|^{p^-} + cN.\end{aligned}$$

Using the preceding inequality and (A_2), we get

$$\Phi_+(u) \leq \frac{N(N+1)^{p^-}}{p^+}\left[(\eta+\varepsilon) - \frac{1}{\left(\sqrt{N}(N+1)\right)^{p^-}}\right]\|u\|^{p^-} + \frac{N+1}{p^+} + cN.$$

Since $\varepsilon < \dfrac{1}{\left(\sqrt{N}(N+1)\right)^{p^-}} - \eta$, then $\Phi_+(u) \to -\infty$ as $\|u\| \to +\infty$. Thus, Φ_+ is anti-coercive and bounded from above, hence there is a maximum point of Φ_+ at some $u_+ \in H_N$ i.e., $\Phi_+(u_+) = \sup\limits_{u \in H_N} \Phi(u)$, which is a critical point of Φ_+.

As a result, the issue (P_+) has a solution that is a non-negative solution to the problem (P) according to the Lemma 8.

We demonstrate that there is also a non-positive solution using Φ_- in a similar manner. The proof of Theorem 1 is now finished.

Proof of Theorem 3. From (G_5), there exist $\rho > 0$ such that

$$G(t,x) \geq \frac{1}{p^-}(G_\star - \varepsilon)|x|^{p^-} \quad \text{for all} \quad (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,\rho],$$

where $\varepsilon > 0$ satisfies

$$\varepsilon < G_\star - 2^{p^-}(N+1)^{\frac{p^-}{2}}. \tag{9}$$

Let $u \in H_N$, $\|u\| \leq r$ with $r = \min\left\{\dfrac{\rho}{\sqrt{N+1}}, 1\right\}$. By (A_4) it follows

$$|u(t)| \leq \max_{t\in[1,N]_{\mathbb{Z}}} |u(t)| \leq \rho, \quad \forall t \in [1,N]_{\mathbb{Z}}.$$

So, by (A_5) and (A_8), we have

$$\sum_{t=1}^{N} G(t,u(t)) \geq \frac{1}{p^-}(G_\star - \varepsilon) \times 2^{-p^-}(N+1)^{\frac{2-p^-}{2}}\|u\|^{p^-}. \tag{10}$$

However, considering that $\|u\| \leq 1$ then $|\Delta u(t-1)| \leq 1$ for any $t \in [1, N+1]_{\mathbb{Z}}$. Therefore, using (A_7) we get

$$\begin{aligned} \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)} &\leq \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p^-} \\ &\leq (N+1)\|u\|^{p^-}. \end{aligned}$$

We combined the previous inequality with (10), we obtain

$$\Phi(u) \geq \frac{2^{-p^-}}{p^-}(N+1)^{\frac{2-p^-}{2}} \left((G_\star - \varepsilon) - 2^{p^-}(N+1)^{\frac{p^-}{2}} \right) \|u\|^{p^-}.$$

Take $\alpha = \dfrac{2^{-p^-}}{p^-}(N+1)^{\frac{2-p^-}{2}} \left((G_\star - \varepsilon) - 2^{p^-}(N+1)^{\frac{p^-}{2}} \right) r^{p^-} > 0$. Then,

$$\Phi(u) \geq \alpha > 0, \quad \forall u \in \partial B_r. \tag{11}$$

The existence of constants $\alpha > 0$ and $r > 0$ such that $\Phi|_{\partial B_r} \geq \alpha$ is also demonstrated by our research. This means that the requirement $\sigma_1)$ of the Mountain Pass Lemma is satisfied by Φ. Clear the $\Phi(0) = 0$ setting. We must confirm all other assumptions in order to use the Mountain Pass Lemma.

We demonstrate the anti-coercive of Φ through contradiction. Let $K \in \mathbb{R}$ and $(u_n) \subset H_N$ such that

$$\|u_n\| \longrightarrow \infty \text{ and } \Phi(u_n) \geq K.$$

Putting $v_n = \dfrac{u_n}{\|u_n\|}$, one has $\|v_n\| = 1$. Since $dim H_N < \infty$, there exists $v \in H_N$ such that

$$\|v_n - v\| \longrightarrow 0, \text{ as } n \to \infty \text{ and } \|v\| = 1.$$

In particular $v \neq 0$, we pose $\Lambda = \{t \in [1, N]_{\mathbb{Z}} /\ v(t) \neq 0\}$.

For $t \in \Lambda$, $|u_n(t)| \longrightarrow \infty$. Using (2), we have

$$\begin{aligned}
K &\le \frac{-1}{p^+}\left[\sum_{|\Delta u_n(t-1)|>1} |\Delta u(t-1)|^{p(t-1)} + \sum_{|\Delta u_n(t-1)|\le 1} |\Delta u(t-1)|^{p(t-1)}\right] \\
&+ \sum_{t=1}^{N} G(t,u_n(t)) \\
&\le \frac{-1}{p^+}\left[\sum_{|\Delta u_n(t-1)|>1} |\Delta u(t-1)|^{p^-} + \sum_{|\Delta u_n(t-1)|\le 1} |\Delta u(t-1)|^{p^+}\right] \\
&+ \sum_{t=1}^{N} G(t,u_n(t)) \\
&= \frac{-1}{p^+}\left[\sum_{t=1}^{N} |\Delta u(t-1)|^{p^-} + \sum_{|\Delta u_n(t-1)|\le 1} |\Delta u(t-1)|^{p^+} - \sum_{|\Delta u_n(t-1)|\le 1} |\Delta u(t-1)|^{p^-}\right] \\
&+ \sum_{t=1}^{N} G(t,u_n(t)) \\
&\le \frac{-1}{p^+}\left[\lambda_1 \sum_{t=1}^{N} |u_n(t)|^{p^+} - (N+1)\right] + \sum_{t=1}^{N}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1 |u_n(t)|^{p^+}\right] \\
&+ \frac{p^-}{(p^+)^2}\lambda_1 \sum_{t=1}^{N} |u_n(t)|^{p^+} \\
&\le \lambda_1 \frac{-1}{p^+}\left(1 - \frac{p^-}{p^+}\right)\sum_{t=1}^{N} |u_n(t)|^{p^+} + \sum_{t=1}^{N}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1 |u_n(t)|^{p^+}\right] \\
&\le \sum_{t\in\Lambda}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1 |u_n(t)|^{p^+}\right] \\
&+ \sum_{t\in[1,N]_{\mathbb{Z}}\setminus\Lambda}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1 |u_n(t)|^{p^+}\right].
\end{aligned}$$

From the condition (G_2), we deduce that

$$\sum_{t\in\Lambda}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1 |u_n(t)|^{p^+}\right] \longrightarrow -\infty, \quad \text{as} \quad n \to \infty.$$

The sequence $(u_n(t))$ is bounded for any $t \in [1,N]_{\mathbb{Z}}\setminus\Lambda$ and G is continuous, then there exists a constant $M \in \mathbb{R}$ such that

$$\sum_{t\in[1,N]_{\mathbb{Z}}\setminus\Lambda}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1 |u_n(t)|^{p^+}\right] \le M.$$

Therefore, we get

$$K \le \sum_{t\in\Lambda}\left[G(t,u_n(t)) - \frac{p^-}{(p^+)^2}\lambda_1|u_n(t)|^{p^+}\right] + M \longrightarrow -\infty, \quad \text{as} \quad n\to\infty.$$

This is absurd. As a result, Φ is anti-coercive toward H_N. In order to guarantee that $\Phi(\overline{u}) < 0$ and that any (PS) sequence (u_n) is bounded, we can choose $\overline{u}$ that is sufficiently large. We can see that Φ fulfills the (PS) requirement since the dimension of H_N is finite. According to the Mountain Pass Lemma, Φ has a critical value

$$c \ge \alpha > 0,$$

where

$$c = \inf_{h\in\Gamma}\max_{s\in[0,1]}\Phi(h(s)),$$

and

$$\Gamma = \{h \in C([0,1], H_N)/\ h(0)=0,\ h(1)=\overline{u}\}.$$

Let $u_1 \in H_N$ be a critical point associated to the critical value c of Φ, i.e., $\Phi(u_1)=c$. Hence, u_1 is nontrivial solution of the problem (P).

Since Φ is anti-coercive and bounded from above, then there is a maximum point of Φ at some $u_2 \in H_N$, i.e., $\Phi(u_2) = \sup_{u\in H_N}\Phi(u)$. Using the preceding equality and (11), we obtain

$$\Phi(u_2) = \sup_{u\in H_N}\Phi(u) \ge \sup_{u\in\partial B_r}\Phi(u) > 0.$$

Hence u_2 is nontrivial solution of the problem (P).

If $u_1 \ne u_2$, then we have two nontrivial solutions u_1 and u_2. Otherwise, suppose $u_1 = u_2$, then $\inf_{h\in\Gamma}\max_{s\in[0,1]}\Phi(h(s)) = \sup_{u\in H_N}\Phi(u)$. Therefore, we have $\Phi(u_1) \le \max_{s\in[0,1]}\Phi(h(s)) \le \Phi(u_2),\ \ \forall h\in\Gamma$.

Since $u_1 = u_2$, we deduce that $\Phi(u_1) = \max_{s\in[0,1]}\Phi(h(s)),\ \forall h\in\Gamma$. By the continuity of $\Phi(h(s))$ with respect to s, $\Phi(0)=0$ and $\Phi(\overline{u})<0$ imply that there exists $s_1 \in\,]0,1[$ such that $\Phi(u_1)=\Phi(h(s_1))$. Choose $h_2, h_3 \in \Gamma$ such that

$$\{h_2(s)\mid\ s\in\,]0,1[\}\cap\{h_3(s)\mid\ s\in[0,1]\} = \varnothing,$$

then there exists $s_2, s_3 \in\,]0,1[$ such that

$$\Phi(h_2(s_2)) = \Phi(h_3(s_3)) = \Phi(u_1) = \max_{s\in[0,1]}\Phi(h(s)).$$

Thus, we get two different critical points of Φ on H_N denoted by $v_2 = h_2(s_2)$, $v_3 = h_3(s_3)$ that are nontrivial solutions of problem (P). The proof of Theorem 3 is complete.

Proof of Theorem 5. Let Φ be defined by (4). Then, it is clear that $\Phi(0) = 0$ and Φ is even by (G_6).
From (G_3), there exists $R > 0$ such that

$$G(t,x) \geq \frac{1}{p^-}(\delta - \varepsilon)|x|^{p^+}, \ \ \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times \left]R, +\infty\right[, \tag{12}$$

where

$$0 < \varepsilon < \delta - 2^{p^+}(N+1)^{\frac{p^+}{2}}. \tag{13}$$

On the other hand, by continuity of $x \longrightarrow G(t,x) - \frac{1}{p^-}(\delta - \varepsilon)|x|^{p^+}$, there exists $d > 0$ such that

$$G(t,x) - \frac{1}{p^-}(\delta - \varepsilon)|x|^{p^+} \geq -d, \ \ \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R]. \tag{14}$$

Thus, we deduce that

$$G(t,x) \geq \frac{1}{p^-}(\delta - \varepsilon)|x|^{p^+} - d, \ \ \forall (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{15}$$

Let $u \in H_N$. According to (A_5), (A_8) and (15), we obtain

$$\begin{aligned} \sum_{t=1}^{N} G(t,u(t)) &\geq \frac{1}{p^-}(\delta - \varepsilon)\sum_{t=1}^{N}|u(t)|^{p^+} - dN \\ &\geq \frac{2^{-p^+}}{p^-}(N+1)^{\frac{2-p^+}{2}}(\delta - \varepsilon)\|u\|^{p^+} - dN. \end{aligned} \tag{16}$$

Using the preceding inequality and (A_6), we get

$$\Phi(u) \geq \frac{N+1}{p^-}\left[2^{-p^+}(N+1)^{\frac{-p^+}{2}}(\delta - \varepsilon) - 1\right]\|u\|^{p^+} - dN - \frac{N+1}{p^-}.$$

Then, in view of (13), $\Phi(u) \to \infty$ as $\|u\| \to \infty$. Thus Φ is bounded from below, coercive and any (PS) sequence (u_n) is bounded. In view of the fact that the dimension of H_N is finite, we see that Φ satisfies the (PS) condition.
Using the condition (G_4), for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$|G(t,x)| \leq \varepsilon|x|^{p^+}, \ \ \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,\eta]. \tag{17}$$

Let $u \in H_N$, $\|u\| \leq \tau$ with $\tau = \min\left\{\frac{\eta}{\sqrt{N+1}}, 1\right\}$. From (A_4) it follows

$$|u(t)| \leq \max_{t \in [1,N]_{\mathbb{Z}}} |u(t)| \leq \eta, \ \ \forall t \in [1,N]_{\mathbb{Z}}.$$

Therefore,

$$|G(t,u(t))| \leq \varepsilon |u(t)|^{p^+}, \ \ \forall t \in [1,N]_{\mathbb{Z}}. \tag{18}$$

Define

$$\Omega = \left\{ u \in H_N \mid \sum_{t=1}^{N} |u(t)|^2 = \frac{\tau^2}{4} \right\}.$$

Let S^{N-1} be the unit sphere in $\mathbb{R}^N$ and define $\Psi : \Omega \longrightarrow S^{N-1}$ by

$$\Psi(u) = \frac{2}{\tau} u.$$

Then, Ψ is an odd homeomorphism between Ω and S^{N-1}. For $u \in \Omega$, by (A_5) clearly $\|u\| \leq \tau$. According to (18), (A_1), (A_3) and (A_7), we obtain

$$\Phi(u) \leq \left[\frac{-1}{p^+} N^{\frac{p^+-2}{2}} + \varepsilon N (N+1)^{p^+} \right] \|u\|^{p^+}.$$

Let us choose $\varepsilon > 0$ such that $\varepsilon < \dfrac{N^{\frac{p^+-4}{2}} (N+1)^{-p^+}}{p^+}$. It follows that $\Phi(u) < 0$ and $\sup\limits_{u \in \Omega} \Phi(u) < 0$.

Hence, all the conditions of Lemma 11 are satisfied, so Φ has at least $2N$ nontrivial critical points, which are nontrivial solutions of the problem (P). The proof of Theorem 5 is now complete. □

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