Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **17** (2023), 249–261.
https://doi.org/10.2298/AADM221211009D

INVERSE PROBLEM FOR DIRAC OPERATORS WITH A CONSTANT DELAY LESS THAN HALF THE LENGTH OF THE INTERVAL

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We study inverse spectral problems for Dirac-type functional-differential operators with a constant delay $a \in [\frac{\pi}{3}, \frac{\pi}{2})$. We consider the asymptotic behavior of eigenvalues and research the inverse problem of recovering operators from two spectra. The main result of the paper refers to the proof that the operator could be recovered uniquely from two spectra in the case $a \in [\frac{2\pi}{5}, \frac{\pi}{2})$, as well as the proof that it is not possible in the case $a \in [\frac{\pi}{3}, \frac{2\pi}{5})$.

1. INTRODUCTION

The paper deals with the boundary value problems $\mathcal{B}_j(a, p, q)$, $j \in \{1, 2\}$, for Dirac-type system of the form

$$By'(x) + Q(x)y(x-a) = \lambda y(x), \quad 0 < x < \pi, \tag{1}$$

$$y_1(0) = y_j(\pi) = 0, \tag{2}$$

where

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \quad Q(x) = \begin{bmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{bmatrix},$$

*Corresponding author. Nebojša Djurić
2020 Mathematics Subject Classification. 34A55, 34K29.
Keywords and Phrases. Dirac-type operator, Constant delay, Functional-differential operator, Inverse spectral problem.

delay $a \in [\frac{\pi}{3}, \frac{\pi}{2})$, $p(x), q(x)$ are complex-valued functions in $L_2(0,\pi)$ and $Q(x) = 0$ on $(0, a)$.

These operators for $a \geq \frac{\pi}{2}$ have been originally studied in the paper [**4**]. The authors have reconstructed complex potentials $p(x), q(x)$ from either complete spectra or from subspectra of two boundary value problems with one common boundary condition and also have established uniform stability of the inverse problem. This paper has opened a new chapter in studying inverse spectral problems for operators with a constant delay, which so far have been mainly related to inverse problems for Sturm-Liouville operators with a constant delay

$$ly := -y''(x) + q(x)y(x-a) = \lambda y(x). \tag{3}$$

Inverse problems for Sturm-Liouville operators have been studied in detail in the papers [**23, 13, 29, 15, 2, 6, 24, 11, 7, 25, 27, 26, 8, 9, 10, 5, 28**], and the extension of these results to other types of operators is of great importance. It has been shown that for $a \in (\frac{2\pi}{5}, \pi)$, two spectra of boundary value problems for equation (3) uniquely determine the complex potential $q(x)$ vanishing on $(0, a)$. Although a similar result was expected to be valid for $a \in (0, \frac{2\pi}{5})$, in the recent papers [**8, 9, 10**], it has been proven that it is not true. This is a quite unexpected result especially considering the results of recovering the potential from two spectra for classical Sturm-Liouville operators, see [**18, 12, 16**]. Therefore, it is important to investigate and explain the obtained results for operators with a delay in comparison to the classical operator. It would be also interesting to investigate whether the same behavior is valid for inverse problems for Dirac operators with a delay.
It is known that in the classical case for Dirac operators, $a = 0$, two spectra are enough to recover potentials $p(x), q(x)$ for boundary value problem (1)-(2), see [**16, 14, 17, 20**]. In the case $a \geq \frac{\pi}{2}$, the dependence of the characteristic functions of the problems $\mathcal{B}_j(a, p, q)$ on $Q(x)$ is linear. As we mentioned before, in the paper [**4**] it has been proven that two spectra or subspectra are enough to recover $Q(x)$. The nonlinear case $a \in [\frac{\pi}{3}, \frac{\pi}{2})$ is more challenging and this is the problem we deal with in this paper.
Differential equations with deviating arguments appear in many problems of natural and technical sciences and main results can be found in [**21, 22, 19**]. Also inverse spectral problems for Dirac-type systems with *integral* delay were studied in [**1**].

The paper is organized as follows. In Section 2 we construct characteristic functions and study asymptotic behavior of eigenvalues. Section 3 is devoted to solving the inverse problem. We show that the solution of inverse problem is not unique in a general case. More precisely, we will show that complex potentials $p(x), q(x)$ for $a \in [\frac{2\pi}{5}, \frac{\pi}{2})$ are uniquely determined from two spectra, but they are not for $a \in [\frac{\pi}{3}, \frac{2\pi}{5})$.

2. CHARACTERISTIC FUNCTIONS

Let us introduce notations

$$A(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad B(t) = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}.$$

Then we have

$$B(0) = B, \quad B \cdot A(t) = B(t), \quad B(t) \cdot Q(x) = Q(x) \cdot B^{-1}(t), \tag{4}$$

where

$$B^{-1}(t) = \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}.$$

Let $\vec{c} = [c_1, c_2]^T$. Equation (1) is equivalent with

$$Y(x,\lambda) = A(\lambda x) \cdot \vec{c} + B \int_a^x A(\lambda(x-t))Q(t)Y(t-a,\lambda)dt. \tag{5}$$

According to (4) we can rewrite equation (5):

$$Y(x,\lambda) = A(\lambda x) \cdot \vec{c} + \int_a^x Q(t)B^{-1}(\lambda(x-t))Y(t-a,\lambda)dt. \tag{6}$$

We denote by $S(x,\lambda)$ the fundamental vector-solution of equation (1) such that

$$S(x,\lambda) = \begin{bmatrix} s_1(x,\lambda) \\ s_2(x,\lambda) \end{bmatrix}, \quad S(0,\lambda) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

From (6) we obtain

$$S(x,\lambda) = S_0(x,\lambda) + \int_a^x Q(t)B^{-1}(\lambda(x-t))S(t-a,\lambda)dt$$

where

$$S_0(x,\lambda) = \begin{bmatrix} \sin \lambda x \\ -\cos \lambda x \end{bmatrix}.$$

We solve this equation by the method of successive approximations and for $a \in [\pi/(N+1), \pi/N)$ obtain the following representation of the fundamental vector solution:

$$\begin{aligned} S(x,\lambda) &= \sum_{k=0}^{N} S_k(x,\lambda), \\ S_k(x,\lambda) &= \int_{ka}^x Q(t)B^{-1}(\lambda(x-t))S_{k-1}(t-a,\lambda)dt, \quad k = \overline{1,N}. \end{aligned} \tag{7}$$

One can calculate

$$S_1(x,\lambda) = \int_a^x Q(t)B^{-1}(\lambda(x-t))S_0(t-a,\lambda)dt = \int_a^x Q(t)\begin{bmatrix}\cos\lambda(x-2t+a)\\ -\sin\lambda(x-2t+a)\end{bmatrix}dt, \tag{8}$$

$$\begin{aligned} S_2(x,\lambda) &= \int_{2a}^x Q(t)B^{-1}(\lambda(x-t))\int_a^{t-a} Q(t_1)\begin{bmatrix}\cos\lambda(t-2t_1)\\ -\sin\lambda(t-2t_1)\end{bmatrix}dt_1dt \\ &= \int_{2a}^x Q(t)\int_a^{t-a} Q(t_1)B(\lambda(x-t))\begin{bmatrix}\cos\lambda(t-2t_1)\\ -\sin\lambda(t-2t_1)\end{bmatrix}dt_1dt \\ &= \int_{2a}^x Q(t)\int_a^{t-a} Q(t_1)\begin{bmatrix}\sin\lambda(x-2t+2t_1)\\ -\cos\lambda(x-2t+2t_1)\end{bmatrix}dt_1dt. \end{aligned} \tag{9}$$

Similar, we obtain

$$S_3(x,\lambda) = \int_{3a}^x Q(t)\int_{2a}^{t-a} Q(t_1)\int_a^{t_1-a} Q(t_2) \begin{bmatrix}\cos\lambda(x-2t+2t_1-2t_2+a)\\ -\sin\lambda(x-2t+2t_1-2t_2+a)\end{bmatrix}dt_2dt_1dt.$$

In this way, using induction, we determine the form of function $S_k(x,\lambda)$ for $k=n$, n even

$$S_k(x,\lambda) = \int_{na}^x Q(t)dt\int_{(n-1)a}^{t-a} Q(t_1)dt_1 ... \int_a^{t_{n-2}-a} Q(t_{n-1})\begin{bmatrix}\sin\lambda(x-2t+2t_1-2t_2+...-2t_{n-2}+2t_{n-1})\\ -\cos\lambda(x-2t+2t_1-2t_2+...-2t_{n-2}+2t_{n-1})\end{bmatrix}dt_{n-1}$$

as well as the form of function $S_k(x,\lambda)$ for $k=m$, m is odd

$$S_m(x,\lambda) = \int_{ma}^x Q(t)dt\int_{(m-1)a}^{t-a} Q(t_1)dt_1 ... \int_a^{t_{m-2}-a} Q(t_{m-1}) \begin{bmatrix}\cos\lambda(x-2t+2t_1-2t_2+...+2t_{m-2}-2t_{m-1}+a)\\ -\sin\lambda(x-2t+2t_1-2t_2+...+2t_{m-2}-2t_{m-1}+a)\end{bmatrix}dt_{m-1}.$$

For $j\in\{1,2\}$, eigenvalues of the problem $\mathcal{B}_j(a,p,q)$ coincide with zeros of the entire function

$$\Delta_j(\lambda) := s_j(\pi,\lambda), \tag{10}$$

which is called *characteristic function* of $\mathcal{B}_j(a,p,q)$. We restrict ourselves to the case $\frac{\pi}{3}\le a<\frac{\pi}{2}$. The case $a<\frac{\pi}{3}$ is more complex and needs separate investigations. Substituting (8),(9) in (7), from (10) we obtain

(11)
$$\Delta_1(\lambda) := s_1(\pi,\lambda) = \sin\lambda\pi + \int_a^\pi p(t)\cos\lambda(\pi-2t+a)\,dt$$
$$-\int_a^\pi q(t)\sin\lambda(\pi-2t+a)\,dt + \int_{2a}^\pi p(t)\int_a^{t-a} p(s)\sin\lambda(\pi-2t+2s)\,ds\,dt$$
$$-\int_{2a}^\pi p(t)\int_a^{t-a} q(s)\cos\lambda(\pi-2t+2s)\,ds\,dt$$
$$+\int_{2a}^\pi q(t)\int_a^{t-a} p(s)\cos\lambda(\pi-2t+2s)\,ds\,dt$$
$$+\int_{2a}^\pi q(t)\int_a^{t-a} q(s)\sin\lambda(\pi-2t+2s)\,ds\,dt,$$

(12)
$$\Delta_2(\lambda) := s_2(\pi,\lambda) = -\cos\lambda\pi + \int_a^\pi p(t)\sin\lambda(\pi-2t+a)\,dt$$
$$+\int_a^\pi q(t)\cos\lambda(\pi-2t+a)\,dt - \int_{2a}^\pi p(t)\int_a^{t-a} p(s)\cos\lambda(\pi-2t+2s)\,ds\,dt$$
$$-\int_{2a}^\pi p(t)\int_a^{t-a} q(s)\sin\lambda(\pi-2t+2s)\,ds\,dt$$
$$+\int_{2a}^\pi q(t)\int_a^{t-a} p(s)\sin\lambda(\pi-2t+2s)\,ds\,dt$$
$$-\int_{2a}^\pi q(t)\int_a^{t-a} q(s)\cos\lambda(\pi-2t+2s)\,ds\,dt.$$

According to (11) and (12), the characteristic functions $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ take the forms

(13)
$$\Delta_1(\lambda) = \sin\lambda\pi + \int_{a/2}^{\pi-a/2} K_1(x)\cos\lambda(\pi-2x)dx - \int_{a/2}^{\pi-a/2} K_2(x)\sin\lambda(\pi-2x)dx,$$

(14)
$$\Delta_2(\lambda) = -\cos\lambda\pi + \int_{a/2}^{\pi-a/2} K_1(x)\sin\lambda(\pi-2x)dx + \int_{a/2}^{\pi-a/2} K_2(x)\cos\lambda(\pi-2x)dx$$

where

(15)
$$K_1(x) = \begin{cases} p(x+a/2), & x\in(a/2,a)\cup(\pi-a,\pi-a/2), \\ p(x+a/2) - \int\limits_{x+a}^{\pi} p(t)q(t-x)dt + \int\limits_{x+a}^{\pi} q(t)p(t-x)dt, & x\in(a,\pi-a), \end{cases}$$

(16)

$$K_2(x) = \begin{cases} q(x + a/2) & , \ x \in (a/2, a) \cup (\pi - a, \pi - a/2), \\ q(x + a/2) - \int\limits_{x+a}^{\pi} p(t)p(t - x)dt - \int\limits_{x+a}^{\pi} q(t)q(t - x)dt, & x \in (a, \pi - a). \end{cases}$$

For the function $f(x) \in L^2(-b, b)$ and $b \in \mathbb{R}$ we will use the following identities:

$$\int_{-b}^{b} f(x) \cos \lambda x \, dx = \frac{1}{2} \int_{-b}^{b} (f(x) + f(-x)) \exp(i\lambda x) \, dx \tag{17}$$

$$\int_{-b}^{b} f(x) \sin \lambda x \, dx = \frac{-i}{2} \int_{-b}^{b} (f(x) - f(-x)) \exp(i\lambda x) \, dx \tag{18}$$

Putting the substitution $\pi - 2x = y$ into (13) and (14) and then using (17) and (18) we obtain

$$\Delta_1(\lambda) = \sin \pi\lambda + \int_{a-\pi}^{\pi-a} w_1(y) \exp(i\lambda y) \, dy,$$

$$\Delta_2(\lambda) = -\cos \pi\lambda + \int_{a-\pi}^{\pi-a} w_2(y) \exp(i\lambda y) \, dy,$$

where

$$w_1(y) = \frac{1}{4}(K_1 + iK_2)\Big(\frac{\pi - y}{2}\Big) + \frac{1}{4}(K_1 - iK_2)\Big(\frac{\pi + y}{2}\Big),$$

$$w_2(y) = \frac{1}{4}(-iK_1 + K_2)\Big(\frac{\pi - y}{2}\Big) + \frac{1}{4}(iK_1 + K_2)\Big(\frac{\pi + y}{2}\Big).$$

Now we consider the asymptotic behavior of eigenvalues of the boundary value problems $\mathcal{B}_j(a, p, q)$, $j \in \{1, 2\}$.

Theorem 1. *The boundary value problems $\mathcal{B}_j(a, p, q)$, $j \in \{1, 2\}$, have infinitely many eigenvalues $\lambda_{n,j}$, $n \in \mathbb{Z}$, of the form*

$$\lambda_{n,j} = n + \frac{1 - j}{2} + \kappa_{n,j}, \tag{19}$$

where for $\kappa_{n,j} \neq 0$ and for $|n| \to \infty$,

(20)

$$\kappa_{n,1} = \frac{1}{\pi} \int_{a/2}^{\pi - a/2} K_1(x) \sin 2nx \, dx + \frac{1}{\pi} \int_{a/2}^{\pi - a/2} K_2(x) \cos 2nx \, dx + \ o(\kappa_{n,1}),$$

$$\kappa_{n,2} = \frac{1}{\pi} \int_{a/2}^{\pi - a/2} K_1(x) \sin(2n - 1)x \, dx + \frac{1}{\pi} \int_{a/2}^{\pi - a/2} K_2(x) \cos(2n - 1)x \, dx + \ o(\kappa_{n,2}).$$

Proof. Using the standard approach involving Rouché's theorem or result from Theorem 1 in [**4**], one can show that eigenvalues $\lambda_{n,j}$, $n \in \mathbb{Z}$, have the form

$$\lambda_{n,j} = n + \frac{1-j}{2} + \zeta_{n,j}, \quad \{\zeta_{n,j}\} \in l_2.$$

Then for $|n| \to \infty$ we have

$$\lambda_{n,1} = n + \zeta_{n,1}, \quad \zeta_{n,1} = o(1). \tag{21}$$

Since $\Delta_1(\lambda_{n,1}) = 0$, from (13) and (21) we obtain

$$\begin{aligned} \sin(n+\zeta_{n,1})\pi + \frac{1}{2}\int_{a-\pi}^{\pi-a} K_1(\frac{\pi-y}{2})\cos(n+\zeta_{n,1})y\,dy \\ -\frac{1}{2}\int_{a-\pi}^{\pi-a} K_2(\frac{\pi-y}{2})\sin(n+\zeta_{n,1})y\,dy = 0. \end{aligned} \tag{22}$$

Further, for $|n| \to \infty$ we have

$$\begin{aligned} \sin(n+\zeta_{n,1})\pi &= (-1)^n\zeta_{n,1}\pi + o(\zeta_{n,1}), \\ \cos(n+\zeta_{n,1})y &= \cos(ny) - \zeta_{n,1}y\sin(ny) + o(\zeta_{n,1}), \\ \sin(n+\zeta_{n,1})y &= \sin(ny) + \zeta_{n,1}y\cos(ny) + o(\zeta_{n,1}). \end{aligned} \tag{23}$$

From (22) and (23) and for $|n| \to \infty$ we obtain

$$\begin{aligned} (-1)^n\zeta_{n,1}\pi + \frac{1}{2}\int_{a-\pi}^{\pi-a} K_1(\frac{\pi-y}{2})\cos ny\,dy \\ -\frac{1}{2}\int_{a-\pi}^{\pi-a} K_2(\frac{\pi-y}{2})\sin ny\,dy + \alpha_n = 0 \end{aligned} \tag{24}$$

where $\alpha_n = o(\zeta_{n,1})$. Then from (24) we obtain

$$\zeta_{n,1} = \frac{1}{\pi}\int_{a/2}^{\pi-a/2} K_1(x)\sin 2nx\,dx + \frac{1}{\pi}\int_{a/2}^{\pi-a/2} K_2(x)\cos 2nx\,dx + o(\zeta_{n,1}),$$

and we conclude that $\lambda_{n,1}$ has the form (19) e.i.

$$\lambda_{n,1} = n + \kappa_{n,1}$$

where $\{\kappa_{n,1}\}$ has the form from (20). In the same one can show that

$$\lambda_{n,2} = n - \frac{1}{2} + \kappa_{n,2}$$

where $\{\kappa_{n,2}\}$ has the form (20). □

The first step in recovering operators from given spectra $\{\lambda_{n,j}\}_{n\in\mathbb{Z}}$ of the problems $\mathcal{B}_j(a,p,q)$, $j = 1, 2$, is construction of characteristic functions. The next lemma holds.

Lemma 2. *The functions $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ are uniquely determined by specifying their zeros. Moreover, following representations hold*

(25)
$$\Delta_1(\lambda) = \pi(\lambda_{0,1}-\lambda)\prod_{|n|\in\mathbb{N}} \frac{\lambda_{n,1}-\lambda}{n}\exp\Big(\frac{\lambda}{n}\Big), \quad \Delta_2(\lambda) = \prod_{n\in\mathbb{Z}} \frac{\lambda_{n,2}-\lambda}{n-1/2}\exp\Big(\frac{\lambda}{n-1/2}\Big).$$

Proof. For $j = 1, 2$, Theorem 5 in [**3**] gives the representation

$$\Delta_j(\lambda) = \alpha_j \exp(\beta_j\lambda)\prod_{n\in\mathbb{Z}} \frac{\lambda_{n,j}-\lambda}{\mu_{n,j}}\exp\Big(\frac{\lambda}{\mu_{n,j}}\Big),$$
$$\mu_{n,0} = \begin{cases} n, & n \neq 0, \\ -1, & n = 0, \end{cases} \qquad \mu_{n,1} = n - \frac{1}{2}, \tag{26}$$

where $\beta_j = 2 - j + \gamma_j$ and

$$\alpha_1 = -\lim_{\lambda\to 0}\frac{\sin\lambda\pi}{\lambda} = -\pi, \quad \alpha_2 = \lim_{\lambda\to 0}\cos\lambda\pi = 1,$$

$$\gamma_1 = \lim_{\lambda\to 0}\frac{d}{d\lambda}\ln\Big(-\frac{\sin\lambda\pi}{\lambda}\Big) = 0, \quad \gamma_2 = \lim_{\lambda\to 0}\frac{d}{d\lambda}\ln\cos\lambda\pi = 0.$$

Hence, formula (25) for $j = 1$ and $j = 2$ takes the forms as in (26), respectively. □

3. SOLUTION OF THE INVERSE PROBLEM

In this section we focus on solving the inverse problem for Dirac-type functional-differential operators with constant delay $a \in [\frac{\pi}{3}, \frac{\pi}{2})$ which we assume to be known.

Inverse Problem 1. Let delay $a \in [\frac{\pi}{3}, \frac{\pi}{2})$. Given the spectra $\{\lambda_{n,j}\}_{n\in\mathbb{Z}}$, $j = 1, 2$, of the problems $\mathcal{B}_j(a, p, q)$, find the functions $p(x)$ and $q(x)$.

In order to solve Inverse problem 1, we need to introduce additional functions

$$\Theta_1(\lambda) = \frac{\Delta_1(\lambda)+\Delta_1(-\lambda)}{2}, \tag{27}$$

$$\Theta_2(\lambda) = \frac{\Delta_2(\lambda)-\Delta_2(-\lambda)}{2}, \tag{28}$$

$$\Theta_3(\lambda) = \frac{-\Delta_1(\lambda)+\Delta_1(-\lambda)}{2} + \sin\lambda\pi, \tag{29}$$

$$\Theta_4(\lambda) = \frac{\Delta_2(\lambda)+\Delta_2(-\lambda)}{2} + \cos\lambda\pi. \tag{30}$$

Theorem 3. *Let delay* $a \in [\frac{2\pi}{5}, \frac{\pi}{2})$. *The spectra* $\lambda_{n,j}$, $j = 1, 2$ *of boundary value problems* $\mathcal{B}_j(a, p, q)$ *uniquely determine functions* $p(x)$ *and* $q(x)$.

Proof. From (27),(28),(29),(30) we can calculate $K_1(x)$ and $K_2(x)$. It holds

$$K_1(x) = \sum_{n\in\mathbb{Z}} \left(\frac{(-1)^n}{\pi} \Theta_1(n) + \frac{i(-1)^n}{\pi} \Theta_2(n) \right) e^{2inx}$$

$$K_2(x) = \sum_{n\in\mathbb{Z}} \left(\frac{(-1)^n}{\pi} \Theta_3(n) + \frac{i(-1)^n}{\pi} \Theta_4(n) \right) e^{2inx}$$

Since $K_1(x)$, $K_2(x)$ are known functions, from (15) and (16) for $x \in (a, \pi)$ we obtain next integral equations:

$$\begin{aligned} K_1(x - a/2) = p(x) &+ \int_a^{\pi - x + a/2} p(y)q(y + x - a/2)dy \, \mathbf{1}_{[3a/2, \pi - a/2]}(x) \\ &- \int_a^{\pi - x + a/2} q(y)p(y + x - a/2)dy \, \mathbf{1}_{[3a/2, \pi - a/2]}(x). \end{aligned} \tag{31}$$

$$\begin{aligned} K_2(x - a/2) = q(x) &- \int_a^{\pi - x + a/2} p(y)p(y + x - a/2)dy \, \mathbf{1}_{[3a/2, \pi - a/2]}(x) \\ &- \int_a^{\pi - x + a/2} q(y)q(y + x - a/2)dy \, \mathbf{1}_{[3a/2, \pi - a/2]}(x). \end{aligned} \tag{32}$$

From (31),(32) we can calculate $p(x)$ and $q(x)$ a.e. on interval $[a, 3a/2] \cup (\pi - a/2, \pi)$. After that, we notice that for $x \in [3a/2, \pi - a/2]$ functions

$$\int_a^{\pi - x + a/2} p(y)q(y + x - a/2)dy, \quad \int_a^{\pi - x + a/2} q(y)p(y + x - a/2)dy,$$
$$\int_a^{\pi - x + a/2} p(y)p(y + x - a/2)dy, \quad \int_a^{\pi - x + a/2} q(y)q(y + x - a/2)dy$$

depend of $p(x)$ and $q(x)$ from intervals $[a, 3a/2] \cup [\pi - a/2, \pi]$. Therefore, we can also calculate the functions $p(x)$ and $q(x)$ a.e. on interval $(3a/2, \pi - a/2)$. □

Remark 4. *Using the idea from the paper [**4**], one can show that for* $a \in [\frac{2\pi}{5}, \frac{\pi}{2})$, *potentials* $p(x)$ *and* $q(x)$ *are uniquely determined by appropriate subspectra* $\lambda_{n_k,j}$ *of boundary value problems* $\mathcal{B}_j(a, p, q)$, $j = 1, 2$.

By virtue of Theorem 3 the uniqueness theorem is valid for the delay $a \in [\frac{2\pi}{5}, \frac{\pi}{2})$. In the following, we will show that this is not the case for the delay $a \in [\frac{\pi}{3}, \frac{2\pi}{5})$. For this purpose we use the idea of the construction of the counterexample from the paper [**9**]. For fixed $a \in [\frac{\pi}{3}, \frac{2\pi}{5})$ let us define the integral operator

$$Mf(x) = \int_{3a/2}^{\pi - x + a/2} f(t)h(t + x - a/2)dt, \ \ x \in (\frac{3a}{2}, \pi - a),$$

where

$$h(x) = \frac{6\pi}{(2\pi - 5a)\sqrt{10}} \sin \frac{\pi\sqrt{10}(\pi - x)}{2\pi - 5a}, \ \ x \in (\frac{3a}{2}, \pi - a).$$

Operator M has eigenvalues $(-1)^\eta, \eta = 0, 1$, with corresponding eigenfunctions

$$e_0(x) = \sin \frac{2\pi(2x - 3a)}{2\pi - 5a} + 2 \sin \frac{\pi(2x - 3a)}{2\pi - 5a},$$

$$e_1(x) = \cos \frac{2\pi(2x - 3a)}{2\pi - 5a} + \cos \frac{\pi(2x - 3a)}{2\pi - 5a}, \ \ x \in (\frac{3a}{2}, \pi - a).$$

Now we construct the family of functions

$$D = \{p_\alpha(x), q_\beta(x) : \alpha, \beta \in \mathbb{C}\}$$

where

$$p_\alpha(x) = \begin{cases} 0, \ \ x \in (0, \frac{3a}{2}) \cup (\pi - a, \pi), \\ \alpha e_1(x), \ \ x \in (\frac{3a}{2}, \pi - a), \end{cases} \tag{33}$$

and

$$q_\beta(x) = \begin{cases} 0, \ \ x \in (0, \frac{3a}{2}) \cup (\pi - a, \frac{5a}{2}), \\ \beta e_0(x), \ \ x \in (\frac{3a}{2}, \pi - a), \\ h(x), x \in (\frac{5a}{2}, \pi). \end{cases} \tag{34}$$

Using this family of functions, we will prove that the solution of Inverse problem 1 is not unique in the general case.

Theorem 5. *Let delay $a \in [\frac{\pi}{3}, \frac{2\pi}{5})$. The spectra $\lambda_{n,j}$ of boundary value problem $\mathcal{B}_j(a, p_\alpha, q_\beta)$ for $j = 1, 2$, is independent of α and β.*

Proof. Taking into account the form of the functions p_α and q_β, from (33) and (34) we obtain

$$K_1(x - a/2) = \begin{cases} 0 \quad , \ \ x \in (0, \frac{3a}{2}) \cup (\pi - a, \pi), \\ p_\alpha(x) + \int\limits_{3a/2}^{\pi - x + a/2} p_\alpha(t)h(t + x - a/2)dt, \ \ x \in (\frac{3a}{2}, \pi - a), \end{cases}$$

$$K_2(x-a/2)=\begin{cases} 0 , & x\in(0,\frac{3a}{2})\cup(\pi-a,\frac{5a}{2}),\\ q_\beta(x)-\int\limits_{3a/2}^{\pi-x+a/2} q_\beta(t)h(t+x-a/2)dt, & x\in(\frac{3a}{2},\pi-a),\\ h(x) , & x\in(\frac{5a}{2},\pi).\end{cases}$$

Since $e_0(x)$ and $e_1(x)$ are eigenfunctions of the operator M corresponding to the eigenvalues 1 and -1 respectively, we obtain

$$K_1(x-a/2)=0,\ \ x\in(a,\pi)$$

and

$$K_2(x-a/2)=\begin{cases} 0, & x\in(a,\frac{5a}{2}),\\ h(x), & x\in(\frac{5a}{2},\pi).\end{cases}$$

Then, using (15) and (16) we conclude that characteristic functions for family of functions

$$D=\{p_\alpha(x),q_\beta(x):\alpha,\beta\in\mathbb{C}\}$$

have the form

$$\Delta_1(\lambda)=\sin\lambda\pi-\int_{5a/2}^{\pi}h(x)\sin\lambda(\pi-2x+a)dx,$$

$$\Delta_2(\lambda)=-\cos\lambda\pi+\int_{5a/2}^{\pi}h(x)\cos\lambda(\pi-2x+a)dx,$$

i.e. they are independent of α and β. □

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(Received 11. 12. 2022.)
(Revised 19. 04. 2023.)

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